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Metaplectic Tensor Products for Automorphic Representation of $\widetilde{GL}(r)$

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Abstract. Let $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k} \subseteq \mathrm{GL}_r$ be a Levi subgroup of GL_r, where $r = r_1 + \cdots + r_k$, and $\widetilde{M}$ its metaplectic preimage in the n-fold metaplectic cover $\widetilde{\mathrm{GL}}_r$ of GL_r. For automorphic representations $\pi_1, \ldots, \pi_k$ of $\widetilde{\mathrm{GL}}_{r_1}(\mathbb{A}), \ldots, \widetilde{\mathrm{GL}}_{r_k}(\mathbb{A})$, we construct (under a certain technical assumption that is always satisfied when $n = 2$) an automorphic representation π of $\widetilde{M}(\mathbb{A})$ that can be considered as the "tensor product" of the representations $\pi_1, \ldots, \pi_k$. This is the global analogue of the metaplectic tensor product defined by P. Mezo in the sense that locally at each place v, π_v is equivalent to the local metaplectic tensor product of $\pi_{1,v}, \ldots, \pi_{k,v}$ defined by Mezo. Then we show that if all of the π_i are cuspidal (resp. square-integrable modulo center), then the metaplectic tensor product is cuspidal (resp. square-integrable modulo center). We also show that (both locally and globally) the metaplectic tensor product behaves in the expected way under the action of a Weyl group element and show the compatibility with parabolic inductions.

1 Introduction

Let F be either a local field of characteristic 0 or a number field. Let R be F if F is local and the ring of adeles $\mathbb{A}$ if F is global. Consider the group $\mathrm{GL}_r(R)$. For a partition $r = r_1 + \cdots + r_k$ of r, one has the Levi subgroup

$$M(R) := \mathrm{GL}_{r_1}(R) \times \cdots \times \mathrm{GL}_{r_k}(R) \subseteq \mathrm{GL}_r(R).$$

Let $\pi_1, \ldots, \pi_k$ be irreducible admissible (resp. automorphic) representations of $\mathrm{GL}_{r_1}(R), \ldots, \mathrm{GL}_{r_k}(R)$, where F is local (resp. global). Then it is a trivial construction to obtain the representation $\pi_1 \otimes \cdots \otimes \pi_k$, which is an irreducible admissible (resp. automorphic) representation of the Levi subgroup $M(R)$. Though highly trivial, this construction is of great importance in the representation theory of $\mathrm{GL}_r(R)$.

Now if one considers the metaplectic n-fold cover $\widetilde{\mathrm{GL}}_r(R)$ constructed by Kazhdan and Patterson [KP], the analogous construction turns out to be far from trivial. Namely, for the metaplectic preimage $\widetilde{M}(R)$ of $M(R)$ in $\widetilde{\mathrm{GL}}_r(R)$ and representations $\pi_1, \ldots, \pi_k$ of the metaplectic n-fold covers $\widetilde{\mathrm{GL}}_{r_1}(R), \ldots, \widetilde{\mathrm{GL}}_{r_k}(R)$, one cannot construct a representation of $\widetilde{M}(R)$ simply by taking the tensor product $\pi_1 \otimes \cdots \otimes \pi_k$. This is because $\widetilde{M}(R)$ is not the direct product of $\widetilde{\mathrm{GL}}_{r_1}(R), \ldots, \widetilde{\mathrm{GL}}_{r_k}(R)$; namely,

$$\widetilde{M}(R) \ncong \widetilde{\mathrm{GL}}_{r_1}(R) \times \cdots \times \widetilde{\mathrm{GL}}_{r_k}(R);$$

and even worse, there is no natural map between them.

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When F is a local field, for irreducible admissible representations $\pi_1, \dots, \pi_k$ of $\widetilde{\mathrm{GL}}_{r_1}(F), \dots, \widetilde{\mathrm{GL}}_{r_k}(F)$, P. Mezo [Me], whose work, we believe, is based on the work by Kable [K2], constructed an irreducible admissible representation of the Levi subgroup $\widetilde{M}(F)$ that can be called the "metaplectic tensor product" of $\pi_1, \dots, \pi_k$, and characterized it uniquely up to certain character twists. (His construction will be reviewed and expanded further in Section 4.)

The theme of the paper is to carry out a construction analogous to Mezo's when F is a number field, and our main theorem is the following.

Main Theorem *Let $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$ be a Levi subgroup of GL_r, and let $\pi_1, \dots, \pi_k$ be unitary automorphic subrepresentations of $\widetilde{\mathrm{GL}}_{r_1}(\mathbb{A}), \dots, \widetilde{\mathrm{GL}}_{r_k}(\mathbb{A})$. Assume that M and n are such that Hypothesis $(*)$ (see Section 3.4) is satisfied, which is always the case if $n = 2$. Then there exists an automorphic representation π of $\widetilde{M}(\mathbb{A})$ such that $\pi \cong \widetilde{\otimes}'_v \pi_v$, where each π_v is the local metaplectic tensor product of Mezo. Moreover, if $\pi_1, \dots, \pi_k$ are cuspidal (resp. square-integrable modulo center), then π is cuspidal (resp. square-integrable modulo center).*

In the above theorem, $\widetilde{\otimes}'_v$ indicates the metaplectic restricted tensor product, the meaning of which will be explained later in the paper. The existence and the local-global compatibility in the main theorem are proved in Theorem 5.9, and the cuspidality and square-integrability are proved in Theorems 5.12 and 5.13, respectively.

Let us note that by π_i unitary, we mean that π_i is equipped with a Hermitian structure invariant under the action of the group. We also require that π_i be an automorphic subrepresentation, so that it is realized in a subspace of automorphic forms and hence each element in π_i is indeed an automorphic form. (Note that an automorphic representation is usually a subquotient.) We need those two conditions for technical reasons, and they are satisfied if π_i is in the discrete spectrum, namely, π_i is either cuspidal or residual.

We should also emphasize that if $n > 2$, we do not know if our construction works unless we impose a technical assumption as in Hypothesis $(*)$. We will show in Appendix A that this assumption is always satisfied if $n = 2$, and if $n > 2$ it is satisfied, for example, if $\gcd(n, r - 1 + 2cr) = 1$, where c is the parameter to be explained. We hope that it is always satisfied even for $n > 2$, though at present we do not know how to prove it.

Strictly speaking the metaplectic tensor product of $\pi_1, \dots, \pi_k$ might not be unique even up to equivalence but is dependent on a character ω on the center $Z_{\widetilde{\mathrm{GL}}_r}$ of $\widetilde{\mathrm{GL}}_r$. Hence we write

$$\pi_\omega := (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$$

for the metaplectic tensor product to emphasize the dependence on ω.

We will also establish a couple of important properties of the metaplectic tensor product both locally, in Section 4, and globally, in Section 5. The first one is that the metaplectic tensor product behaves in the expected way under the action of the Weyl group.

Theorem (4.8 and 5.19) *Let $w \in W_M$ be a Weyl group element of GL_r that only permutes the GL_{r_i}-factors of M. Namely for each $(g_1, \dots, g_k) \in \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$, we have $w(g_1, \dots, g_k)w^{-1} = (g_{\sigma(1)}, \dots, g_{\sigma(k)})$ for some permutation $\sigma \in S_k$ of k letters. Then both locally and globally, we have*

$$^{w}(\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega \cong (\pi_{\sigma(1)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{\sigma(k)})_\omega,$$

where the left-hand side is the twist of $(\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$ by w.

The second important property we establish is the compatibility of the metaplectic tensor product with parabolic inductions.

Theorem (4.11 and 5.22) *Both locally and globally, let $P = MN \subseteq \mathrm{GL}_r$ be the standard parabolic subgroup whose Levi part is $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$. Further, for each $i = 1, \dots, k$ let $P_i = M_i N_i \subseteq \mathrm{GL}_{r_i}$ be the standard parabolic of GL_{r_i} whose Levi part is $M_i = \mathrm{GL}_{r_{i,1}} \times \cdots \times \mathrm{GL}_{r_{i,l_i}}$. For each i, we are given a representation*

$$\sigma_i := (\tau_{i,1} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{i,l_i})_{\omega_i}$$

of $\widetilde{M}_i$, which is given as the metaplectic tensor product of the representations $\tau_{i,1}, \dots, \tau_{i,l_i}$ of $\widetilde{\mathrm{GL}}_{r_{i,1}}, \dots, \widetilde{\mathrm{GL}}_{r_{i,l_i}}$. Assume that π_i is an irreducible constituent of the induced representation $\mathrm{Ind}_{\widetilde{P}_i}^{\widetilde{\mathrm{GL}}_{r_i}} \sigma_i$. Then the metaplectic tensor product

$$\pi_\omega := (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$$

is an irreducible constituent of the induced representation

$$\mathrm{Ind}_{\widetilde{Q}}^{\widetilde{M}} (\tau_{1,1} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{1,l_1} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{k,1} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{k,l_k})_\omega,$$

where Q is the standard parabolic subgroup of M whose Levi part is $M_1 \times \cdots \times M_k$.

