MAJORANTS IN VARIATIONAL INTEGRATION

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In Perron integration, majorants are usually functions of points. If the domain of definition is a Euclidean space of n dimensions, we can define a finitely additive n-dimensional majorant rectangle function by taking suitable differences of the majorant point function with respect to each of the n co-ordinates. The way is then open to a generalization, in that we need only suppose that the majorant rectangle function is finitely superadditive. Similarly, we need only suppose that a minorant rectangle function is finitely subadditive. These kinds of rectangle functions were used by J. Mařík **(5)** to prove the Fubini theorem for Perron integrals in Euclidean space of $m + n$ dimensions. He also proved that for a function that is Perron, and absolutely Perron, integrable, the majorant and minorant rectangle functions can be taken to be finitely additive. As a result he posed the following problem.

(4, 9.1**)**. Does there exist a two-variable function f that is Perron-integrable using finitely superadditive majorants and finitely subadditive minorants, but that is not Perron-integrable using finitely additive majorants and minorants only?

A further problem was posed by K. Karták.

(4, 9.2**)**. Can the Perron integral fail to exist when we restrict the majorants and minorants to be continuous?

In one dimension the question corresponding to **(4**, 9.2**)** has been answered in the negative by Saks **(6**, pp. 250–251, Theorems 3.9, 3.11**)**. The question now arises of whether these last proofs could be shortened by omitting all reference to Denjoy integration.

We can put questions of this type into a more general setting by replacing the Perron integral by the variational integral. This is possible in one dimension, for the Perron integral of f is equivalent to the Ward integral of f with respect to x **(7**, p. 587**)**, which in turn corresponds to the variational integral of $f.m(.)$, where mI is the length of the interval I **(3**, pp. 123–126 or **1**, pp. 45–46**)**. The proofs assume finitely additive majorant and minorant interval functions, but can easily be extended to deal with finitely superadditive majorant and finitely subadditive minorant interval functions. Then the question of whether we need use only finitely additive (or continuous) majorant and minorant interval functions for the Perron integral of f is equivalent to the question of whether

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we need only use finitely additive (or continuous) interval functions χ in the definition of the variational integral of $f.m(.)$.

For more general spaces we have similar results. The χ for two dimensions is a finitely superadditive rectangle function, as in (**3**, Chapter 6), and the general χ of (**2**, p. 114) is a finitely superadditive set function, and it is easy to show the connection between the variational integrals of special kinds of set functions and the corresponding Perron integrals. Thus we can generalize (**4**, 9.1 and 9.2) in the form of the following questions.

(1) What is the class of variationally integrable **h** for which the interval or set function χ can be taken to be finitely additive?

(2) What is the class of variationally integrable **h** for which χ can be taken to be continuous?

(3) What is the class of variationally integrable **h** for which χ can be taken to be finitely additive and continuous?

In (2, 3) we naturally have to specify the kind of continuity required.

The basic definitions for variational integration in one dimension are as follows. First, we use intervals closed on the left and open on the right, as in (**3**, pp. 17–18), but clashing with (**1**, p. 44) and (**2**, pp. 129–130). This disagreement makes no difference to the integration theory, provided that we define our divisions suitably. Here, a *division* $\mathfrak{D}$ of a closed interval $[a, b]$ is a finite family

$$[x_{j-1}, x_j) \qquad (x_{j-1} < x_j, \quad j = 1, 2, \ldots, n)$$

of intervals such that $x_0 = a$ and $x_n = b$.

A family $\mathfrak{L}$ of intervals $[t, x)$, with *associated points* x, is *left-complete* in $[a, b]$ if to each x in $a < x \leqslant b$ there is a $\delta_1(x) > 0$ such that $[t, x)$ is in $\mathfrak{L}$ for all t in $x - \delta_1(x) \leqslant t < x$. A family $\mathfrak{R}$ of intervals $[x, u)$, with *associated points* x, is *right-complete* in $[a, b]$ if to each x in $a \leqslant x < b$ there is a $\delta_2(x) > 0$ such that $[x, u)$ is in $\mathfrak{R}$ for all u in $x < u \leqslant x + \delta_2(x)$. If $\mathfrak{L}$ is left-complete and $\mathfrak{R}$ right-complete in $[a, b]$, we say that $\mathbf{A} = \{\mathfrak{L}, \mathfrak{R}\}$ is *complete* in $[a, b]$.

If **A** is complete in $[a, b]$, then we can construct a division of $[a, b]$ from the intervals of $\mathfrak{L}$ and $\mathfrak{R}$ (**3**, Theorem 16.1, p. 22 or **2**, Theorem 16, p. 129).

We use functions of intervals $[v, w)$, so that we can write $h(v, w)$ in place of $h([v, w))$. An interval function χ is *finitely superadditive* in $[a, b]$ if

$$\chi(u, v) + \chi(v, w) \leqslant \chi(u, w) \qquad (\textit{all } u, v, w \textit{ with } a \leqslant u < v < w \leqslant b).$$

If equality always occurs, we say that χ is *finitely additive.* If $-\chi$ is finitely superadditive, we say that χ is *finitely subadditive.*

A pair $\mathbf{h} = \{h_l, h_r\}$ of interval functions is of *bounded variation* (VB*) in $[a, b]$ if there are an **A** complete in $[a, b]$ and a non-negative finitely superadditive interval function χ such that $\chi(a, b)$ is finite and

$$(4) \quad |h_s(I)| \leqslant \chi(I) \qquad (I \subseteq [a, b];\ I \in \mathfrak{L} \textit{ if } s = l, \textit{ and } I \in \mathfrak{R} \textit{ if } s = r).$$

Let $[v, w]$ be an interval contained in $[a, b]$, and let $\mathfrak{D}$ be a division of $[v, w]$

that uses only intervals of $\mathfrak{L}, \mathfrak{R}$. By (4) and finite superadditivity,

$$(\mathfrak{D}) \sum |h_s| \leqslant (\mathfrak{D}) \sum \chi \leqslant \chi(v, w).$$

It follows that

$$\chi_1(v, w) = \sup(\mathfrak{D}) \sum |h_s| \leqslant \chi(v, w), \tag{5}$$

the supremum being taken over all such $\mathfrak{D}$ for the fixed **A**. It is easily shown that χ_1 is non-negative and finitely superadditive, so that χ_1 is the smallest χ for the given **A**, if χ_1 is finite.

The *variation* of **h** in $[a, b]$ is

$$V(\mathbf{h}; [a, b]) = \inf \chi(a, b) = \inf \chi_1(a, b) \tag{6}$$

for all such χ, **A**. If there is no such χ, we write the right-hand side symbolically as $+\infty$. If the infimum is 0 we say that **h** is of *variation zero* in $[a, b]$. Two pairs $\mathbf{h}, \mathbf{h}^*$ of interval functions are *variationally equivalent* in $[a, b]$ if

$$\{h_l - h_l^*, h_r - h_r^*\}$$

is of variation zero in $[a, b]$. If also $h_l^* = h_r^* = H$, finitely additive, then $H(a, b)$ is called the *variational integral* of **h** in $[a, b]$ and written

$$(V) \int_a^b \{h_l, h_r\} = (V) \int_a^b \mathbf{h},$$

and we say that **h** is *variationally integrable* in $[a, b]$.

A pair **h** of interval functions is of *generalized bounded variation* (VBG*) in $[a, b]$, if $[a, b]$ is the union of sets X_n $(n = 1, 2, \ldots)$ for which the pairs of interval functions

$$\{h_l(t, x)\mathrm{ch}(X_n, x), h_r(x, u)\mathrm{ch}(X_n, x)\} \qquad (n = 1, 2, \ldots)$$

are all VB* in $[a, b]$, where $\mathrm{ch}(X, x)$ is the characteristic function of the set X.

The continuity in which we are interested is of the type

$$\chi(v, w) \to 0 \textit{ as } w - v \to 0 \textit{ with } a \leqslant v < w \leqslant b \textit{ and } v \leqslant x \leqslant w, \tag{7}$$

for each fixed x in $[a, b]$.

To show that problem (1) is trivial for interval functions, we put

$$\chi(a, a) = 0, \qquad \chi_2(v, w) = \chi(a, w) - \chi(a, v) \qquad (a \leqslant v < w \leqslant b), \tag{8}$$

so that χ_2 is finitely additive. By the finite superadditivity of χ,

$$\chi_2(v, w) - \chi(v, w) = \chi(a, w) - \chi(a, v) - \chi(v, w) \geqslant 0,$$

$$\chi_2(v, w) \geqslant \chi(v, w), \tag{9}$$

so that χ_2 can replace χ in (4). The infimum in (6) is unaltered since

$$\chi_2(a, b) = \chi(a, b). \tag{10}$$

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Not all variationally integrable $\mathbf{h}$ can have a χ continuous as in (7). For let the sequence $\{a_n\}$ be everywhere dense in a perfect set P^* contained in $[a, b]$, and take

$$b_n > 0, \qquad \sum_{n=1}^{\infty} b_n < \infty,$$

$$h_{l1}(t, x) = \begin{cases} b_n & (t = a_n < x, n = 1, 2, \ldots), \\ 0 & (\textit{otherwise}), \end{cases}$$

$$h_{r1}(x, u) = \begin{cases} b_n & (u = a_n > x, n = 1, 2, \ldots), \\ 0 & (\textit{otherwise}). \end{cases}$$

Given $\mathbf{A}$ complete in $[a, b]$, and an integer n, we put

$$\delta(x) = \min(\delta_1(x), \delta_2(x), \tfrac{1}{2}|x - a_m|),$$

for all integers m in $1 \leqslant m \leqslant n$ such that $a_m \neq x$. Then $\delta(x) > 0$, and can be used instead of δ_1, δ_2 to define $\mathbf{A}_1$ complete in $[a, b]$. Sums over divisions of $[a, b]$ from the corresponding $\mathfrak{L}_1, \mathfrak{R}_1$ will then be not greater than

$$\sum_{m=n+1}^{\infty} 2b_m.$$

As $n \to \infty$, this tends to 0, so that $\mathbf{h}_1$ is of variation zero in $[a, b]$, and its variational integral is 0.