In the above two theorems, it is implicitly assumed that if $n > 2$ and F is global, the metaplectic tensor products in the theorems exist in the sense that Hypothesis $(*)$ is satisfied for the relevant Levi subgroups.

Finally, we will discuss the behavior of the global metaplectic tensor product when restricted to a smaller Levi subgroup. Namely, for each automorphic form $\varphi \in (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$ in the metaplectic tensor product, we would like to know which space the restriction $\varphi|_{\widetilde{M}_2}$ belongs to, where $M_2 = \{I_{r_1}\} \times \mathrm{GL}_{r_2} \times \cdots \times \mathrm{GL}_{r_k} \subset M$, viewed as a subgroup of M, is the Levi subgroup for the smaller group GL_{r-r_1}. Similarly to the non-metaplectic case, the restriction $\varphi|_{\widetilde{M}_2}$ belongs to the metaplectic tensor product of $\pi_2, \dots, \pi_k$. But the precise statement is a bit more subtle. Indeed, we will prove the following theorem.

***Theorem* 5.28** *Assume Hypothesis $(**)$ (see Section 5.6) is satisfied, which is always the case if $n = 2$ or $\gcd(n, r-1+2cr) = \gcd(n, r-r_1-1+2c(r-r_1)) = 1$. Then there exists a realization of the metaplectic tensor product $\pi_\omega = (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$ such that if we let*

$$\pi_\omega\|_{\widetilde{M}_2(\mathbb{A})} = \{\widetilde{\varphi}|_{\widetilde{M}_2(\mathbb{A})} : \widetilde{\varphi} \in \pi_\omega\},$$

then

$$\pi_\omega\|_{\widetilde{M}_2(\mathbb{A})} \subseteq \bigoplus_\delta m_\delta (\pi_2 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_{\omega_\delta},$$

as a representation of $\widetilde{M}_2(\mathbb{A})$, where $(\pi_2\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_{\omega_\delta}$ is the metaplectic tensor product of $\pi_2,\ldots,\pi_k$, ω_δ is a certain character twisted by δ that runs through a finite subset of $\mathrm{GL}_{r_1}(F)$*, and $m_\delta\in\mathbb{Z}^{\geq 0}$ is a multiplicity.*

The precise meanings of the notation will be explained in Section 5.6.

Even though the theory of metaplectic groups is an important subject in representation theory and automorphic forms and used in various important literatures such as [B, F, BBL, BFH, BH, S] to name a few, and most importantly for the purpose of this paper [BG], which concerns the symmetric square L-function on $\mathrm{GL}(r)$, it has an unfortunate history of numerous technical errors and as a result published literatures in this area are often marred by those errors which compromise their reliability. As is pointed out in [BLS], this is probably due to the deep and subtle nature of the subject. At any rate, this has made people who work in the area particularly wary of inaccuracies in new works. For this reason, especially considering the foundational nature of this paper, we tried to provide detailed proofs for most of our assertions at the expense of the length of the paper. Furthermore, in large part, we rely only on the two fundamental works, namely the work on the metaplectic cocycle by Banks, Levy and Sepanski ([BLS]) and the local metaplectic tensor product by Mezo ([Me]), both of which are written carefully enough to be reliable.

Finally, let us mention that the result of this paper will be used in our forthcoming [T2], which will improve the main result of [T1].

Notation

Throughout the paper, F is a local field of characteristic zero or a number field. If F is a number field, we denote the ring of adeles by $\mathbb{A}$. As we did in the introduction we often use the notation

$$R=\begin{cases}F & \text{if } F \text{ is local,}\\ \mathbb{A} & \text{if } F \text{ is global.}\end{cases}$$

The symbol $R^\times$ has the usual meaning, and we set

$$R^{\times n}=\{a^n : a\in R^\times\}.$$

Both locally and globally, we denote by $\mathcal{O}_F$ the ring of integers of F. For each algebraic group G over a global F and each $g\in G(\mathbb{A})$, by g_v we mean the v-th component of g, and so $g_v\in G(F_v)$.

For a positive integer r, we denote by I_r the $r\times r$ identity matrix. Throughout we fix an integer $n\geq 2$, and we let μ_n be the group of n-th roots of unity in the algebraic closure of the prime field. We always assume that $\mu_n\subseteq F$, where F is either local or global. So in particular if $n\geq 3$, for archimedean F, we have $F=\mathbb{C}$, and for global F, F is totally complex.

If F is local, the symbol $(\,\cdot\,,\,\cdot\,)_F$ denotes the n-th order Hilbert symbol of F , which is a bilinear map

$$(\,\cdot\,,\,\cdot\,)_F\colon F^\times\times F^\times\longrightarrow\mu_n.$$

If F is global, we let $(\,\cdot\,,\,\cdot\,)_{\mathbb{A}}:=\prod_v(\,\cdot\,,\,\cdot\,)_{F_v}$, where the product is finite. We sometimes write $(\,\cdot\,,\,\cdot\,)$ for $(\,\cdot\,,\,\cdot\,)_R$ when there is no danger of confusion. Let us recall that both locally and globally the Hilbert symbol has the following properties:

$$(a,b)^{-1} = (b,a), \quad (a^n, b) = (a, b^n) = 1, \quad (a, -a) = 1$$

for $a, b \in R^\times$. Also, for the global Hilbert symbol, we have the product formula $(a,b)_{\mathbb{A}} = 1$ for all $a, b \in F^\times$.

We fix a partition $r_1 + \cdots + r_k = r$ of r, and we let

$$M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k} \subseteq \mathrm{GL}_r$$

and assume it is embedded diagonally as usual. We often denote an element $m \in M$ by

$$m = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix} \quad \text{or} \quad m = \mathrm{diag}(g_1, \ldots, g_k),$$

or sometimes simply $m = (g_1, \ldots, g_k)$, where $g_i \in \mathrm{GL}_{r_i}$.

For GL_r, we let $B = TN_B$ be the Borel subgroup with the unipotent radical N_B and the maximal torus T.

If π is a representation of a group G, we denote the space of π by V_π, though we often equate π with V_π when there is no danger of confusion. We say π is unitary if V_π is equipped with a Hermitian structure invariant under the action of G, but we do not necessarily assume that the space V_π is complete. Now assume that V_π is a space of functions or maps on the group G and π is the representation of G on V_π defined by right translation. (This is the case, for example, if π is an automorphic subrepresentation.) Let $H \subseteq G$ be a subgroup. Then we define $\pi\|_H$ to be the representation of H realized in the space

$$V_{\pi\|_H} := \{ f|_H : f \in V_\pi \}$$

of restrictions of $f \in V_\pi$ to H, on which H acts by right translation. Namely, $\pi\|_H$ is the representation obtained by restricting the functions in V_π. Occasionally, we equate $\pi\|_H$ with its space when there is no danger of confusion. Note that there is an H-intertwining surjection $\pi|_H \to \pi\|_H$, where $\pi|_H$ is the (usual) restriction of π to H.

For any group G and elements $g, h \in G$, we define ${}^g h = ghg^{-1}$. For a subgroup $H \subseteq G$ and a representation π of H, we define ${}^g\pi$ to be the representation of gHg^{-1} defined by ${}^g\pi(h') = \pi(g^{-1}h'g)$ for $h' \in gHg^{-1}$.

We let W be the set of all $r \times r$ permutation matrices, so for each element $w \in W$ each row and each column has exactly one 1 and all the other entries are 0. The Weyl group of GL_r is identified with W. Also for our Levi subgroup M, we let W_M be the subset of W that only permutes the GL_{r_i}-blocks of M. Namely, W_M is the collection of block matrices

$$W_M := \{ (\delta_{\sigma(i),j} I_{r_j}) \in W : \sigma \in S_k \},$$

where S_k is the permutation group of k letters. Though W_M is not a group in general, it is in bijection with S_k. Note that if $w \in W_M$ corresponds to $\sigma \in S_k$, we have

$${}^w\,\mathrm{diag}(g_1, \ldots, g_k) = w\,\mathrm{diag}(g_1, \ldots, g_k)w^{-1} = \mathrm{diag}(g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(k)}).$$

In addition to W, in order to use various results from [BLS], which give a detailed description of the 2-cocycle σ_r defining our metaplectic group $\widetilde{\mathrm{GL}}_r$, one sometimes needs to use another set of representatives of the Weyl group elements, which we denote by $\mathfrak{M}$ as in [BLS]. The set $\mathfrak{M}$ is chosen to be such that for each element $\eta \in \mathfrak{M}$ we have $\det(\eta) = 1$. To be more precise, each η with length l is written as

$$\eta = w_{\alpha_1} \cdots w_{\alpha_l},$$

where w_{α_i} is a simple root reflection corresponding to a simple root α_i and is the matrix of the form

$$w_{\alpha_i} = \begin{pmatrix} \ddots & & & \\ & & -1 & \\ & 1 & & \\ & & & \ddots \end{pmatrix}.$$

Though the set $\mathfrak{M}$ is not a group, it has the advantage that we can compute the cocycle σ_r in a systematic way, as one can see in [BLS]. For each $w \in W$, we denote by η_w the corresponding element in $\mathfrak{M}$. If $w \in W_M$, one can see that η_w is of the form $(\varepsilon_j \delta_{\sigma(i),j} I_{r_j})$ for $\varepsilon_j \in \{\pm 1\}$. Namely, η_w is a $k \times k$ block matrix in which the non-zero entries are either I_{r_j} or $-I_{r_j}$.

2 The Metaplectic Cover $\widetilde{\mathrm{GL}}_r$ of GL_r

In this section, we review the theory of the metaplectic n-fold cover $\widetilde{\mathrm{GL}}_r$ of GL_r for both local and global cases, which was originally constructed by Kazhdan and Patterson [KP].

2.1 The Local Metaplectic Cover $\widetilde{\mathrm{GL}}_r(F)$

Let F be a (not necessarily non-archimedean) local field of characteristic 0 that contains all the n-th roots of unity. In this paper, by the metaplectic n-fold cover $\widetilde{\mathrm{GL}}_r(F)$ of $\mathrm{GL}_r(F)$ with a fixed parameter $c \in \{0, \dots, n-1\}$, we mean the central extension of $\mathrm{GL}_r(F)$ by μ_n as constructed by Kazhdan and Patterson [KP]. To be more specific, let us first recall that the n-fold cover $\widetilde{\mathrm{SL}}_{r+1}(F)$ of $\mathrm{SL}_{r+1}(F)$ was constructed by Matsumoto [Mat], and there is an embedding

$$l_0\colon \mathrm{GL}_r(F) \longrightarrow \mathrm{SL}_{r+1}(F), \quad g \longmapsto \begin{pmatrix} \det(g)^{-1} & \\ & g \end{pmatrix}. \tag{2.1}$$

Our metaplectic n-fold cover $\widetilde{\mathrm{GL}}_r(F)$ with $c = 0$ is the preimage of $l_0(\mathrm{GL}_r(F))$ via the canonical projection $\widetilde{\mathrm{SL}}_{r+1}(F) \to \mathrm{SL}_{r+1}(F)$. Then $\widetilde{\mathrm{GL}}_r(F)$ is defined by a 2-cocycle

$$\sigma_r\colon \mathrm{GL}_r(F) \times \mathrm{GL}_r(F) \longrightarrow \mu_n.$$

For arbitrary parameter $c \in \{0, \dots, n-1\}$, we define the twisted cocycle $\sigma_r^{(c)}$ by

$$\sigma_r^{(c)}(g, g') = \sigma_r(g, g')\big(\det(g), \det(g')\big)_F^c$$

for $g, g' \in \mathrm{GL}_r(F)$, where recall from the notation section that $(-,-)_F$ is the n-th order Hilbert symbol for F. The metaplectic cover with a parameter c is defined by this cocycle. In [KP], the metaplectic cover with parameter c is denoted by $\widetilde{\mathrm{GL}}_r^{(c)}(F)$, but we avoid this notation. This is because we will later introduce the notation $\widetilde{\mathrm{GL}}_r^{(n)}(F)$, which has a completely different meaning. We also suppress the superscript (c) from the notation of the cocycle and always agree that the parameter c is fixed throughout the paper.

By carefully studying Matsumoto's construction, Banks, Levy, and Sepanski [BLS] gave an explicit description of the 2-cocycle σ_r and showed that their 2-cocycle is "block-compatible" in the following sense: for the standard $(r_1, \dots, r_k)$-parabolic of GL_r, so that its Levi subgroup M is of the form $\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$, which is embedded diagonally into GL_r, we have

$$\sigma_r\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \begin{pmatrix} g_1' & & \\ & \ddots & \\ & & g_k' \end{pmatrix}\right) = \prod_{i=1}^{k} \sigma_{r_i}(g_i, g_i') \prod_{1 \le i < j \le k} \big(\det(g_i), \det(g_j')\big)_F \prod_{i \ne j} \big(\det(g_i), \det(g_j')\big)_F^c, \tag{2.2}$$

for all $g_i, g_i' \in \mathrm{GL}_{r_i}(F)$. (See [BLS, Theorem 11, §3]. Strictly speaking, in [BLS] only the case $c = 0$ is considered, but one can derive the above formula using the bilinearity of the Hilbert symbol.) This 2-cocycle generalizes the well-known cocycle given by Kubota [Kub] for the case $r = 2$. We should also note that if $r = 1$, this cocycle is trivial. Note that $\widetilde{\mathrm{GL}}_r(F)$ is not the F-rational points of an algebraic group, but this notation seems to be standard.

Let us list some other important properties of the cocycle σ_r that we will use in this paper.

Proposition 2.1 *Let $B = TN_B$ be the Borel subgroup of GL_r, where T is the maximal torus and N_B the unipotent radical. The cocycle σ_r satisfies the following properties:*

(i) $\sigma_r(g, g')\sigma_r(gg', g'') = \sigma_r(g, g'g'')\sigma_r(g', g'')$ *for* $g, g', g'' \in \mathrm{GL}_r$.

(ii) $\sigma_r(ng, g'n') = \sigma_r(g, g')$ *for* $g, g' \in \mathrm{GL}_r$ *and* $n, n' \in N_B$, *and so in particular* $\sigma_r(ng, n') = \sigma_r(n, g'n') = 1$.

(iii) $\sigma_r(gn, g') = \sigma_r(g, ng')$ *for* $g, g' \in \mathrm{GL}_r$ *and* $n \in N_B$.

(iv) $\sigma_r(\eta, t) = \prod_{\substack{\alpha=(i,j)\in\Phi^+ \\ \eta\alpha<0}} (-t_j, t_i)$ *for* $\eta \in \mathfrak{W}$ *and* $t = \mathrm{diag}(t_1, \dots, t_r) \in T$,

where Φ^+ is the set of positive roots and each root $\alpha \in \Phi^+$ is identified with a pair of integers (i, j) with $1 \le i < j \le r$ as usual.

(v) $\sigma_r(t, t') = \prod_{i<j} (t_i, t_j')(\det(t), \det(t'))^c$ *for* $t = \mathrm{diag}(t_1, \dots, t_r) \in T$ *and* $t' = \mathrm{diag}(t_1', \dots, t_r') \in T$.

(vi) $\sigma_r(t, \eta) = 1$ *for* $t \in T$ *and* $\eta \in \mathfrak{W}$.

Proof The first one is simply the definition of 2-cocycle, and all of the others are some of the properties of σ_r listed in [BLS, Theorem 7, p. 153]. ■

We need to recall how this cocycle is constructed. As mentioned earlier, Matsumoto constructed $\widetilde{\mathrm{SL}}_{r+1}(F)$. It is shown in [BLS] that $\widetilde{\mathrm{SL}}_{r+1}(F)$ is defined by a cocycle $\sigma_{\mathrm{SL}_{r+1}}$ that satisfies the block-compatibility in a much stronger sense as in [BLS, Theorem 7, §2, p. 145]. (Note that our SL_{r+1} corresponds to $\mathbb{G}^\flat$ of [BLS].)

Then the cocycle σ_r is defined by

$$\sigma_r(g,g') = \sigma_{\mathrm{SL}_{r+1}}\big(l(g), l(g')\big)\,\big(\det(g),\det(g')\big)_F\big(\det(g),\det(g')\big)_F^c,$$

where l is the embedding defined by

$$l\colon\ \mathrm{GL}_r(F) \longrightarrow \mathrm{SL}_{r+1}(F), \quad g \longmapsto \begin{pmatrix} g & \\ & \det(g)^{-1} \end{pmatrix}.$$

See [BLS, p. 146]. (Note the difference between this embedding and the one in (2.1). This is the reason we have the extra Hilbert symbol in the definition of σ_r.)