However, for this $\mathbf{h}_1$, and each $\mathbf{A}$, the χ_1 of (5), and so every χ, is discontinuous at some points of P^*. For let Y_n be the set of all x in P^* for which

$$\delta_j(x) \geqslant 1/n \qquad (j = 1, 2).$$

Then P^* is the union of the Y_n, so that by Baire's density theorem there are an interval (v, w) containing points of P^* and an integer n such that Y_n is everywhere dense in $(v, w) \cap P^*$. Each point a_p in $(v, w) \cap P^*$ is therefore a limit-point of Y_n, and either $[a_p, x) \in \mathfrak{L}$ *for* $x \in Y_n, x \to a_p+$, *or* $[x, a_p) \in \mathfrak{R}$ *for* $x \in Y_n, x \to a_p-$, or both. Thus χ_1 is discontinuous at all a_p in $(v, w) \cap P^*$.

If h_s is continuous in the sense of (7), for $s = l, r$, then χ_1 is also continuous (see **3**, Theorem 24.2, pp. 41, 42). But in the simple case corresponding to ordinary Perron integration,

$$h_{l2}(t, x) = f(x)(x - t), \qquad h_{r2}(x, u) = f(x)(u - x);$$

the former need not be continuous as $x \to t+$, since $f(x)$ might conceivably tend to infinity sufficiently rapidly to nullify $x - t \to 0$; and similarly for h_{r2}. To answer (**4**, 9.2), a special proof of the continuity of a finitely additive χ for suitable $\mathbf{A}$ will be needed, and it is contained in Theorem 2. The example of $\mathbf{h}_1$ shows that we cannot prove the continuity of χ_1 for every variationally integrable $\mathbf{h}$ of bounded variation, and also shows that in some sense the conditions imposed in Theorem 1 are the best possible, in order to obtain continuous χ_1.

THEOREM 1. *Let* $\mathbf{h} = \{h_l, h_r\}$ *be a pair of interval functions with variational integral* H *in* $[a, b]$, *and let* $[a, b]$ *be the union of sets* Z_n *with the following properties. For each* t, u *in* (a, b), *apart possibly from a countable set, with a countable closure, and for some complete set* $\mathbf{A}$,

(11) $h_l(t, x) - H(t, x) \to 0$ *as* $x \to t+$, $x \in Z_n$ $(n = 1, 2, \ldots)$, $[t, x) \in \mathfrak{L}$;

(12) $h_r(x, u) - H(x, u) \to 0$ *as* $x \to u-$, $x \in Z_n$ $(n = 1, 2, \ldots)$, $[x, u) \in \mathfrak{R}$;

(13) $h_s(v, w) - H(v, w) \to 0$ *as* $v \to t-$, $w \to t+$, $a < t < b$,

where the associated point lies in Z_n $(n = 1, 2, \ldots)$, *and where* $[v, w) \in \mathfrak{L}$ *when* $s = l$, *or* $[v, w) \in \mathfrak{R}$ *when* $s = r$. *Then in the definition of* H *we need only use continuous majorants* χ.

In particular, conditions (11), (12), (13) *are true if*

(14) $h_l(t, x) \to 0$ *as* $x \to t+$, $x \in Z_n$, $a \leqslant t < b$ $(n = 1, 2, \ldots)$,

and as $t \to x-$, $a < x \leqslant b$, *with* $[t, x) \in \mathfrak{L}$;

(15) $h_r(x, u) \to 0$ *as* $x \to u-$, $x \in Z_n$, $a < u \leqslant b$ $(n = 1, 2, \ldots)$,

and as $u \to x+$, $a \leqslant x < b$, *with* $[x, u) \in \mathfrak{R}$;

(16) $h_s(v, w) \to 0$ *as* $v \to t-$, $w \to t+$, $a < t < b$,

where the associated point lies in Z_n $(n = 1, 2, \ldots)$, *and where* $[v, w) \in \mathfrak{L}$ *when* $s = l$, *or* $[v, w) \in \mathfrak{R}$, *when* $s = r$.

In particular, if χ_3 *is a continuous non-negative finitely superadditive interval function, if* $k(x) \geqslant 1$ *is a point function, if* $\mathbf{A}_2$ *is complete in* $[a, b]$, *and if*

(17) $|h_s(I)| \leqslant k(x)\chi_3(I)$ $\quad (I \in \mathfrak{L}_2, s = l;$ *and* $I \in \mathfrak{R}_2, s = r)$,

the x *being the associated point of* I, *then* (14), (15), (16) *are true.*

Condition (17), with the continuity of χ_3 deleted, is the necessary and sufficient condition in order that $\mathbf{h}$ be VBG* in $[a, b]$ (cf. **3**, Theorem 29.1, p. 56).

By definition of H, for each integer n there are a non-negative finitely superadditive interval function $\chi_{4,n}$ and an $\mathbf{A}_{3,n}$ complete in $[a, b]$, and defined by $\delta_{1,n}(x) > 0$, $\delta_{2,n}(x) > 0$, such that

(18) $|H(I) - h_s(I)| \leqslant \chi_{4,n}(I)$ $\quad (I \in \mathfrak{L}_{3,n}, s = l;$ *and* $I \in \mathfrak{R}_{3,n}, s = r)$,

(19) $$\chi_{4,n}(a, b) < 2^{-n}.$$

We first assume that (11), (12), (13) are respectively true for all t, u in $a \leqslant t < b$, $a < u \leqslant b$, $a < t < b$. We define

(20) $$\chi_{5,n}(v, w) = \sup\{0; (\mathfrak{P}) \textstyle\sum |h_s(I) - H(I)|\} \qquad (a \leqslant v < w \leqslant b),$$

for each finite collection $\mathfrak{P}$ of non-overlapping intervals I in $[v, w)$, such that if $s = l$, then $I = [t, x)$ in $\mathfrak{L} \cap \mathfrak{L}_{3,n+m}$ and x is in Z_n; while if $s = r$, then

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$I = [x, u)$ in $\mathfrak{R} \cap \mathfrak{R}_{3,n+m}$ and x is in Z_n. By (18), (19), (20),

$$0 \leqslant \chi_{5,n}(v, w) \leqslant \chi_{5,n}(a, b) < 2^{-n-m}. \tag{21}$$

We can define $\mathbf{A}_4$ complete in $[a, b]$ by using $\min(\delta_1(x); \delta_{1,n+m}(x))$ for $\mathfrak{L}_4$, and $\min(\delta_2(x); \delta_{2,n+m}(x))$ for $\mathfrak{R}_4$, where n is such that x is in Z_n; and by (20), (21),

$$\chi_6 = \sum_{n=1}^{\infty} \chi_{5,n}$$

can replace $\chi_{4,n}$ in (18), (19), with n replaced by m, and $\mathfrak{L}_{3,n}$, $\mathfrak{R}_{3,n}$ replaced by $\mathfrak{L}_4$, $\mathfrak{R}_4$, respectively. To complete the proof we show that χ_6 is continuous. By (21) we need only prove that each $\chi_{5,n}$ is continuous, and we can use a proof similar to that of (**3**, Theorem 24.2, pp. 41–43).

We now suppose that there is an exceptional set X with $\bar{X}$ countable, such that (11), (12), (13) need not be true if t, u are in X. Let G be the union of all *admissible* intervals (v, w) in (a, b), i.e. those intervals such that $\mathbf{h}$ is variationally integrable with continuous χ in $[v, w]$. Then by the first part,

$$(a, b) \cap \mathfrak{C}G \subseteq \bar{X}, \tag{22}$$

where $\mathfrak{C}G$ is the complement of G. We prove the following.

(23) *If $x_0 = b$, and if $\{x_n\}$ is strictly decreasing in $(a, b]$, with limit a, such that (x_j, x_{j-1}) is admissible for $j = 1, 2, \ldots$, then (a, b) is admissible.*

From (23), from a similar result with strictly increasing $\{x_n\}$, and from Borel's covering theorem, we see that

(24) *each interval of G is admissible.*

It follows that $\mathfrak{C}G$ contains no isolated points, and so is perfect. Since $\bar{X}$ is countable, it can contain no perfect component, so that (22) then implies that $G = (a, b)$, which then is admissible from (24).

To prove (23), let χ_{7j} be a suitable continuous non-negative finitely super-additive χ majorizing $|h_s - H|$ in $[x_j, x_{j-1}]$, with

$$\chi_{7j}(a, b) < \epsilon . 2^{-j-1},$$

and put

$$\chi_8 = \sum_{j=1}^{\infty} \chi_{7j}.$$

Then χ_8 is a suitable continuous χ for $|h_s - H|$ in $(a, b]$, with

$$\chi_8(a, b) < \tfrac{1}{2}\epsilon.$$

We construct a continuous finitely additive χ_9 that is suitable at the point a in $[a, b]$, with

$$\chi_9(a, b) < \tfrac{1}{2}\epsilon.$$

By (**3**, Theorem 21.2 (21.13), p. 33), there is a $\delta > 0$ such that

$$|H(a, u) - h_r(a, u)| < \tfrac{1}{2}\epsilon \qquad (a < u \leqslant a + \delta \leqslant b).$$

Thus we can put

$$\chi_9(a, v) = \sup_{a<u\leqslant a+\delta} \min\left(\frac{v-a}{u-a}\,;1\right)|H(a, u) - h_r(a, u)|, \qquad \chi_9(a, b) < \tfrac{1}{2}\epsilon,$$

and prove (23).

To show that (14), (15), (16) imply (11), (12), (13), we use (18), (19), obtaining

$$\lim_{t\to x-} \sup|H(t, x)| \leqslant 2^{-n} \qquad (a < x \leqslant b),$$

and hence that $H(t, x) \to 0$ as $t \to x-$; and similarly that $H(x, u) \to 0$ as $u \to x+, a \leqslant x < b$. Then

$$H(t, u) = H(t, x) + H(x, u) \to 0 \qquad (a < x < b),$$

and (11), (12), (13) follow, there being no exceptional set of t, u. This set X, with a countable $\bar{X}$, could be added if desired.

To show that (17) implies (14), (15), (16), we need only note the continuity of χ_3, and put

$$x \in Z_n \text{ if } n \leqslant k(x) < n + 1 \qquad (n = 1, 2, \ldots).$$

There remains question (3), in which we require χ to be continuous and finitely additive. Theorem 1 is not strong enough to show the existence of such a χ, and we have to impose a slightly stronger condition than (17).