Since we would like to emphasize the cocycle being used, we denote $\widetilde{\mathrm{GL}}_r(F)$ by ${}^{\sigma}\widetilde{\mathrm{GL}}_r(F)$ when the cocycle σ is used. Namely, ${}^{\sigma}\widetilde{\mathrm{GL}}_r(F)$ is the group whose underlying set is

$${}^{\sigma}\widetilde{\mathrm{GL}}_r(F) = \mathrm{GL}_r(F) \times \mu_n = \{(g,\xi) : g \in \mathrm{GL}_r(F), \xi \in \mu_n\},$$

and the group law is defined by

$$(g,\xi)\cdot(g',\xi') = \big(gg', \sigma_r(g,g')\xi\xi'\big).$$

Using the block-compatible 2-cocycle of [BLS] has obvious advantages. In particular, it has been explicitly computed, and, of course, it is block-compatible. Indeed, when we consider purely local problems, we always assume that the cocycle σ_r is used.

However, it does not allow us to construct the global metaplectic cover $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. Namely, one cannot define the adelic block-combatible 2-cocycle simply by taking the product of the local block-combatible 2-cocycles over all the places. Namely for $g, g' \in \mathrm{GL}_r(\mathbb{A})$, the product

$$\prod_v \sigma_{r,v}(g_v, g'_v)$$

is not necessarily finite. This can be already observed for the case $r = 2$. (See [F, p. 125].)

For this reason, we will use a different 2-cocycle τ_r, which works nicely with the global metaplectic cover $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. To construct such τ_r, first assume F is non-archimedean. It is known that an open compact subgroup K splits in $\widetilde{\mathrm{GL}}_r(F)$, and, moreover, if $|n|_F = 1$, we have $K = \mathrm{GL}_r(\mathcal{O}_F)$. (See [KP, Proposition 0.1.2].) Also, for $k, k' \in K$, a property of the Hilbert symbol gives $(\det(k), \det(k'))_F = 1$. Hence, one has a continuous map $s_r\colon \mathrm{GL}_r(F) \to \mu_n$ such that $\sigma_r(k,k')s_r(k)s_r(k') = s_r(kk')$ for all $k, k' \in K$. Then we define our 2-cocycle τ_r by

$$\tau_r(g,g') := \sigma_r(g,g')\cdot\frac{s_r(g)s_r(g')}{s_r(gg')} \tag{2.3}$$

for $g, g' \in \mathrm{GL}_r(F)$. If F is archimedean, we set $\tau_r = \sigma_r$.

The choice of s_r and hence τ_r is not unique. However, when $|n|_F = 1$, there is a canonical choice with respect to the splitting of K in the following sense. Assume that F is such that $|n|_F = 1$. Then the Hilbert symbol $(\,\cdot\,,\,\cdot\,)_F$ is trivial on $\mathcal{O}_F^\times \times \mathcal{O}_F^\times$, and hence, when restricted to $\mathrm{GL}_r(\mathcal{O}_F)\times\mathrm{GL}_r(\mathcal{O}_F)$, the cocycle σ_r is the restriction of $\sigma_{\mathrm{SL}_{r+1}}$ to the image of the embedding l. It is known that the compact group $\mathrm{SL}_{r+1}(\mathcal{O}_F)$ also splits in $\widetilde{\mathrm{SL}}_{r+1}(F)$, and hence there is a map $\mathfrak{s}_r\colon\ \mathrm{SL}_{r+1}(F) \to \mu_n$ such that the section $\mathrm{SL}_{r+1}(F) \to \widetilde{\mathrm{SL}}_{r+1}(F)$ given by $(g, \mathfrak{s}_r(g))$ is a homomorphism on $\mathrm{SL}_{r+1}(\mathcal{O}_F)$. (Here we are assuming $\widetilde{\mathrm{SL}}_{r+1}(F)$ is realized as $\mathrm{SL}_{r+1}(F) \times \mu_n$ as a set and the group structure is

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defined by the cocycle $\sigma_{\mathrm{SL}_{r+1}}$.) Moreover, $\mathfrak{s}_r|_{\mathrm{SL}_{r+1}(\mathcal{O}_F)}$ is determined up to twists by the elements in $H^1(\mathrm{SL}_{r+1}(\mathcal{O}_F),\mu_n)=\mathrm{Hom}(\mathrm{SL}_{r+1}(\mathcal{O}_F),\mu_n)$. But $\mathrm{Hom}(\mathrm{SL}_{r+1}(\mathcal{O}_F),\mu_n)=1$, because $\mathrm{SL}_{r+1}(\mathcal{O}_F)$ is a perfect group and μ_n is commutative. Hence, $\mathfrak{s}_r|_{\mathrm{SL}_{r+1}(\mathcal{O}_F)}$ is unique. (See also [KP, p. 43] for this matter.) We choose s_r so that

$$s_r|_{\mathrm{GL}_r(\mathcal{O}_F)}=\mathfrak{s}_r|_{l(\mathrm{GL}_r(\mathcal{O}_F))}. \tag{2.4}$$

With this choice, we have the commutative diagram

$$\begin{tikzcd} {}^{\sigma}\widetilde{\mathrm{GL}}_r(\mathcal{O}_F) \arrow[r] & \widetilde{\mathrm{SL}}_{r+1}(\mathcal{O}_F) \\ K \arrow[u, "k\mapsto(k,\,s_r(k))"] \arrow[r] & \mathrm{SL}_{r+1}(\mathcal{O}_F), \arrow[u, "k\mapsto(k,\,\mathfrak{s}_r(k))"'] \end{tikzcd} \tag{2.5}$$

where the top arrow is $(g,\xi)\mapsto(l(g),\xi)$, the bottom arrow is l, and all the arrows can be seen to be homomorphisms. This choice of s_r will be crucial for constructing the metaplectic tensor product of automorphic representations. Also note that the left vertical arrow in the above diagram is what is called the canonical lift in [KP] and denoted by κ^* there. (Although we do not need this fact in this paper, if $r=2$ one can show that τ_r can be chosen to be block compatible, and is the cocycle used in [F].)

Using τ_r, we realize $\widetilde{\mathrm{GL}}_r(F)$ as

$$\widetilde{\mathrm{GL}}_r(F)=\mathrm{GL}_r(F)\times\mu_n,$$

as a set and the group law is given by

$$(g,\xi)\cdot(g',\xi')=\left(gg',\tau_r(g,g')\xi\xi'\right).$$

Note that we have the exact sequence

$$0\longrightarrow\mu_n\longrightarrow\widetilde{\mathrm{GL}}_r(F)\xrightarrow{\ p\ }\mathrm{GL}_r(F)\longrightarrow 0$$

given by the obvious maps, where we call p the canonical projection.

We define a set theoretic section

$$\kappa\colon\ \mathrm{GL}_r(F)\longrightarrow\widetilde{\mathrm{GL}}_r(F),\quad g\longmapsto(g,1).$$

Note that κ is not a homomorphism, but by our construction of the cocycle τ_r, $\kappa|_K$ is a homomorphism if F is non-archimedean and K is a sufficiently small open compact subgroup. Moreover, if $|n|_F=1$, one has $K=\mathrm{GL}_r(\mathcal{O}_F)$.

Also, we define another set theoretic section

$$\mathbf{s}_r\colon\ \mathrm{GL}_r(F)\longrightarrow\widetilde{\mathrm{GL}}_r(F),\quad g\longmapsto\left(g,s_r(g)^{-1}\right)$$

where $s_r(g)$ is as above, and then we have the isomorphism

$$\widetilde{\mathrm{GL}}_r(F)\longrightarrow{}^{\sigma}\widetilde{\mathrm{GL}}_r(F),\quad(g,\xi)\longmapsto\left(g,s_r(g)\xi\right),$$

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which gives rise to the commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{GL}}_r(F) \arrow[rr] & & {}^{\sigma}\widetilde{\mathrm{GL}}_r(F) \\ & \mathrm{GL}_r(F) \arrow[ul, "\mathbf{s}_r"] \arrow[ur, "g\mapsto(g,1)"'] & \end{tikzcd}$$

of set theoretic maps. Also note that the elements in the image $\mathbf{s}_r(\mathrm{GL}_r(F))$ "multiply via σ_r" in the sense that for $g, g' \in \mathrm{GL}_r(F)$, we have

$$(g, s_r(g)^{-1})\,(g', s_r(g')^{-1}) = (gg', \sigma_r(g,g')s_r(gg')^{-1})\,. \tag{2.6}$$

Lemma 2.2 *Assume F is non-archimedean with $|n|_F = 1$. We have*

$$\kappa|_{T\cap K} = \mathbf{s}_r|_{T\cap K}, \quad \kappa|_W = \mathbf{s}_r|_W, \quad \kappa|_{N_B\cap K} = \mathbf{s}_r|_{N_B\cap K}, \tag{2.7}$$

where W is the Weyl group and $K = \mathrm{GL}_r(\mathcal{O}_F)$. In particular, this implies that $s_r|_{T\cap K} = s_r|_W = s_r|_{N_B\cap K} = 1$.