THEOREM 2. *In Theorem* 1 *let* (17) *be true for a continuous non-negative finitely additive* χ_3. *Then in the definition of the variational integral we need only use continuous finitely additive majorants.*

We have (18), (19) for suitable $\chi_{4,n}$, $\mathbf{A}_{3,n}$. Using a difference as in (8), if necessary, we can assume that $\chi_{4,n}$ is finitely additive. But as in (8), the continuity of χ in the sense (7) does not imply the continuity of χ_2; we cannot assume that $\chi_{4,n}$ is continuous. By (19), it has an at most countable number of discontinuities, so that the union of the sets of discontinuities, for $n = 1, 2, \ldots$, is an at most countable set Y, which can be enumerated as a sequence $\{y_j\}$. We cut out open sets containing the discontinuities.

We write

$$\chi_{4,n} = \chi_{10,n} + \sum_{j=1}^{\infty} J_{j,n}, \tag{25}$$

where $\chi_{10,n}$ is continuous and finitely additive, and where $J_{j,n}$ is finitely additive, and zero except for a possible singularity at y_j. As $\chi_{4,n}$ is bounded and non-negative, we take intervals round y_j with lengths tending to 0, to show that $J_{j,n} \geqslant 0$ for all j, n. By taking intervals around the first k points of Y, we obtain in the limit

$$\chi_{4,n} - \sum_{j=1}^{k} J_{j,n} \geqslant 0, \qquad \text{so that } \chi_{10,n} \geqslant 0.$$

There is no need to alter $\chi_{10,n}$, but we have to majorize the $J_{j,n}$ by continuous finitely additive interval functions. We have

$$J_{j,n}(a, x) = \begin{cases} 0 & (x < y_j), \\ q_{j,n} \geqslant 0 & (x > y_j), \end{cases}$$

and we take $t_{j,n} < y_j < u_{j,n}$, defining

$$K_{j,n}(a, x) = \begin{cases} 0 & (x \leqslant t_{j,n}), \\ q_{j,n} & (x \geqslant y_j), \end{cases}$$

$$L_{j,n}(a, x) = \begin{cases} 0 & (x \leqslant y_j), \\ q_{j,n} & (x \geqslant u_{j,n}). \end{cases}$$

We define $K_{j,n}(a, x)$ to be linear in $t_{j,n} \leqslant x \leqslant y_j$, $L_{j,n}(a, x)$ linear in

$$y_j \leqslant x \leqslant u_{j,n},$$

and then construct $K_{j,n}(v, w)$, $L_{j,n}(v, w)$ by using differences; and we obtain the following:

$$\text{(26)} \qquad \begin{cases} J_{j,n}(v, w) \leqslant K_{j,n}(v, w) & (v \leqslant t_{j,n}, v < w), \\ J_{j,n}(v, w) \leqslant L_{j,n}(v, w) & (w \geqslant u_{j,n}, v < w), \\ J_{j,n}(v, w) \leqslant K_{j,n}(v, w) + L_{j,n}(v, w), \end{cases}$$

if at least one of v, w is outside the interval $(t_{j,n}, u_{j,n})$. The open set

$$G_n = \bigcup_{j=1}^{\infty} (t_{j,n}, u_{j,n})$$

encloses Y, and if the associated point of the interval $[v, w)$ lies outside G_n we see by (18), (25), (26) and the convergence of

$$\sum_{j=1}^{\infty} q_{j,n}$$

that the continuous, non-negative, and finitely additive function

$$\text{(27)} \qquad \chi_{10,n} + \sum_{j=1}^{\infty} (K_{j,n} + L_{j,n})$$

majorizes $\chi_{4,n}$, and so $|H - h_s|$, for intervals of $\mathfrak{L}_{3,n}$ and $\mathfrak{R}_{3,n}$. By (19) we see that (27) is bounded by 2^{1-n}.

By continuity of χ_3, the points $t_{j,n}$, $u_{j,n}$ can now be chosen so that

$$\text{(28)} \qquad \chi_3(t_{j,n}, u_{j,n}) < 2^{-2n-j}.$$

For each interval $[v, w)$ we define

$$V_n(v, w) = V(\chi_3 \operatorname{ch}(G_n; .); [v, w]).$$

Then since χ_3 is continuous, non-negative, and finitely additive, $V_n(v, w)$ is the sum of the differences of χ_3 over the intervals of $G_n \cap [v, w]$, so that, by (28),

(29) $\quad V_n < 2^{-2n}$, *and* V_n *majorizes* χ_3 *in the intervals of* G_n.

By (29) we can further define

$$\chi_{11} = \sum_{n=p}^{\infty} 2^n V_n, \qquad M = \bigcap_{n=p}^{\infty} G_n. \tag{30}$$

To each x in M there corresponds $k(x)$, and so an $n = n(x) \geqslant p$, such that

$$k(x) \leqslant 2^n, \tag{31}$$

and also a $\delta_3(x) > 0$ such that

$$(x - \delta_3(x), x + \delta_3(x)) \subseteq G_n. \tag{32}$$

Let $\mathfrak{L}_5$ be the set of $[t, x)$ in $[a, b]$ such that

$$x \in M, \quad n = n(x), \quad x - t < \delta_3(x), \quad [t, x) \in \mathfrak{L}_2 \cap \mathfrak{L}_{3,n}, \tag{33}$$

and let $\mathfrak{R}_5$ be the set of $[x, u)$ in $[a, b]$ such that

$$x \in M, \quad n = n(x), \quad u - x < \delta_3(x), \quad [x, u) \in \mathfrak{R}_2 \cap \mathfrak{R}_{3,n}. \tag{34}$$

By (17) and (29)–(34),

$$|h_s(v, w)| \leqslant \chi_{11}(v, w) \qquad (s = l, [v, w) \in \mathfrak{L}_5, \textit{or } s = r, [v, w) \in \mathfrak{R}_5). \tag{35}$$

We now have to majorize $|H|$ for the same intervals. First we note that by definition the variational integral H is finitely additive. The continuity of H follows, as in Theorem 1, the deduction of (11), (12), (13) from (14), (15), (16). We define

$$\chi_{12}(a, w) = \sup(\mathfrak{Q}) \sum |H| \tag{36}$$

for all finite sets $\mathfrak{Q}$ of non-overlapping intervals from $\mathfrak{L}_5 \cap \mathfrak{R}_5$ and in $[a, w]$. Clearly χ_{12} is monotone increasing as w increases, and as in (9), the difference of χ_{12} majorizes $|H|$ for all intervals of $\mathfrak{L}_5 \cap \mathfrak{R}_5$. To show that $\chi_{12}(a, b)$ is finite, we use (18), (19), and (33)–(35). Then

$$|H| \leqslant |h_s| + \sum_{n=p}^{\infty} \chi_{4,n} \leqslant \chi_{11} + \sum_{n=p}^{\infty} \chi_{4,n},$$

where $s = l$ for intervals of $\mathfrak{L}_5$, and $s = r$ for intervals of $\mathfrak{R}_5$. Thus we obtain

$$\chi_{12} \leqslant \chi_{11} + \sum_{n=p}^{\infty} \chi_{4,n}, \qquad \chi_{12}(a, b) \leqslant 2^{2-p}. \tag{37}$$

This result does not clash with the fact that usually H is not VB*. For by (17), h_s, and so H, are ACG* with respect to χ_3, while M has "χ_3-measure zero," in the older notation. Thus (37) need not be unexpected.

Defining $\chi_{12}(v, w)$ as a difference, χ_{12} is non-negative and finitely additive. Finally, to show that χ_{12} is continuous in $[a, b]$ we use proofs similar to that of (**3**, Theorem 24.2, pp. 41–42).

For example, from

$$\lim_{t \to x-} \chi_{12}(a, t) < \chi_{12}(a, x) - \epsilon$$

for some $\epsilon > 0$, $a < x \leqslant b$, we deduce that either

$$\limsup_{t\to x-} |H(t, x)| \geqslant \epsilon, \quad \textit{or} \limsup_{t\to x-, v\to w} |H(c, t)| \leqslant |H(w, x)| - \epsilon,$$

for some w, x in $a \leqslant w < x \leqslant b$. Both results are false by the continuity and finite additivity of H.

Combining the above with (35), we see that in $\mathfrak{L}_5 \cup \mathfrak{R}_5$,

$$|h_s - H| \leqslant \chi_{11} + \chi_{12}. \tag{38}$$

Using (27) and (38),

$$\chi_{13} = \sum_{n=p}^{\infty} \chi_{10,n} + \sum_{n=p}^{\infty} \sum_{j=1}^{\infty} (K_{j,n} + L_{j,n}) + \chi_{11} + \chi_{12}$$

majorizes $|h_s - H|$ for some complete set in $[a, b]$, where χ_{13} is continuous and finitely additive, with $\chi_{13}(a, b) < 10.2^{-p}$, thus proving the theorem.

As in Theorem 1 we could have allowed an exceptional set X with countable closure, since in the proof of (23), χ_9 is continuous and finitely additive. But for simplicity in Theorem 2 we omitted mention of X.

It is now a matter of taste whether the given proof answers the last question of the Introduction. The proof omits all mention of the Denjoy integral, and even of the Denjoy extension of (**3**, §48, pp. 118–120), and goes back to the basic definitions. Whether the proof is shorter than the proof given in Saks (**6**) of a special case, together with proofs of the relevant properties of the Denjoy integral, is a debatable point.

Turning now to the case of the plane of points (x_1, x_2), the basic definitions are as follows. In our divisions we use half-open rectangles $([u_1, v_1); [u_2, v_2))$, i.e. $u_\alpha \leqslant x_\alpha < v_\alpha$ $(\alpha = 1, 2)$. Such a rectangle has four vertices that can be numbered clockwise from the vertex (u_1, v_2) and can be used as associated points. For $j = 1, 2, 3, 4$, a family $\mathfrak{C}^j$ of half-open rectangles is *j-complete* in $R = ([a_1, b_1]; [a_2, b_2])$, i.e. the rectangle $a_\alpha \leqslant x_\alpha \leqslant b_\alpha$ $(\alpha = 1, 2)$, if to each point (x_1, x_2) that is the jth vertex of some half-open rectangle in R there corresponds a half-open rectangle $R_j(x_1, x_2)$ in R with jth vertex (x_1, x_2), and called the *defining rectangle* of $\mathfrak{C}^j$ at (x_1, x_2), such that every half-open rectangle in $R_j(x, x_2)$ with jth vertex (x_1, x_2) lies in $\mathfrak{C}^j$. From these families $\mathfrak{C}^j (j = 1, 2, 3, 4)$ it is possible to construct divisions of the main interval (cf. **3**, p. 101, Theorem 41.1). Using divisions of this kind we define the variational integral in the plane. Our functions are functions p of the rectangles $([u_1, v_1); [u_2, u_2))$, so that we can write p as

$$p(u_1, v_1; u_2, v_2).$$

Such a rectangle function p is *finitely superadditive* in R if for each division $\mathfrak{D}$ of each closed rectangle R^* contained in R, and with sides parallel to the axes, we have

$$(\mathfrak{D}) \sum p(u_1, v_1; u_2, v_2) \leqslant p(a_1^*, b_1^*; a_2^*, b_2^*).$$

Here, the half-open rectangles $([u_1, v_1); [u_2, v_2))$ are disjoint, with union $([a_1^*, b_1^*); [a_2^*, b_2^*))$, the closure of this being R^*.