Proof See [KP, Proposition 0.I.3]. ■

Remark 2.3 Though we do not need this fact in this paper, it should be noted that $\mathbf{s}_r$ splits the Weyl group W if and only if $(-1,-1)_F = 1$. So, in particular, it splits W if $|n|_F = 1$. See [BLS, §5].

If P is a parabolic subgroup of GL_r whose Levi subgroup is $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$, we often write

$$\widetilde{M}(F) = \widetilde{\mathrm{GL}}_{r_1}(F)\widetilde{\times}\cdots\widetilde{\times}\widetilde{\mathrm{GL}}_{r_k}(F)$$

for the metaplectic preimage of $M(F)$. Next, let

$$\mathrm{GL}_r^{(n)}(F) = \{g \in \mathrm{GL}_r(F) : \det g \in F^{\times n}\},$$

and let $\widetilde{\mathrm{GL}}_r^{(n)}(F)$ be its metaplectic preimage. Also, we define

$$M^{(n)}(F) = \left\{(g_1, \ldots, g_k) \in M(F) : \det g_i \in F^{\times n}\right\}$$

and often denote its preimage by

$$\widetilde{M}^{(n)}(F) = \widetilde{\mathrm{GL}}_{r_1}^{(n)}(F)\widetilde{\times}\cdots\widetilde{\times}\widetilde{\mathrm{GL}}_{r_k}^{(n)}(F).$$

The group $\widetilde{M}^{(n)}(F)$ is a normal subgroup of finite index. Indeed, we have the exact sequence

$$1 \longrightarrow \widetilde{M}^{(n)}(F) \longrightarrow \widetilde{M}(F) \longrightarrow \underbrace{F^{\times n}\backslash F^{\times} \times \cdots \times F^{\times n}\backslash F^{\times}}_{k \text{ times}} \longrightarrow 1, \tag{2.8}$$

where the third map is given by $(\mathrm{diag}(g_1, \ldots, g_k), \xi) \mapsto (\det(g_1), \ldots, \det(g_k))$. We should mention the explicit isomorphism $F^{\times n}\backslash F^{\times} \times \cdots \times F^{\times n}\backslash F^{\times} \to \widetilde{M}^{(n)}(F)\backslash\widetilde{M}(F)$ is defined as follows. First, for each $i \in \{1, \ldots, k\}$, define a map $\iota_i \colon F^{\times} \to \mathrm{GL}_{r_i}$ by

$$\iota_i(a) = \begin{pmatrix} a & \\ & I_{r_i-1} \end{pmatrix}. \tag{2.9}$$

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Then the map given by

$$(a_1,\dots,a_k)\longmapsto\left(\begin{pmatrix}\iota_1(a_1)&&\\&\ddots&\\&&\iota_k(a_k)\end{pmatrix},1\right)$$

is a homomorphism. Clearly, the map is well defined and one-to-one. Moreover, it is surjective, because each element $g_i\in\mathrm{GL}_{r_i}$ is written as

$$g_i=g_i\iota_i\big(\det(g_i)^{n-1}\big)\,\iota_i\big(\det(g_i)^{1-n}\big)$$

and $g_i\iota_i(\det(g_i)^{n-1})\in\mathrm{GL}_{r_i}^{(n)}$.

The following should be mentioned.

Lemma 2.4 *The groups $F^{\times n}$, $M^{(n)}(F)$, and $\widetilde{M}^{(n)}(F)$ are closed subgroups of $F^\times$, $M(F)$ and $\widetilde{M}(F)$, respectively.*

Proof It is well known that $F^{\times n}$ is closed and of finite index in $F^\times$. Hence the group $F^{\times n}\backslash F^\times\times\cdots\times F^{\times n}\backslash F^\times$ is discrete, and in particular, Hausdorff. But both $\widetilde{M}^{(n)}(F)\backslash\widetilde{M}(F)$ and $M^{(n)}(F)\backslash M(F)$ are, as topological groups, isomorphic to this Hausdorff space. This completes the proof. ∎

Remark 2.5 If $F=\mathbb{C}$, clearly $\widetilde{M}^{(n)}(F)=\widetilde{M}(F)$. If $F=\mathbb{R}$, then necessarily $n=2$ and $\mathrm{GL}_r^{(2)}(\mathbb{R})$ consists of the elements of positive determinants, which is usually denoted by $\mathrm{GL}_r^+(\mathbb{R})$. Accordingly, one may denote $\widetilde{\mathrm{GL}}_r^{(n)}(\mathbb{R})$ and $\widetilde{M}^{(n)}(\mathbb{R})$ by $\widetilde{\mathrm{GL}}_r^+(\mathbb{R})$ and $\widetilde{M}^+(\mathbb{R})$, respectively. Both $\widetilde{\mathrm{GL}}_r^+(\mathbb{R})$ and $\widetilde{\mathrm{GL}}_r(\mathbb{R})$ share the identity component, and hence they have the same Lie algebra. The same applies to $\widetilde{M}^+(\mathbb{R})$ and $\widetilde{M}(\mathbb{R})$.

Let us mention the following important fact. Let $Z_{\mathrm{GL}_r}(F)\subseteq\mathrm{GL}_r(F)$ be the center of $\mathrm{GL}_r(F)$. Then its metaplectic preimage $\widetilde{Z_{\mathrm{GL}_r}}(F)$ is not the center of $\widetilde{\mathrm{GL}}_r(F)$ in general. (It might not even be commutative for $n>2$.) The center, which we denote by $Z_{\widetilde{\mathrm{GL}}_r}(F)$, is

$$\begin{aligned}Z_{\widetilde{\mathrm{GL}}_r}(F)&=\{(aI_r,\xi):a^{r-1+2rc}\in F^{\times n},\xi\in\mu_n\}\\&=\{(aI_r,\xi):a\in F^{\times\frac{n}{d}},\xi\in\mu_n\},\end{aligned}\tag{2.10}$$

where $d=\gcd(r-1+2c,n)$. (The second equality is proved in [GO, Lemma 1].) Note that $Z_{\widetilde{\mathrm{GL}}_r}(F)$ is a closed subgroup.

Let π be an admissible representation of a subgroup $\widetilde{H}\subseteq\widetilde{\mathrm{GL}}_r(F)$, where $\widetilde{H}$ is the metaplectic preimage of a subgroup $H\subseteq\mathrm{GL}_r(F)$. We say π is *genuine* if each element $(1,\xi)\in\widetilde{H}$ acts as multiplication by ξ, where we view ξ as an element of $\mathbb{C}$ in the natural way.

2.2 The Global Metaplectic Cover $\widetilde{\mathrm{GL}}_r(\mathbb{A})$

In this subsection we consider the global metaplectic group. So we let F be a number field that contains all the n-th roots of unity and $\mathbb{A}$ the ring of adeles. Note that if $n>2$, then F must be totally complex. We shall define the n-fold metaplectic cover $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ of $\mathrm{GL}_r(\mathbb{A})$. (As in the local case, we write $\widetilde{\mathrm{GL}}_r(\mathbb{A})$, even though it is not the adelic points of an algebraic group.) The construction of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ has been done in various places, such as [KP, FK].

First, define the adelic 2-cocycle τ_r by

$$\tau_r(g,g') := \prod_v \tau_{r,v}(g_v, g'_v),$$

for $g, g' \in \mathrm{GL}_r(\mathbb{A})$, where $\tau_{r,v}$ is the local cocycle defined in the previous subsection. By definition of $\tau_{r,v}$, we have $\tau_{r,v}(g_v, g'_v) = 1$ for almost all v, and hence the product is well defined.

We define $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ to be the group whose underlying set is $\mathrm{GL}_r(\mathbb{A}) \times \mu_n$ and the group structure is defined via τ_r as in the local case, *i.e.*,

$$(g,\xi)\cdot(g',\xi') = \left(gg', \tau_r(g,g')\xi\xi'\right),$$

for $g, g' \in \mathrm{GL}_r(\mathbb{A})$, and $\xi, \xi' \in \mu_n$. As in the local case, we have

$$0 \longrightarrow \mu_n \longrightarrow \widetilde{\mathrm{GL}}_r(\mathbb{A}) \xrightarrow{p} \mathrm{GL}_r(\mathbb{A}) \longrightarrow 0,$$

where we call p the canonical projection. Define a set theoretic section $\kappa\colon \mathrm{GL}_r(\mathbb{A}) \to \widetilde{\mathrm{GL}}_r(\mathbb{A})$ by $g \mapsto (g, 1)$.