If equality always occurs, we say that p is *finitely additive*, while if $-p$ is finitely superadditive, we say that p is *finitely subadditive*. Note that we cannot restrict our divisions to consist of only two rectangles, since a situation as illustrated by (**3**, p. 103, Figure 1) might occur.

In place of the pair $\mathbf{h} = \{h_l, h_r\}$ of interval functions we use a *rectangle vector* $\mathbf{p} = \{p_j\}$ $(j = 1, 2, 3, 4)$, each component p_j being a rectangle function. Then $\mathbf{p}$ is of *variation zero* in R if, given $\epsilon > 0$, there are an $\mathbf{E} = \{\mathfrak{E}^j\}$ *complete in* R (i.e. $\mathfrak{E}^j$ j-complete in R, for $j = 1, 2, 3, 4$) and a non-negative finitely superadditive rectangle function ξ such that for half-open rectangles R_1,

$$\xi(R) < \epsilon, \tag{39}$$

$$|p_j(R_1)| \leqslant \xi(R_1) \qquad (R_1 \subseteq R, R_1 \in \mathfrak{E}^j, j = 1, 2, 3, 4). \tag{40}$$

The rectangle vector $\mathbf{p}$ is *variationally integrable* in R with *variational integral* P, if P is a finitely additive rectangle function with $\{p_j - P\}$ of variation zero.

The *diameter* $\operatorname{diam}(R_1)$ of a rectangle R_1 is the supremum of the distance between any two of its points. Then the continuity in which we are interested is of the following type:

(41) *For each fixed* (x_1, x_2) *in* R, $\xi(R_1) \to 0$ *as* $\operatorname{diam}(R_1) \to 0$, *with* $R_1 \subseteq R$ *and* $(x_1, x_2) \in \bar{R}_1$.

Given a perfect set P^* in the plane, we can construct an example similar to that for one dimension, for which each ξ is discontinuous at some points of P^*. It is clear also that a theorem and proof analogous to Theorem 1 and its proof are possible, so that I need only give the enunciation of the theorem.

THEOREM 3. *Let* $\mathbf{p}$ *be a rectangle vector with variational integral* P *in* R, *let* X *be a set in* R *with a countable closure, and let* R *be the union of sets* Z_n, *with the following properties. For some complete set* $\mathbf{E}$ *in* R, *each* $j = 1, 2, 3, 4$, *and each* (t_1, t_2) *in* $\mathfrak{C}X$,

$$p_j(R_1) - P(R_1) \to 0 \text{ as } \operatorname{diam}(R_1) \to 0,\ R_1 \in \mathfrak{E}^j, \tag{42}$$

when the jth *vertex of* R_1 *lies in* Z_n, *and with* (t_1, t_2) *fixed in* $\bar{R}_1$. *Then in the definition of* P *we need only use continuous non-negative finitely superadditive majorants* ξ.

In particular, (42) *is true if for each* $j = 1, 2, 3, 4$, *and each* (t_1, t_2) *in* $\mathfrak{C}X$,

$$p_j(R_1) \to 0 \text{ as } \operatorname{diam}(R_1) \to 0,\ R_1 \in \mathfrak{E}^j, \tag{43}$$

when the jth *vertex of* R_1 *lies in* Z_n, *and with* (t_1, t_2) *fixed in* $\bar{R}_1$; *and also when the* jth *vertex of* R_1 *is fixed*.

In particular, if ξ_1 *is a continuous non-negative finitely superadditive rectangle function, if* $k(x_1, x_2) \geqslant 1$ *is a point function, and if*

$$|p_j(R_1)| \leqslant k(x_1, x_2)\xi_1(R_1) \qquad (R_1 \in \mathfrak{E}^j, j = 1, 2, 3, 4), \tag{44}$$

where (x_1, x_2) *is the* jth *vertex of* R_1, *then* (43) *is true*.

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Further, there is a theorem analogous to Theorem 2.

THEOREM 4. *In Theorem* 3 *let* (44) *be true for a continuous non-negative finitely additive* ξ_1. *If in the definition of P we need only use non-negative finitely additive* ξ, *then we can also assume that they are continuous.*

A ξ that is non-negative and finitely additive (even finitely superadditive) with $\xi(R)$ finite can have an at most countable number of points, where ξ is not continuous in the sense of (41). We proceed as in Theorem 2, taking a suitable sequence $\{\xi_{2,n}\}$, and the union of the sets of discontinuities, for $n = 1, 2, 3, \ldots$, is an at most countable set that can be enumerated as a sequence

$$\{(y_{1,m}, y_{2,m})\}.$$

We again cut out open sets containing the discontinuities, and follow the proof of Theorem 2, except that we replace $K_{j,n}$, $L_{j,n}$ by rectangle functions. The use of $K_{j,n}$ in Theorem 2 ensures that for points z not in G_n, and for y_j tending to $z+$, the rise in $K_{j,n}(a, x)$ occurs as x tends to y_j-. Thus the jump of $J_{j,n}$ at y_j is spread over an interval that lies between z and y_j, and not beyond y_j, resulting in the first of the two inequalities lying above (26). Similarly for $L_{j,n}$ and the second inequality, for y_j tending to $z-$.

To obtain similar results in two dimensions, for $(y_{1,m}, y_{2,m})$ approaching a point (z_1, z_2) in a rectangle with jth vertex (z_1, z_2), we have to spread the discontinuity of $\xi_{2,n}$ at $(y_{1,m}, y_{2,m})$ linearly and continuously over a rectangle with j'th vertex $(y_{1,m}, y_{2,m})$ and jth vertex $(t_{1,m}, t_{2,m})$, where the j'th vertex of a rectangle is the one opposite to the jth ,i.e.

$$j' \equiv j + 2 \pmod 4.$$

For simplicity we could assume that the union of the four rectangles so chosen is a square with centre $(y_{1,m}, y_{2,m})$. For example, when $j = 4, j' = 2$, we can put

$$\eta_{n,m}([t_{1,m}, t_{1,m} + \delta(y_{1,m} - t_{1,m})); [t_{2,m}, t_{2,m} + \delta(y_{2,m} - t_{2,m}))) = \delta q_{n,m}$$

for $0 < \delta \leqslant 1$, where $q_{n,m}$ is the jump of $\xi_{2,n}$ at $(y_{1,m}, y_{2,m})$, with $\eta_{n,m} = 0$ for rectangles that do not cross the diagonal from $(t_{1,m}, t_{2,m})$ to $(y_{1,m}, y_{2,m})$, and with $\eta_{n,m}$ finitely additive. The proof then proceeds as before.

The preceding theory shows that in two dimensions the continuity of type (41) does not pose great problems. But even if (41) is satisfied, $\xi(a_1, x_1; a_2, x_2)$ can be discontinuous as a point function; for example, its graph could have a continuous cliff or escarpment. Such a discontinuity, however, does not seem so relevant to the theory as the crude discontinuity implied by the failure of (41).

Also problem (1) does not have a trivial solution like that shown by (8), (9), (10) for interval functions. The corresponding results in two dimensions would include

$$(45)\quad \begin{aligned} p(a_1, v_1; a_2, v_2) - p(a_1, u_1; a_2, v_2) - p(a_1, v_1; a_2, u_2) + p(a_1, u_1; a_2, u_2) \\ \geqslant p(u_1, v_1; u_2, v_2) \qquad (a_\alpha < u_\alpha < v_\alpha \leqslant b_\alpha, \alpha = 1, 2), \end{aligned}$$

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with $p = 0$ for vanishing rectangles. But not every non-negative finitely super-additive p satisfies (45), since for the five successive rectangles in (45), p could take the values 5, 3, 3, 1, 1, with

$$p(u_1, v_1; a_2, u_2) = 1 = p(a_1, u_1; u_2, v_2), \qquad p(a_1, v_1; u_2, v_2) = 2 = p(u_1, v_1; a_2, v_2),$$

these being consistent with finite superadditivity.

To construct a finitely additive rectangle function from ξ we could begin with x_1 alone, writing

$$\xi_3(u_1, v_1; u_2, v_2) = \xi(a_1, v_1; u_2, v_2) - \xi(a_1, u_1; u_2, v_2). \tag{46}$$

By (46), ξ_3 is finitely additive in the x_1-direction, and the finite superadditivity of ξ gives

$$\xi_3 \geqslant \xi. \tag{47}$$

But ξ_3 need not be finitely superadditive. For in the example in which nine values of p are given, the corresponding ξ_3 has values 5, 3, 3, 1, 1, 2, 1, 2, 2, and, in particular,

$$\xi_3(u_1, v_1; a_2, v_2) = 2 < 3 = \xi_3(u_1, v_1; a_2, u_2) + \xi_3(u_1, v_1; u_2, v_2).$$

To remove difficulties of this kind, we could put

$$\xi_4(u_1, v_1; u_2, v_2) = \sup(\mathfrak{D}) \sum \xi_3(x_1, y_1; x_2, y_2), \tag{48}$$

the supremum being taken for all sums of ξ_3 over divisions $\mathfrak{D}$ of $([u_1, v_1]; [u_2, v_2])$, with general rectangle $([x_1, y_1); [x_2, y_2))$. However, ξ_4 could be infinite, and so useless. For example, put

$$p_5(0, n/(n+1); 0, 1/n) = 1 \qquad (n = 1, 2, 3, \ldots),$$

while $p_5 = 0$ otherwise; and let

$$\xi_5(u_1, v_1; u_2, v_2) = \sup(\mathfrak{D}) \sum p_5,$$

with supremum over all divisions $\mathfrak{D}$ of $([u_1, v_1]; [u_2, v_2])$. Clearly ξ_5 is non-negative and finitely superadditive, since

$$\xi_5(u_1, v_1; u_2, v_2) = \begin{cases} 1 & (u_1 \leqslant 0, u_2 \leqslant 0, v_1 \geqslant n/(n+1), v_2 \geqslant 1/n; n = 1, 2, \ldots), \\ 0 & (\textit{otherwise}). \end{cases}$$