It is well known that $\mathrm{GL}_r(F)$ splits in $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. However, the splitting is not via κ. In what follows, we will see that the splitting is via the product of all the local $\mathbf{s}_r$.

Let us start with the following "product formula" for σ_r.

Proposition 2.6 *For $g, g' \in \mathrm{GL}_r(F)$, we have $\sigma_{r,v}(g,g') = 1$ for almost all v, and*

$$\prod_v \sigma_{r,v}(g,g') = 1.$$

Proof From the explicit description of the cocycle $\sigma_{r,v}(g,g')$ given at the end of [BLS, §4], one can see that $\sigma_{r,v}(g,g')$ is written as a product of Hilbert symbols of the form $(t,t')_{F_v}$ for $t, t' \in F^\times$. This proves the first part of the proposition. The second part follows from the product formula for the Hilbert symbol. ∎

Proposition 2.7 *If $g \in \mathrm{GL}_r(F)$, then we have $s_{r,v}(g) = 1$ for almost all v, where $s_{r,v}$ is the map $s_{r,v}\colon \mathrm{GL}(F_v) \to \mu_n$ defining the local section $\mathbf{s}_r\colon \mathrm{GL}(F_v) \to \widetilde{\mathrm{GL}}_r(F_v)$.*

Proof By the Bruhat decomposition, we have $g = bwb'$ for some $b, b' \in B(F)$ and $w \in W$. Then for each place v,

$$\begin{aligned} s_{r,v}(g) &= s_{r,v}(bwb') \\ &= \sigma_{r,v}(b,wb')s_{r,v}(b)s_{r,v}(wb')/\tau_{r,v}(b,wb') \quad \text{by (2.3)} \\ &= \sigma_{r,v}(b,wb')s_{r,v}(b)\sigma_{r,v}(w,b')s_{r,v}(w)s_{r,v}(b')/\tau_{r,v}(w,b')\tau_{r,v}(b,wb') \;\; \text{by (2.3)}. \end{aligned}$$

By the previous proposition, $\sigma_{r,v}(b,wb') = \sigma_{r,v}(w,b') = 1$ for almost all v. By (2.7) we know $s_{r,v}(b) = s_{r,v}(w) = s_{r,v}(b') = 1$ for almost all v. Finally, by definition of $\tau_{r,v}$, $\tau_{r,v}(w,b') = \tau_{r,v}(b,wb') = 1$ for almost all v. ∎

This proposition implies that the expression

$$s_r(g) := \prod_v s_{r,v}(g)$$

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makes sense for all $g \in \mathrm{GL}_r(F)$, and one can define the map

$$\mathbf{s}_r\colon\ \mathrm{GL}_r(F) \to \widetilde{\mathrm{GL}}_r(\mathbb{A}), \quad g \mapsto (g, s_r(g)^{-1}).$$

Moreover, this is a homomorphism because of Proposition 2.6 and (2.6).

Unfortunately however, the expression $\prod_v s_{r,v}(g_v)$ does not make sense for every $g \in \mathrm{GL}_r(\mathbb{A})$, because one does not know whether $s_{r,v}(g_v) = 1$ for almost all v. Yet, we have the following proposition.

Proposition 2.8 *The expression $s_r(g) = \prod_v s_{r,v}(g_v)$ makes sense when g is in $\mathrm{GL}_r(F)$ or $N_B(\mathbb{A})$, so $\mathbf{s}_r$ is defined on $\mathrm{GL}_r(F)$ and $N_B(\mathbb{A})$. Moreover, $\mathbf{s}_r$ is indeed a homomorphism on $\mathrm{GL}_r(F)$ and $N_B(\mathbb{A})$. Also if $g \in \mathrm{GL}_r(F)$ and $n \in N_B(\mathbb{A})$, both $s_r(gn)$ and $s_r(ng)$ make sense, and further we have $\mathbf{s}_r(gn) = \mathbf{s}_r(g)\mathbf{s}_r(n)$ and $\mathbf{s}_r(ng) = \mathbf{s}_r(n)\mathbf{s}_r(g)$.*

Proof We already know $s_r(g)$ is defined and $\mathbf{s}_r$ is a homomorphism on $\mathrm{GL}_r(F)$. Also, $s_r(n)$ is defined thanks to (2.7), and $\mathbf{s}_r$ is a homomorphism on $N_B(\mathbb{A})$ thanks to Proposition 2.1(ii). Moreover, for all places v, we have $\sigma_{r,v}(g_v, n_v) = 1$ again by Proposition 2.1(ii). Hence for all v, $s_{r,v}(gn_v) = s_{r,v}(g)s_{r,v}(n_v)/\tau_{r,v}(g, n_v)$. For almost all v, the right-hand side is 1. Hence the global $s_r(gn)$ is defined. Also this equality shows that $\mathbf{s}_r(gn) = \mathbf{s}_r(g)\mathbf{s}_r(n)$. The same argument works for ng. ■

If $H \subseteq \mathrm{GL}_r(\mathbb{A})$ is a subgroup on which $\mathbf{s}_r$ is not only defined but also a group homomorphism, we write $H^* := \mathbf{s}_r(H)$. In particular, we have

$$\mathrm{GL}_r(F)^* := \mathbf{s}_r(\mathrm{GL}_r(F)) \quad \text{and} \quad N_B(\mathbb{A})^* := \mathbf{s}_r(N_B(\mathbb{A})). \tag{2.11}$$

We define the groups like $\widetilde{\mathrm{GL}}_r^{(n)}(\mathbb{A})$, $\widetilde{M}(\mathbb{A})$, $\widetilde{M}^{(n)}(\mathbb{A})$, etc., completely analogously to the local case.

Lemma 2.9 *The groups $\mathbb{A}^{\times n}$, $M^{(n)}(\mathbb{A})$, and $\widetilde{M}^{(n)}(\mathbb{A})$ are closed subgroups of $\mathbb{A}^{\times}$, $M(\mathbb{A})$, and $\widetilde{M}(\mathbb{A})$, respectively.*

Proof That $\mathbb{A}^{\times n}$ and $M^{(n)}(\mathbb{A})$ are closed follows from the following lemma together with Lemma 2.4. Once one knows $M^{(n)}(\mathbb{A})$ is closed, one will know $\widetilde{M}^{(n)}(\mathbb{A})$ is closed, because it is the preimage of the closed $M^{(n)}(\mathbb{A})$ under the canonical projection, which is continuous. ■

Lemma 2.10 *Let G be an algebraic group over F and let $G(\mathbb{A})$ be its adelic points. Let $H \subseteq G(\mathbb{A})$ be a subgroup such that H is written as $H = \prod_v' H_v$ (algebraically), where for each place v, $H_v := H \cap G(F_v)$ is a closed subgroup of $G(F_v)$. Then H is closed.*

Proof Let $(x_i)_{i \in I}$ be a net in H that converges in $G(\mathbb{A})$, where I is some index set. Let $g = \lim_{i \in I} x_i$. Assume $g \notin H$. Then there exists a place w such that $g_w \notin H_w$. Since H_w is closed, the set $U_w := G(F_w) \backslash H_w$ is open. Then there exists an open neighborhood U of g of the form $U = \prod_v U_v$, where U_v is some open neighborhood of g_v and at $v = w$, $U_v = U_w$. But for any $i \in I$, $x_i \notin U$, because $x_{i,w} \notin U_w$, which contradicts the assumption that $g = \lim_{i \in I} x_i$. Hence, $g \in H$, which shows H is closed. ■

As in the local case, the preimage $\widetilde{Z_{\mathrm{GL}_r}}(\mathbb{A})$ of the center $Z_{\mathrm{GL}_r}(\mathbb{A})$ of $\mathrm{GL}_r(\mathbb{A})$ is in general not the center of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. The center, which we denote by $Z_{\widetilde{\mathrm{GL}}_r}(\mathbb{A})$, is

$$\begin{aligned} Z_{\widetilde{\mathrm{GL}}_r}(\mathbb{A}) &= \{(aI_r, \xi) : a^{r-1+2rc} \in \mathbb{A}^{\times n}, \xi \in \mu_n\} \\ &= \{(aI_r, \xi) : a \in \mathbb{A}^{\times \frac{n}{d}}, \xi \in \mu_n\}, \end{aligned}$$

where $d = \gcd(r-1+2c, n)$. The center is a closed subgroup of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$.

We can also describe $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ as a quotient of a restricted direct product of the groups $\widetilde{\mathrm{GL}}_r(F_v)$ as follows. Consider the restricted direct product $\prod'_v \widetilde{\mathrm{GL}}_r(F_v)$ with respect to the groups $\kappa(K_v) = \kappa(\mathrm{GL}_r(\mathcal{O}_{F_v}))$ for all v with $v \nmid n$ and $v \nmid \infty$. If we denote each element in this restricted direct product by $\Pi'_v(g_v, \xi_v)$ so that $g_v \in K_v$ and $\xi_v = 1$ for almost all v, we have the surjection

$$\rho\colon \prod_v{}' \widetilde{\mathrm{GL}}_r(F_v) \longrightarrow \widetilde{\mathrm{GL}}_r(\mathbb{A}), \quad \Pi'_v(g_v, \xi_v) \longmapsto (\Pi'_v g_v, \Pi_v \xi_v), \tag{2.12}$$

where the product $\Pi_v \xi_v$ is literally the product inside μ_n. This is a group homomorphism, because $\tau_r = \prod_v \tau_{r,v}$ and the groups $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ and $\widetilde{\mathrm{GL}}_r(F_v)$ are defined, respectively, by τ_r and $\tau_{r,v}$. We have

$$\prod_v{}' \widetilde{\mathrm{GL}}_r(F_v) / \ker\rho \cong \widetilde{\mathrm{GL}}_r(\mathbb{A}),$$

where $\ker\rho$ consists of the elements of the form $(1, \xi)$ with $\xi \in \prod'_v \mu_n$ and $\Pi_v \xi_v = 1$.

Let π be a representation of $\widetilde{H} \subseteq \widetilde{\mathrm{GL}}_r(\mathbb{A})$, where $\widetilde{H}$ is the metaplectic preimage of a subgroup $H \subseteq \mathrm{GL}_r(\mathbb{A})$. As in the local case, we call π genuine if $(1, \xi) \in \widetilde{H}(\mathbb{A})$ acts as multiplication by ξ for all $\xi \in \mu_n$. Also we have the notion of automorphic representation as well as automorphic form on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ or $\widetilde{M}(\mathbb{A})$. In this paper, by an *automorphic form*, we mean a smooth automorphic form instead of a K-finite one, namely an automorphic form is K_f-finite, $\mathcal{Z}$-finite, and of uniformly moderate growth. (See [C, p. 17].) Hence, if π is an automorphic representation of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ (or $\widetilde{M}(\mathbb{A})$), the full group $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ (or $\widetilde{M}(\mathbb{A})$) acts on π. An automorphic form f on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ (or $\widetilde{M}(\mathbb{A})$) is said to be genuine if $f(g, \xi) = \xi f(g, 1)$ for all $(g, \xi) \in \widetilde{\mathrm{GL}}_r(\mathbb{A})$ (or $\widetilde{M}(\mathbb{A})$). In particular every automorphic form in the space of a genuine automorphic representation is genuine.

Suppose we are given a collection of irreducible admissible representations π_v of $\widetilde{\mathrm{GL}}_r(F_v)$ such that π_v is $\kappa(K_v)$-spherical for almost all v. Then we can form an irreducible admissible representation of $\prod'_v \widetilde{\mathrm{GL}}_r(F_v)$ by taking a restricted tensor product $\otimes'_v \pi_v$ as usual. Suppose further that $\ker\rho$ acts trivially on $\otimes'_v \pi_v$, which is always the case if each π_v is genuine. Then it descends to an irreducible admissible representation of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$, which we denote by $\widetilde{\otimes}'_v \pi_v$, and call the "metaplectic restricted tensor product". Let us emphasize that the space for $\widetilde{\otimes}'_v \pi_v$ is the same as that for $\otimes'_v \pi_v$. Conversely, if π is an irreducible admissible representation of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$, it is written as $\widetilde{\otimes}'_v \pi_v$ where π_v is an irreducible admissible representation of $\widetilde{\mathrm{GL}}_r(F_v)$, and for almost all v, π_v is $\kappa(K_v)$-spherical. (To see it, view π as a representation of the restricted product $\prod'_v \widetilde{\mathrm{GL}}_r(F_v)$ by pulling it back by ρ as in (2.12) and applying the usual tensor product theorem for the restricted direct product. This gives the restricted tensor product $\otimes'_v \pi_v$, where each π_v is genuine, and hence descends to $\widetilde{\otimes}'_v \pi_v$.)

Finally in this section, let us mention that we define

$$\mathrm{GL}_r^{(n)}(F) := \mathrm{GL}_r(F) \cap \mathrm{GL}_r^{(n)}(\mathbb{A}),$$

namely, $\mathrm{GL}_r^{(n)}(F) = \{g \in \mathrm{GL}_r(F) : \det g \in \mathbb{A}^{\times n}\}$. But since F contains μ_n, one can easily show that

$$\mathrm{GL}_r^{(n)}(F) = \{g \in \mathrm{GL}_r(F) : \det g \in F^{\times n}\}.$$

(See, for example, [AT, Chap. 9, Theorem 1]. Also, for $n = 2$, this is a consequence of the Hasse–Minkowski theorem.) Similarly, we define

$$M^{(n)}(F) = M(F) \cap M^{(n)}(\mathbb{A}).$$

3 The Metaplectic Cover $\widetilde{M}$ of the Levi Subgroup M

Both locally and globally, one cannot show that the cocycle τ_r has the block-compatibility as in (2.2) (except when $r = 2$). Yet, in order to define the metaplectic tensor product, it seems to be necessary to have the block-compatibility of the cocycle. To get around it, we will introduce another cocycle τ_M, but this time it is a cocycle only on the Levi subgroup M, and will show that τ_M is cohomologous to the restriction $\tau_r|_{M\times M}$ of τ_r to $M \times M$ both for the local and global cases.

3.1 The Cocycle τ_M

In this subsection, we assume that all the groups are over F if F is local and over $\mathbb{A}$ if F is global, and suppress it from our notation.

We define the cocycle $\tau_M \colon M \times M \to \mu_n$, by

$$\tau_M\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \begin{pmatrix} g'_1 & & \\ & \ddots & \\ & & g'_k \end{pmatrix}\right) = \\ \prod_{i=1}^{k} \tau_{r_i}(g_i, g'_i) \prod_{1\le i<j\le k} \big(\det(g_i), \det(g'_j)\big) \prod_{i\ne j} \big(\det(g_i), \det(g'_j)\big)^{c},$$

where $(\,\cdot\,,\,\cdot\,)$ is the local or global Hilbert symbol. Note that the definition makes sense both locally and globally. Moreover, the global τ_M is the product of the local ones.

We define the group ${}^c\widetilde{M}$ to be ${}^c\widetilde{M} = M \times \mu_n$ as a set and the group structure is given by τ_M. The superscript ^{c} is for "compatible". One advantage to working with ${}^c\widetilde{M}$ is that each $\widetilde{\mathrm{GL}}_{r_i}$ embeds into ${}^c\widetilde{M}$ via the natural map

$$(g_i, \xi) \mapsto \left(\begin{pmatrix} I_{r_1+\cdots+r_{i-1}} & & \\ & g_i & \\ & & I_{r_{i+1}+\cdots+r_k} \end{pmatrix}, \xi\right).$$

Indeed, the cocycle τ_M is so chosen that we have this embedding.

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Also recall our notation

$$M^{(n)} = \mathrm{GL}_{r_1}^{(n)} \times \cdots \times \mathrm{GL}_{r_k}^{(n)} \quad \text{and} \quad \widetilde{M}^{(n)} = \widetilde{\mathrm{GL}}_{r_1}^{(n)} \widetilde{\times} \cdots \widetilde{\times} \widetilde{\mathrm{GL}}_{r_k}^{(n)}.$$

We define ${}^c\widetilde{M}^{(n)}$ analogously to ${}^c\widetilde{M}$; namely, the group structure of ${}^c\widetilde{M}^{(n)}$ is defined via the cocycle τ_M. Of course, ${}^c\widetilde{M}^{(n)}$ is a subgroup of ${}^c\widetilde{M}$. Note that each $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$ naturally embeds into ${}^c\widetilde{M}^{(n)}$ as above.