Corresponding to ξ_3, we write

$$\xi_6(u_1, v_1; u_2, v_2) = \xi_5(0, v_1; u_2, v_2) - \xi_5(0, u_1; u_2, v_2).$$

In particular,

$$\xi_6(u_1, v_1; 0, 1/n) = 1 \qquad (u_1 < n/(n+1) \leqslant v_1; n = 1, 2, 3, \ldots),$$

and we can take $u_1 = w_n, v_1 = w_{n+1}$, where

$$w_j = \tfrac{1}{2}\{(j^2 - 1) + j^2\}/\{j(j+1)\}.$$

If a division $\mathfrak{D}_1$ of $([0, 1]; [0, 1])$ uses such rectangles, for $n = 1, \ldots, m$, and other rectangles, then for ξ_7 corresponding to ξ_4,

$$(\mathfrak{D}_1) \sum \xi_6 \geqslant m, \qquad \xi_7 = +\infty.$$

However, p_5 is majorized by the finitely additive function

$$\xi_8(u_1, v_1; u_2, v_2) = \begin{cases} 1 & (u_j \leqslant x_j < v_j, j = 1, 2), \\ 0 & (\textit{otherwise}), \end{cases}$$

where (x_1, x_2) is a common point of the rectangles $([0, n/(n+1)); [0, 1/n))$. Thus $x_2 = 0$, $0 \leqslant x_1 < \frac{1}{2}$. This same ξ_8 will serve as a majorant for similar p_5, constructed for rectangles $([y_1, 1]; [0, 1])$ with $0 < y_1 \leqslant x_1$, in place of the original rectangle with $y_1 = 0$, and so for the supremum of such p_5; and there are more complicated examples.

The preceding results are not quite good enough for variational integration, since by choice of $\mathbf{E}$ we can avoid the use of divisions such as $\mathfrak{D}_1$. Thus we put

$$R_k = ([0, 1/k]; [0, 1/k]),$$

$$p_{6,k}(u_1, v_1; u_2, v_2) = 2^{-k} p_5(ku_1, kv_1; ku_2, kv_2),$$

$$p_{7,4} = \sum_{k=1}^{\infty} p_{6,k}, \qquad p_{7,j} = 0 \quad (j \neq 4).$$

Let $\mathbf{E}$ be complete in $([0, 1]; [0, 1])$. Then the only non-zero $p_{7,j}$ are those with $j = 4$ and $(0, 0)$ as fourth vertex of the rectangles. There are two integers K', K such that R_k lies in the defining rectangle R of $\mathfrak{E}^4$ at $(0, 0)$, for all $k \geqslant K'$, but if $k \leqslant K$, the rectangles $([0, n/\{k(n+1)\}]; [0, 1/kn])$ $(n = 1, 2, \ldots)$ cannot lie in R. Thus sums of $p_{7,j}$ over divisions using $\mathfrak{E}^j$ $(j = 1, 2, 3, 4)$ are

$$\leqslant \sum_{k=K}^{\infty} 2^{-k} = 2^{1-K},$$

and the variational integral of $\mathbf{p}_7$ is zero. But if we restrict divisions to be formed from rectangles from $\mathfrak{E}^j$ $(j = 1, 2, 3, 4)$, changing p_5 to $\mathbf{p}_7$, then the corresponding ξ_5 is the least majorant, relative to $\mathbf{E}$ and $\mathbf{p}_7$, that is used in variational integration. From this ξ_5, we construct an ξ_6. As $p_{6,k} \geqslant 0$, it follows that the ξ_6 is not less than the ξ_6 corresponding to a $p_{6,k}$ with $k \geqslant K'$, and the new $\xi_7 = +\infty$. Thus $\mathbf{p}_7$ provides the required *gegenbeispiel* to show that we cannot construct a finitely additive majorant for $\mathbf{p}_7$ by first taking a difference with respect to x_1. However, $\mathbf{p}_7$ is majorized by a finitely additive rectangle function constructed from ξ_8.

Similar results are obtained on taking a difference with respect to x_2. For let

$$\xi_9(u_1, v_1; u_2, v_2) = \xi_5(u_1, v_1; a_2, v_2) - \xi_5(u_1, v_1; a_2, u_2),$$

where $a_2 \leqslant 0$. Then if $L(x)$ denotes the integer part of $x/(1-x)$,

$$\xi_9(u_1, v_1; u_2, v_2) = \begin{cases} 1 & (u_1 \leqslant 0 < v_1, u_2 < 1/L(v_1) \leqslant v_2), \\ 0 & (\textit{otherwise}). \end{cases}$$

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The function corresponding to ξ_7 that is constructed from this is again infinite.

Thus two obvious ways of constructing a finitely additive rectangle function from a finitely superadditive one sometimes lead to useless results. Further, even though in the given example it is clear where to put a discontinuity, to obtain a ξ_8, it may not be clear where to put a discontinuity in a more complicated example, and problem (1) does not have an obvious solution.

A little progress has been made in the simpler situation where a point function is integrated with respect to a simple rectangle function. The difficulty is to avoid the use of the inner variation, since a two-dimensional set of inner variation 0 is not always of variation 0. We begin as follows.

THEOREM 5. *If a pair* $|\mathbf{h}|$ *of interval functions is integrable in* $[a, b]$, *and if* $\mathbf{h}_c$ *is the continuous part of* $\mathbf{h}$, *while* X *is a set such that*

$$IV(\mathbf{h}; [a, b]; X) = 0, \tag{49}$$

then it follows that

$$V(\mathbf{h}_c; [a, b]; X) = 0. \tag{50}$$

For proof we put together (**3**, Ex. 26.1, p. 47, and Theorems 31.2, p. 60; 32.1, p. 65; 32.2, p. 67; 32.3, p. 68).

THEOREM 6. *If* $|\mathbf{h}|$ *is integrable, and if* f *is integrable with respect to* $\mathbf{h}$, *with integral* K, *both in* $[a, b]$, *then the derivative of* K *with respect to* $\mathbf{h}$ *is* f *except in a set* X *satisfying* (50); *and a one-sided derivative of* K *with respect to* $\mathbf{h}$ *is* f *on that side of a singularity of* $\mathbf{h}$ *for which the* h_s *does not tend to* 0 *as the interval length tends to* 0.

We use Theorem 5, together with (**3**, Theorems 35.1, p. 78; 21.2, p. 33).

If $\mathbf{h}_n$ $(n = 1, 2)$ are two pairs of interval functions, their (*Cartesian*) *product* $\mathbf{k}_1 = (\mathbf{h}_1, \mathbf{h}_2)$, a vector with four component rectangle functions, can be defined to be given by

$$k_{1j}(I_1, I_2) = h_{1s}(I_1)h_{2\sigma}(I_2) \qquad (j = 1, s = r, \sigma = l; j = 2, s = l = \sigma; \\ j = 3, s = l, \sigma = r; j = 4, s = r = \sigma).$$

We also define the following interval and rectangle functions:

$$|\mathbf{k}_1| = (|\mathbf{h}_1|, |\mathbf{h}_2|), \qquad V_n(I) = V(\mathbf{h}_n; I),$$

$$q_{ns} = \begin{cases} h_{ns}\, V_n/|h_{ns}| & (h_{ns} \neq 0), \\ V_n & (h_{ns} = 0), \end{cases}$$

$$\mathbf{k}_2 = (\mathbf{q}_1, \mathbf{q}_2), \qquad \mathbf{k}_3 = (\mathbf{h}_{1c}, \mathbf{h}_{2c}),$$

where $\mathbf{h}_{nc}$ is the continuous part of $\mathbf{h}_n$ $(n = 1, 2)$, as in (**3**, Theorem 32.2, p. 67).

THEOREM 7. *If* $\mathbf{h}_n$ *is variationally integrable and* VB* *in* $[a_n, b_n]$ $(n = 1, 2)$, *then* $\mathbf{k}_1$ *is variationally integrable in* $R = ([a_1, b_1]; [a_2, b_2])$, *with the integral the product of the integrals* H_n *of* $\mathbf{h}_n$ $(n = 1, 2)$.

Let χ_{14}^n majorize $|h_{ns} - H_n|$ $(n = 1, 2)$. By (8) we can assume χ_{14}^n to be finitely additive. Then for intervals I_1, I_2 in the appropriate left-complete and right-complete families, and by (**3**, Theorem 31.1 (31.1), p. 59),

$$|h_{1s}(I_1)h_{2\sigma}(I_2) - H_1(I_1)H_2(I_2)| = |(h_{1s} - H_1)(h_{2\sigma} - H_2) + H_1(h_{2\sigma} - H_2) + H_2(h_{1s} - H_1)|$$

$$\leqslant |h_{1s} - H_1| . |h_{2\sigma} - H| + |H_1| . |h_{2\sigma} - H_2| + |H_2| . |h_{1s} - H_1|$$

$$\leqslant \chi_{14}^1(I_1)\chi_{14}^2(I_2) + V_1(I_1)\chi_{14}^2(I_2) + V_2(I_2)\chi_{14}^1(I_1),$$

the last sum being a non-negative finitely additive rectangle function with value for R as small as we please. Hence the result.

Note that if, for example, R is divided as in (**3**, Figure 1, p. 103), and if χ_{14}^n is only finitely superadditive, then we cannot show that the final sum is finitely superadditive for this division.

THEOREM 8. *Let* $|\mathbf{h}_n|$ *be variationally integrable in* $[a_n, b_n]$ $(n = 1,2)$. *Then* $|\mathbf{k}_1|$ *is variationally integrable in* R *with integral*

$$K_1^+(I_1, I_2) = V_1(I_1)V_2(I_2),$$

and $\mathbf{k}_1$ *is variationally equivalent to* $\mathbf{k}_2$.

Theorem 7 and (**3**, Theorem 31.2, p. 60) give the results, noting that

$$|k_{1j} - k_{2j}| = ||k_{1j}| - V_1 V_2| = ||k_{1j}| - K_1^+|.$$

THEOREM 9. *Let* $|\mathbf{h}_n|$ *be variationally integrable in* $[a_n, b_n]$ $(n = 1, 2)$. *If the components of* $\mathbf{k}_4$ *are rectangle functions, then*

$$D(\mathbf{k}_4, \mathbf{k}_1; R; (x_1, x_2)) = D(\mathbf{k}_4, \mathbf{k}_2; R; (x_1, x_2)), \tag{51}$$

or else both do not exist, except for (x_1, x_2) *in a set* C *with*

$$V(\mathbf{k}_3; R; C) = 0. \tag{52}$$

The D-functions in (51) are two-dimensional strong derivatives analogous to derivatives of (**3**, Chapter 4). The result is analogous to a special case of (**3**, Theorem 34.2, p. 75), avoiding inner variation.

First, if X_n is a set with

$$V(\mathbf{h}_{nc}; [a_n, b_n]; X_n) = 0, \tag{53}$$

then $|\mathbf{h}_{nc}|\mathrm{ch}(X_n; .)$ is variationally integrable to 0 in $[a_n, b_n]$, where $\mathrm{ch}(X; .)$ is the characteristic function of X. By Theorem 8, the set C_n of (x_1, x_2) with $x_n \in X_n, a_j \leqslant x_j \leqslant b_j$ $(j \neq n)$ satisfies

$$V(\mathbf{k}_3; R; C_n) = 0 \qquad (n = 1, 2). \tag{54}$$

Secondly, $\mathbf{h}_n$ is variationally equivalent to $\mathbf{q}_n$, by (**3**, Theorem 31.2, and

$$|h_{ns} - q_{ns}| = ||h_{ns}| - V_n|.$$

Hence by Theorem 5 and (**3**, Theorem 34.1 (34.5), p. 74), given $0 < \epsilon < 1$, there are sets X_n^1 satisfying (53), such that for intervals of some $\mathfrak{L}_6^n \cup \mathfrak{R}_6^n$ with associated points not in X_n^1, then

$$m_{ns} \equiv |h_{ns} - q_{ns}|/|h_{ns}| \leqslant \epsilon.$$

Thus if the associated point of the rectangle in question is not in the corresponding $C = C_1^1 \cup C_2^1$, which satisfies (52) by (54), we have

$$\begin{aligned} |k_{1j} - k_{2j}| &= |h_{1s} h_{2\sigma} - q_{1s} q_{2\sigma}| \\ &= |(h_{1s} - q_{1s})(h_{2\sigma} - q_{2\sigma}) + h_{2\sigma}(q_{1s} - h_{1s}) + h_{1s}(q_{2\sigma} - h_{2\sigma})| \\ &\leqslant |h_{1s} h_{2\sigma}|(m_{1s} m_{2\sigma} + m_{1s} + m_{2\sigma}) \leqslant 3\epsilon|k_{1j}|, \end{aligned}$$

for I_n in $\mathfrak{L}_6^n \cup \mathfrak{R}_6^n$. If, now, we have

$$|k_{4j} - D.k_{2j}| \leqslant \epsilon|k_{2j}|,$$

it follows that

$$|k_{4j} - D.k_{1j}| \leqslant \epsilon|k_{1j}| + (|D| + \epsilon)|k_{2j} - k_{1j}| \leqslant (4 + 3|D|)\epsilon|k_{1j}|.$$

Hence if the right-hand side of (51) exists, so does the other side, with equality, except possibly in C. Similarly, if the left-hand side of (51) exists, so does the other side, with equality, except possibly in C.

THEOREM 10. *Let f be a point function in $[a_n, b_n]$ such that*

$$(V)\int_{a_n}^{b_n} f(.)d\mathbf{q}_n, \qquad (V)\int_{a_n}^{b_n} |f(.)|dV_n$$

exist. Then there are a complete set $\mathbf{A}_7$, a function $g_n(x_n) = \pm 1$ of x_n alone, and a set X_n^2 satisfying

(55) $$V(V_n; [a_n, b_n]; X_n^2) = 0,$$

such that if x_n is the associated point of I in $\mathfrak{L}_7 \cup \mathfrak{R}_7$, with the appropriate s, and if x_n is not a singularity of V_n,

(56) $$\text{either } h_{ns}(I) = |h_{ns}(I)|g_n(x_n), \text{ or } f(x_n) = 0, \text{ or } x_n \in X_n^2.$$

If x_n is a singularity of V_n, so that $x_n \notin X_n^2$, then (56) *holds on the side of x_n on which a discontinuity of V_n occurs.*

By (**3**, Theorem 34.3, p. 76 and Example 34.2, p. 78),

$$D(f\mathbf{q}_n; V_n; [a_n, b_n]; x_n)$$

exists for all x_n save those of a set X_n^3 satisfying

$$IV(V_{nc}; [a_n, b_n]; X_n^3) = 0.$$

Removing from X_n^3 the singularities of V_n, and using Theorem 5, we obtain (55). But as $h_{ns}(I)$ is real,

$$f(x_n)q_{ns}(I)/V_n(I) = \begin{cases} f(x_n)h_{ns}(I)/|h_{ns}(I)| = \pm f(x_n) & (h_{ns}(I) \neq 0), \\ f(x_n) & (h_{ns}(I) = 0). \end{cases}$$

This gives (56) when x_n is not a singularity of V_n.

Now the variational integral $H_{3,n}$ of f with respect to $\mathbf{q}_n$ is finitely additive and VB*, so that $H_{3,n}(t, x)$ and $H_{3,n}(x, u)$ tend to finite limits for $t < x < u$, as $t, u \to x$. Hence by using (**3**, Theorem 21.2 (21.12, 21.13), p. 33), we finish the proof.

THEOREM 11. *Let* $|\mathbf{h}_n|$ *be variationally integrable in* $[a_n, b_n]$, *for* $n = 1, 2$, *and let* f *be a point function in* R. *If* f *is variationally integrable in* R *with respect to* $\mathbf{k}_1$, *with integral* K_2, *then the point functions*

$$J_1(x_1) = (V)\int_{a_2}^{b_2} f(x_1, .)\, d\mathbf{h}_2, \qquad J_2(x_2) = (V)\int_{a_1}^{b_1} f(., x_2)\, d\mathbf{h}_1$$

exist, except for x_n *in some* X_n^4 *satisfying*

$$V(\mathbf{h}_n; [a_n, b_n]; X_n^4) = 0 \qquad (n = 1, 2), \tag{57}$$

and also

$$(V)\int_{a_n}^{b_n} J_n\, d\mathbf{h}_n = K_2 \qquad (n = 1, 2).$$

We use (**3**, Theorems 31.2, p. 60; 44.2, pp. 109–110).

THEOREM 12. *Let* $|\mathbf{h}_n|$ *be variationally integrable in* $[a_n, b_n]$ $(n = 1, 2)$, *and let* f *be a point function in* R. *If*

$$K_2(.) = (V)\int f\, d\mathbf{k}_1, \qquad K_2^+(.) = (V)\int |f|\, dV_1\, V_2$$

exist in R, *then for the* g_n *of Theorem* 10, *in an obvious notation*,

$$K_2 = (V)\int f . g_1 . g_2\, dV_1\, V_2.$$

By Theorem 8 and the two-dimensional analogue of (**3**, Theorem 31.3, p. 62), we can replace $\mathbf{k}_1$ by $\mathbf{k}_2$ in K_2. Also, if

$$R^* = ([a_1, b_1]; [\alpha, \beta]) \subseteq R,$$

then Theorem 11 gives

$$K_2(R^*) = (V)\int_{a_1}^{b_1} \left\{(V)\int_\alpha^\beta f(x_1, .)\, d\mathbf{q}_2\right\} d\mathbf{q}_1,$$

$$(V)\int_{a_1}^{b_1} \left|(V)\int_\alpha^\beta f(x_1, .)\, dq_2\right| dV_1 \leqslant (V)\int_{a_1}^{b_1}\left\{(V)\int_\alpha^\beta |f(x_1, .)|\, dV_2\right\} dV_1,$$

the integral on the left existing by (**3**, Theorem 25.2, p. 45). Hence by Theorem 10 there is a set X_1^5 satisfying (55) for $n = 1$, such that if x_1 is not a singularity of V_1, and is not in X_1^5, then for intervals in a neighbourhood of x_1, with x_1 as associated point, and with the appropriate s, either

$$h_{1s}(I) = |h_{1s}(I)| g_1(x_1), \tag{58}$$

or

$$(V)\int_\alpha^\beta f(x_1, .)\, d\mathbf{q}_2 = 0. \tag{59}$$

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We have a similar result when $x_1 \notin X_1{}^5$, x_1 a singularity of V_1 in $[a_1, b_1]$, if we restrict s to correspond to the side or sides on which a discontinuity of V_1 occurs.

We now take a countable number of $\alpha < \beta$, each everywhere dense in $[a_2, b_2]$, and including all the discontinuities of V_2 in $[a_2, b_2]$. The union of the corresponding sets $X_1{}^5$ is another set $X_1{}^6$ satisfying (55) for $n = 1$. Also, as (58) is independent of α, β, (58) is true if at least one choice of the given α, β falsifies (59). If, however, (59) is true for all the chosen pairs $\alpha < \beta$, we use Theorem 5 with (**3**, Theorems 34.2, p. 75; 34.3, p. 76). Then, except in a set $X_2{}^6(x_1)$ satisfying

$$V(V_{2c}; [a_2, b_2]; X_2{}^6(x_1)) = 0,$$

we have

$$D(f(x_1, x_2)\mathbf{q}_2, V_2; [a_2, b_2]; x_2) = 0, \qquad \pm f(x_1, x_2) \to 0, \qquad f(x_1, x_2) = 0.$$

Each discontinuity x_2 of V_2 in $[a_2, b_2]$ occurs as an α and as a β, unless x_2 is at an end of $[a_2, b_2]$, when only one choice holds. Taking $[\alpha, \beta]$ on one side in $[a_2, b_2]$ of x_2 where a discontinuity of V_2 occurs, with $\alpha = x_2$ or $\beta = x_2$, using (**3**, Theorem 21.2 (21.12; 21.13), p. 33), and the finite additivity and VB* property of the integral in (59), we again find that $f(x_1, x_2) = 0$. Thus we prove: (60) Result (58) is true for $x_1 \notin X_1{}^6$, with a possible restriction of s to that side of x_1 where a discontinuity of V_1 occurs in $[a_1, b_1]$; unless $f(x_1, x_2) = 0$ except in a set $X_2{}^6(x_1)$ depending on x_1 and satisfying (55) for $n = 2$.

We now examine the set C_3 of (x_1, x_2) where $f = 0$. By the two-dimensional analogue of (**3**, Theorem 38.2, pp. 90–91), the characteristic function of C_3 is variationally integrable with respect to $V_1 V_2$ in R. Also using Theorem 11, if

$$X_2{}^7(x_1) = \{x_2 \colon (x_1, x_2) \in C_3\}, \qquad f_1(x_1) = V(V_2; [a_2, b_2]; X_2{}^7(x_1)),$$

then f_1 is variationally integrable with respect to V_1 in $[a_1, b_1]$. Hence by (**3**, Theorem 38.2, pp. 90–91), the characteristic function of the set $X_1{}^7$ where

(61) $$f_1(x_1) = V_2(a_2, b_2),$$

is variationally integrable with respect to V_1 in $[a_1, b_1]$. Hence, using Theorem 7 and the two-dimensional analogue of (**3**, Theorem 25.1, p. 43), putting $-h_{sj}$ for h_{sj}, if C_4 is the set where $x_1 \in X_1{}^7$ and $(x_1, x_2) \in C_3$, the characteristic function of C_4 is variationally integrable with respect to $V_1 V_2$. Subtracting this characteristic function from 1, we see that if C_5 is the set where $x_1 \in X_1{}^7$ and $(x_1, x_2) \notin C_3$, the characteristic function of C_5 is also variationally integrable with respect to $V_1 V_2$, and so with integral equal to $V(V_1 V_2; R; C_5)$ over R. For each $x_1 \in X_1{}^7$ let $X_2{}^8(x_1)$ be the set of x_2 where $f(x_1, x_2) \neq 0$, and so where $(x_1, x_2) \in C_5$. Then by Theorem 11, the characteristic function of$X_2{}^8(x_1))$ is variationally integrable, except possibly for a set $X_1{}^8$ of x_1 satisfying (55) with $n = 1$. Since

$$X_2^7(x_1) \cup X_2^8(x_1) = [a_2, b_2], \qquad X_2^7(x_1) \cap X_2^8(x_1) = \emptyset,$$

we use (61) and (**3**, Theorems 19.1, p. 27; 31.2, pp. 60–61) to show that $X_2^8(x_1)$ satisfies (55) with $n = 2$. Thus we can identify $X_2^8(x_1)$ with $X_2^6(x_1)$, and we also have

$$V(V_1 V_2; R; C_5) = 0. \tag{62}$$

From (60 and 62) and the two-dimensional analogue of (**3**, Theorem 31.3, pp. 62–63), we obtain the replacement of $\mathbf{q}_1$ with $g_1(x_1) V_1$. No trouble occurs with the discontinuities on the sides of the lines in two dimensions where the V_n are continuous, since the lines are countable in number.

We can now repeat the proof, interchanging x_1 and x_2, etc., finally showing that we can also replace $\mathbf{q}_2$ by $g_2(x_2) V_2$. Hence the theorem holds.

THEOREM 13. *Under the conditions of Theorem* 12, *suppose that*

$$D(K_2, V_1 V_2; R; (x_1, x_2)) = f(x_1, x_2) g_1(x_1) g_2(x_2),$$

except for a set C_6 *satisfying* (52). *Then*

$$D(K_2, \mathbf{k}_1; R; (x_1, x_2)) = f(x_1, x_2),$$

except for a set C_7 *satisfying* (52).

We use Theorems 9 and 10 to show that, except for a set C_7, if $f(x_1, x_2) \neq 0$,

$$\begin{aligned} D(K_2, \mathbf{k}_1; R; (x_1, x_2)) &= D(K_2, \mathbf{k}_2; R; (x_1, x_2)) \\ &= D(K_2, V_1 V_2; R; (x_1, x_2))/\{g_1(x_1) g_2(x_2)\}. \end{aligned}$$

Hence the result follows.

We have therefore reduced the problem of strong differentiation to one in which the integrator $\mathbf{k}_1$ is of the form $V_1 V_2$, non-negative and finitely additive. In the case when $V_n(u, w) = w - u$ $(n = 1, 2)$ we have the theorem of Jessen, Marcinkiewicz, and Zygmund, (cf. **6**, pp. 147–149). To reduce our problem to this, we put

$$x_{n+2}(a_n) = 0, \quad x_{n+2}(x_n) = V_n(a_n, x_n) \qquad (a_n < x_n \leqslant b_n, n = 1, 2).$$

Note that if $V_n(u_n, w_n) = 0$, then $x_{n+2}(u_n) = x_{n+2}(w_n)$. But nothing is lost since by Theorem 8 the $V_1 V_2$ for

$$([u_1, w_1], [a_2, b_2]), \qquad ([a_1, b_1], [u_2, w_2])$$

are 0. The transformation is the two-dimensional analogue of that in (**3**, Theorem 23.1, p. 35), while the integral is simpler. The ϕ there corresponds to the V_n here, and conditions corresponding to (23.3) are satisfied. But a critical examination of the proof shows that it is assumed that ϕ is continuous. As the

V_n can have discontinuities, we have to allow for these in the transformation. We define

$$f_3(x_3, x_4) = f_3(x_3(x_1), x_4(x_2)) = f(x_1, x_2)g_1(x_1)g_2(x_2),$$

where if (u_n, w_n) is a maximal interval with $x_{n+2}(u_n) = x_{n+2}(w_n)$, the value of f_3 is taken with $x_n = u_n$, disregarding the points of (u_n, w_n), which contribute nothing to the integral. This defines f_3 except for strips due to discontinuities in the V_n.

For example, let x_1 be a discontinuity of V_1 on the right. Then

$$x_3(x_1+) - x_3(x_1) = \lim_{h\to 0+} V_1(x_1, x_1 + h), = V_1(x_1, x_1+),$$

say, and there is a gap in the transformed plane of width $V_1(x_1, x_1+)$. Similarly for discontinuities of V_1 on the left, and discontinuities of V_2. In the gap we define

$$f_3(x_3, x_4(x_2)) = f(x_1, x_2)g_1(x_1)g_2(x_2) \qquad (x_3(x_1) < x_3 \leqslant x_3(x_1+)),$$

so that, for fixed x_2, f_3 is constant in the gap. Similarly for other gaps. Then the contribution of the gap to

$$(V)\int_{R^+} f_3\, d(x_1, x_2), \qquad R^+ = ([0, V_1(a_1, b_1)], [0, V_2(a_2, b_2)]),$$

in an obvious notation, is

$$(V)\int_{a_2}^{b_2} f(x_1,.)g_2(.)dV_2.g_1(x_1).\ V_1(x_1, x_1+),$$

which is the contribution of the discontinuity of V_1 on the right of x_1 to

$$(V)\int_R f.g_1.g_2\, dV_1\, V_2.$$

Summing over the discontinuities, the sum being absolutely convergent since the integral is an absolute integral, and using a proof like that of (**3**, Theorem 23.1, p. 35), we find that

$$(V)\int f.g_1.g_2\, dV_1\, V_2 = (V)\int_{R^+} f_3\, d(x_1, x_2).$$

The integral on the right is then equal to the corresponding Lebesgue integral, since it is an absolute integral. To apply the strong differentiation theorem we need a stronger condition than this, namely, the absolute integrability of $f.\log^+|f|$, where

$$\log^+|f| = \max(\log|f|, 0).$$

Further, rectangles tending to (x_1, x_2), where one of the x_n is a discontinuity of the corresponding V_n, sometimes transform into rectangles whose diameters do not tend to 0, because of a gap. Thus we have to deal separately with dis-

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continuities in a way analogous to Theorem 6. Using also Theorems 12, 13, and the strong differentiation theorem, we obtain the following theorem.

THEOREM 14. *Let* $|\mathbf{h}_n|$ *be variationally integrable in* $[a_n, b_n]$ $(n = 1, 2)$, *and let* f *be a point function in* R. *If*

$$K_2(.) = (V)\int f\, d\mathbf{k}_1, \qquad (V)\int |f| \log^+ |f|\, d V_1\, V_2$$

exist in R, *then*

$$D(K_2, \mathbf{k}_1; R; (x_1, x_2)) = f(x_1, x_2)$$

except for a set C_8 *satisfying*

$$V(\mathbf{k}_3; R; C_8) = 0.$$

Also, a single-quadrant derivative of K_2 *with respect to* $\mathbf{k}_1$ *is* f *in that quadrant near to a singularity of* $\mathbf{k}_1$ *for which the* $h_{1s}\, h_{2\sigma}$ *does not tend to* 0 *as the diameter of the rectangle tends to* 0.

Theorem 14 is of interest in itself, since we have widened the scope of the strong differentiation theorem. Note that we have not needed the integrability of the $\mathbf{h}_n$ themselves, so that our result is not a simple transformation of the strong differentiation theorem for Lebesgue integrals.

We use Theorem 14 to obtain finitely additive ξ in some cases.

THEOREM 15. *Under the conditions of Theorem* 14,

$$|K_2 - f.h_{1s}h_{2\sigma}| \tag{63}$$

is majorized by non-negative finitely additive rectangle functions ξ_{10} *with* $\xi_{10}(R)$ *arbitrarily small.*

As $|h_n|$ is variationally integrable, we have

$$||h_{ns}| - V_n| \leqslant \chi_{15}{}^n \qquad (n = 1, 2) \tag{64}$$

for appropriate intervals, where $\chi_{15}{}^n$ is non-negative, and finitely additive with $\chi_{15}{}^n\ (a_n, b_n)$ arbitrarily small, by (8), (9), (10). Now let $\epsilon > 0$. Then by (64) and Theorems 12 and 14, except in a set C_8 satisfying (52), and for rectangles in j-complete families, we have

$$|K_2 - f.h_{1s}\, h_{2\sigma}| \leqslant \epsilon |h_{1s}\, h_{2\sigma}| \leqslant \epsilon (V_1 + \chi_{15}{}^1)(V_2 + \chi_{15}{}^2). \tag{65}$$

The last product is clearly an arbitrarily small non-negative finitely additive rectangle function. Also (65) is true in a quadrant connected to a discontinuity (x_1, x_2) of $\mathbf{k}_1$, in which $h_{1s}\, h_{2\sigma}$ does not tend to 0 as the rectangle tends to its associated point (x_1, x_2).

For the quadrants where $h_{1s}\, h_{2\sigma} \to 0$ we enumerate the discontinuities of $\mathbf{k}_1$ and at the mth discontinuity (x_1, x_2) we put $j_m(R') = \epsilon . 2^{-m}$ when R' has (x_1, x_2) as associated point, and otherwise $j_m(R') = 0$. Then

$$V\left(\sum_{m=1}^{\infty} j_m; R' \right)$$

is a non-negative finitely additive rectangle function majorizing the expression in (63) for these quadrants, and is arbitrarily small.

Hence to prove (63) we need only deal with the points of C_9, those points of the set C_8 satisfying (52) that are not singularities of $\mathbf{k}_1$. Thus C_9 satisfies

(66) $$V(V_1\,V_2; R; C_9) = 0.$$

From (66), given $\epsilon > 0$, there are an $\mathbf{E}_1$ complete in R and a ξ_{11}, non-negative and finitely superadditive, with $\xi_{11}(R) < \epsilon$, such that if R' is in $\mathfrak{E}_1{}^j$ with associated point in C_9, then

(67) $$V_1\,V_2(R') \leqslant \xi_{11}(R').$$

If U is the union of these R', there is an open set G with

(68) $$C_9 \subseteq G \subseteq U, \qquad V(V_1\,V_2; R; G) \leqslant \xi_{11}(R) < \epsilon.$$

For take a mesh like $x_n = (2m+1)2^{-\alpha}$, for $n = 1, 2$, and integer values of m, α, that avoids the singularities of V_1 and of V_2. Rectangles formed from lines of this mesh can be put as a sequence $\{R_m\}$. Let $X_m{}^9$ be the set of points (x_1, x_2) of C_9 such that $R_m \cap R$ lies in the union of the four (or less) defining intervals at (x_1, x_2) with (x_1, x_2) in R_m. If rectangles $R'_1 = R_1, R'_2, \ldots, R'_m$ have been defined, with open union U_m and complement $\mathfrak{C}U_m$ that is the closure of an open set, and if $C_9 \subseteq U_m$, then $U_m = G$ is a suitable set. Otherwise, let m_1 be the first integer such that some point of $X_{m_1}{}^9$ is in $\mathfrak{C}U_m$. Since U_m is a finite union of rectangles, with $\mathfrak{C}U_m$ the closure of an open set, $\mathfrak{C}U_m \cap R_{m_1}$ can be divided into a finite number of rectangles, where, inductively, the sides belong to the mesh. Let $R'_{m+1}, R'_{m+2}, \ldots$ be those rectangles of the finite number that contain points of $X_{m_1}{}^9$ in their closures, including such boundary points with $R'_{m+1}, \ldots$ so that the unions $U_{m+1}, \ldots$ are open and the $\mathfrak{C}U_{m+1}, \ldots$ closures of open sets. Then if (x_1, x_2) is a point of $X_{m_1}{}^9$ in $\bar{R}'_{m+1}$, this rectangle splits into 1, 2, or 4 rectangles by lines through (x_1, x_2) and parallel to the axes, and by the finite superadditivity of ξ_{11}, R'_{m+1} satisfies (67). Thus by induction, $\{U_m\}$ is defined, the limit set being an open set G satisfying (68), since

$$V(V_1\,V_2; R; G) = \lim_{m\to\infty} V(V_1\,V_2; R; U_m) = \sum_{m=1}^{\infty} V_1\,V_2(R_m{}')$$
$$\leqslant \sum_{m=1}^{\infty} \xi_{11}(R_m{}') \leqslant \xi_{11}(R) < \epsilon.$$

In the notation we have disregarded boundary points of R'_m.

Similarly, we can prove that if ξ_{12} is non-negative and finitely additive, if $\epsilon > 0$, and if $X^{10} \subseteq R$, there is an open set $G_1 \supseteq X^{10}$ such that

(69) $$V(\xi_{12}; R; G_1) < V(\xi_{12}; R; X^{10}) + \epsilon.$$

It follows also that by an analogue of (**3**, Theorem 49.1, p. 121), *the outer ξ_{12}-measure of X^{10} is equal to $V(\xi_{12}; R; X^{10})$.*

By (64; 68), $h_{1s}\, h_{2\sigma}(R')$ is majorized at points of C_9 by

$$\xi_{13}(\delta; R') = V(V_1\, V_2; R; G \cap R') + V_1\, \chi_{15}{}^2(R') + \chi_{15}{}^1 V_2(R') + \chi_{15}{}^1\chi_{15}{}^2(R'),$$

this being a non-negative finitely additive rectangle function with arbitrarily small value $\delta \geqslant 0$ for R. To majorize $fh_{1s}\, h_{2\sigma}$ in C_9 we use

$$\xi_{14}(.) = \sum_{m=1}^{\infty} 2^m \xi_{13}(\delta_m; .) \qquad (\delta_m \leqslant \epsilon . 4^{-m}).$$

Also, since $K_2{}^+$ is AC*, by the analogue of **(3**, Theorem 40.1, p. 98**)**, we obtain $V(K_2{}^+; R; C_9) = 0$ from (52), and then from (69) there is an open set $G_2 \supseteq C_9$ such that

$$V(K_2{}^+; R; G_2) < \epsilon.$$

Hence K_2 is majorized in C_9 by

$$V(K_2{}^+; R; G_2 \cap R'), \qquad as\ |K_2| \leqslant K_2{}^+.$$

It follows that the expression in (63) is majorized in C_9 by a non-negative finitely additive rectangle function that is arbitrarily small for R, completing the proof of Theorem 15.

THEOREM 16. *Let* $|\mathbf{h}_n|$ *be variationally integrable in* $[a_n, b_n]$ $(n = 1, 2)$, *and let the point function f in R be such that*

$$K_2(.) = (V)\int f\, d\mathbf{k}_1, \qquad K_2{}^+(.) = (V)\int |f|\, dV_1\, V_2$$

exist in R. Then (63) *is true.*

We replace $|k_{1j}| = |h_{1s}\, h_{2\sigma}|$ by $V_1\, V_2$ in K_2 and (63), by using Theorem 12 and (64). For at the points where $|f| \leqslant 2^m$ we use the $\chi_{15}{}^n$ with

$$\chi_{15}{}^n(\alpha_n; \beta_n) < \epsilon . 4^{-m}.$$

This needs a combination of j-complete families ($j = 1, 2, 3, 4$) analogous to that arranged in **(3**, Example 16.9, p. 24**)**, there being a countable number of j-complete families involved. The combination is also j-complete. This construction has been used several times in the paper. Hence we can assume in the rest of the proof that $h_{ns} = V_n$ $(n = 1, 2)$. Let us put

$$f_1 = m \qquad (m \leqslant f < m + 1; m = 0, \pm 1, \pm 2, \ldots).$$

Then by the analogue of **(3**, Theorem 38.2, pp. 90–91**)**, f_1 is variationally integrable with respect to $V_1\, V_2$, while $0 \leqslant f - f_1 < 1$. Hence $f - f_1$ satisfies the conditions of Theorem 15, so that we can concentrate on f_1. Let $X_m{}^{11}$ be the set where $f_1 = m$. If for a point (x_1, x_2), $f_1 = M$, then for rectangles lying in some $\mathfrak{E}_{2,m}$, with associated point (x_1, x_2), and for non-negative finitely additive rectangle functions $\xi_{15}{}^m$, we have

$$|(V)\int \mathrm{ch}\,(X_M{}^{11}; .)d V_1\, V_2 - V_1\, V_2| \leqslant \xi_{15}{}^M,$$

$$|(V)\int \mathrm{ch}\,(X_m{}^{11}; .)d V_1\, V_2| \leqslant \xi_{15}{}^m \qquad (m \neq M),$$

$$\xi_{15}{}^m(R) < \epsilon/\{(|m| + 1)2^{|m|}\} \qquad (m = 0, \pm 1, \pm 2, \ldots).$$

These results follow from Theorem 15. If, now, we put

$$\mathfrak{S}^j_{3,M} = \bigcap_{m=-N}^{N} \mathfrak{S}^j_{2,m} \cap \mathfrak{S}^j_{2,M}$$

and combine the $\mathfrak{S}_3{}^j{}_{,M}$ in sets $X_M{}^{11}$ to obtain an $\mathfrak{S}_4{}^j$, and if $X_N{}^{12}$ is the set where $|f_1| > N$, we obtain

$$|K_2 - f_1 V_1 V_2| \leqslant \sum_{\substack{m=-N\\ m\neq M}}^{N} |m| \left| (V) \int \mathrm{ch}(X_m{}^{11}; .) d V_1 V_2 \right|$$

$$+ (V) \int |f_1| \mathrm{ch}(X_N{}^{12}; .) d V_1 V_2 + |M| \left| (V) \int \mathrm{ch}(X_M{}^{11}; .) d V_1 V_2 - V_1 V_2 \right|$$

$$\leqslant \sum_{m=-N}^{N} |m| . \xi_{15}{}^m + (V) \int |f_1| \mathrm{ch}(X_N{}^{12}; .) d V_1 V_2 + |M| \xi_{15}{}^M.$$

The last sum is non-negative and finitely additive, while the value for R is less than

$$4\epsilon + (V) \int_R |f_1| \mathrm{ch}(X_N{}^{12}; .) d V_1 V_2.$$

As $N \to \infty$, this last integral tends to 0, so that we have proved the result.

Theorem 16 gives a non-trivial extension of Mařík's result for Lebesgue integrals, non-trivial since we do not assume that the $\mathbf{h}_n$ are variationally integrable. It seems a very difficult question to extend (63) to the case when K_2 exists as a non-absolute integral. One difficulty is the majorization of

$$(V) \int f_1 \, \mathrm{ch}(X_N{}^{12}; .) d V_1 V_2,$$

and a second is the obtaining of a result like Theorem 12. It is possible to extend the results of this paper to higher dimensions. In fact, if a result corresponding to Theorem 16 is true in two spaces, then it seems likely to be true in the Cartesian product of the two spaces, a crucial theorem probably being Theorem 7.

Two results especially have independent interest, namely, the extension of the strong differentiation theorem given in Theorem 14, and the connection between outer measure and variation in special cases, given just after (69).

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