Lemma 3.1 *The subgroups $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$ and $\widetilde{\mathrm{GL}}_{r_j}^{(n)}$ in ${}^c\widetilde{M}^{(n)}$ commute pointwise for $i \neq j$.*

Proof Locally or globally, it suffices to show $\tau_M(g_i, g_j) = \tau_M(g_j, g_i)$ for $g_i \in \mathrm{GL}_{r_i}^{(n)}$ and $g_j \in \mathrm{GL}_{r_j}^{(n)}$. For the block-compatibility of the 2-cocycle τ_M, we have $\tau_M(g_i, g_j) = \tau_{r_i}(g_i, I_{r_j})\tau_{r_j}(I_{r_j}, g_j) = 1$, and similarly we have $\tau_M(g_j, g_i) = 1$. ∎

Lemma 3.2 *There is a surjection $\widetilde{\mathrm{GL}}_{r_1}^{(n)} \times \cdots \times \widetilde{\mathrm{GL}}_{r_k}^{(n)} \to {}^c\widetilde{M}^{(n)}$ given by the map*

$$\big((g_1, \xi_1), \ldots, (g_k, \xi_k)\big) \longmapsto \left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \xi_1 \cdots \xi_k \right),$$

whose kernel is

$$\mathcal{K}_P := \left\{ \big((1, \xi_1), \ldots, (1, \xi_k)\big) : \xi_1 \cdots \xi_k = 1 \right\},$$

so that ${}^c\widetilde{M}^{(n)} \cong \widetilde{\mathrm{GL}}_{r_1}^{(n)} \times \cdots \times \widetilde{\mathrm{GL}}_{r_k}^{(n)} / \mathcal{K}_P$.

Proof The block-compatibility of τ_M guarantees that the map is indeed a group homomorphism. The description of the kernel is immediate. ∎

3.2 The Relation Between τ_M and τ_r

Note that for the group $\widetilde{M}$ (instead of ${}^c\widetilde{M}$), the group structure is defined by the restriction of τ_r to $M \times M$, and hence each $\widetilde{\mathrm{GL}}_{r_i}$ might not embed into $\widetilde{\mathrm{GL}}_r$ in the natural way because of the possible failure of the block-compatibility of τ_r unless $r = 2$. To make explicit the relation between ${}^c\widetilde{M}$ and $\widetilde{M}$, the discrepancy between τ_M and $\tau_r|_{M \times M}$ (which we denote simply by τ_r) has to be clarified.

Local case:

Assume F is local. Then we have

$$\tau_M\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \begin{pmatrix} g_1' & & \\ & \ddots & \\ & & g_k' \end{pmatrix} \right) = \sigma_r\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \begin{pmatrix} g_1' & & \\ & \ddots & \\ & & g_k' \end{pmatrix} \right) \prod_{i=1}^{k} \frac{s_{r_i}(g_i)s_{r_i}(g_i')}{s_{r_i}(g_i g_i')},$$

so τ_M and $\sigma_r|_{M \times M}$ are cohomologous via the function $\prod_{i=1}^k s_{r_i}$. Here, recall from Section 2.2 that the map $s_{r_i}\colon \mathrm{GL}_{r_i} \to \mu_n$ relates τ_{r_i} with σ_{r_i} by

$$\sigma_{r_i}(g_i, g_i') = \tau_{r_i}(g_i, g_i') \cdot \frac{s_{r_i}(g_i, g_i')}{s_{r_i}(g_i)s_{r_i}(g_i')},$$

for $g_i, g_i' \in \mathrm{GL}_{r_i}$. Moreover, if $|n|_F = 1$, then s_{r_i} is chosen to be "canonical" in the sense that (2.4) is satisfied.

The block-compatibility of σ_r implies

$$\tau_r(m,m')\cdot\frac{s_r(mm')}{s_r(m)s_r(m')}=\sigma_r(m,m')=\tau_M(m,m')\cdot\prod_{i=1}^{k}\frac{s_{r_i}(g_ig_i')}{s_{r_i}(g_i)s_{r_i}(g_i')}$$

for

$$m=\begin{pmatrix}g_1&&\\&\ddots&\\&&g_k\end{pmatrix}\quad\text{and}\quad m'=\begin{pmatrix}g_1'&&\\&\ddots&\\&&g_k'\end{pmatrix}.$$

Hence, if we define $\widehat{s}_M\colon M\to\mu_n$ by

$$\widehat{s}_M(m)=\frac{\prod_{i=1}^{k}s_{r_i}(g_i)}{s_r(m)},\tag{3.1}$$

we have

$$\tau_M(m,m')=\tau_r(m,m')\cdot\frac{\widehat{s}_M(m)\widehat{s}_M(m')}{\widehat{s}_M(mm')};\tag{3.2}$$

namely, τ_r and τ_M are cohomologous via $\widehat{s}_M$. Therefore, we have the isomorphism

$$\alpha_M\colon {}^c\widetilde{M}\longrightarrow\widetilde{M},\quad (m,\xi)\longmapsto(m,\widehat{s}_M(m)\xi).$$

The following lemma will be crucial later for showing that the global τ_M is also cohomologous to $\tau_r|_{M(\mathbb{A})\times M(\mathbb{A})}$.

Lemma 3.3 *Assume F is such that $|n|_F=1$. Then for all $k\in M(\mathcal{O}_F)$, we have $\widehat{s}_M(k)=1$.*

Proof First note that if $k,k'\in M(\mathcal{O}_F)$, then $\tau_r(k,k')=\tau_M(k,k')=1$, and so by (3.2) we have

$$\widehat{s}_M(kk')=\widehat{s}_M(k)\widehat{s}_M(k'),$$

i.e., $\widehat{s}_M$ is a homomorphism on $M_M(\mathcal{O}_F)$. Hence, it suffices to prove the lemma only for the elements $k\in M(\mathcal{O}_F)$ of the form

$$k=\begin{pmatrix}I_{r_1+\cdots+r_{i-1}}&&\\&k_i&\\&&I_{r_{i+1}+\cdots+r_k}\end{pmatrix},$$

where $k_i\in\mathrm{GL}_{r_i}$ is in the i-th place on the diagonal. Namely, we need to prove

$$\frac{s_{r_i}(k_i)}{s_r(k)}=1.$$

In the sequel, we will show that this follows from the "canonicality" of s_r and s_{r_i}, and the fact that the cocycle for SL_{r+1} is block-compatible in a very strong sense as in [BLS, Lemma 5, Theorem 7 §2, p. 145]. Recall from (2.4) that s_r has been chosen to satisfy $s_r|_{\mathrm{GL}_r(\mathcal{O}_F)}=\mathfrak{s}_r|_{l(\mathrm{GL}_r(\mathcal{O}_F))}$, where $\mathfrak{s}_r$ is the map on $\mathrm{SL}_{r+1}(F)$ that makes diagram (2.5) commute, and similarly for s_{r_i} with r replaced by r_i. Let us write

$$l_i\colon\ \mathrm{GL}_{r_i}(F)\longrightarrow\mathrm{SL}_{r_i+1}(F),\quad g_i\longmapsto\begin{pmatrix}g_i&\\&\det(g_i)^{-1}\end{pmatrix}$$

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for the embedding that is used to define the cocycle σ_{r_i}. Define the embedding

$$F\colon \mathrm{SL}_{r_i+1}(F)\to \mathrm{SL}_{r+1}(F),\qquad \begin{pmatrix} A & b\\ c & d\end{pmatrix}\mapsto \begin{pmatrix} I_{r_1+\cdots+r_{i-1}} & & & \\ & A & & b\\ & & I_{r_{i+1}+\cdots+r_k} & \\ & c & & d\end{pmatrix},$$

where A is an $r_i\times r_i$-block and accordingly b is $r_i\times 1$, c is $1\times r_i$ and d is 1×1. Note that this embedding is chosen, so that we have

$$F\big(l_i(k_i)\big) = l(k). \tag{3.3}$$

By the block compatibility of $\sigma_{\mathrm{SL}_{r+1}}$ we have

$$\sigma_{\mathrm{SL}_{r+1}}|_{F(\mathrm{SL}_{r_i+1})\times F(\mathrm{SL}_{r_i+1})} = \sigma_{\mathrm{SL}_{r_i+1}}.$$

This is simply [BLS, Lemma 5, §2]. (The reader has to be careful in that the image $F(\mathrm{SL}_{r_i+1})$ is not a standard subgroup in the sense defined in [BLS, p.143] if one chooses the set Δ of simple roots of SL_{r+1} in the usual way. One can, however, choose Δ differently so that $F(\mathrm{SL}_{r_i+1})$ is indeed a standard subgroup. And all the results of [BLS, §2] are totally independent of the choice of Δ.) This implies that the map $(g_i,\xi)\mapsto(F(g_i),\xi)$ for $(g_i,\xi)\in\widetilde{\mathrm{SL}}_{r_i+1}$ is a homomorphism. Hence the canonical section $\mathrm{SL}_{r+1}(\mathcal{O}_F)\to\widetilde{\mathrm{SL}}_{r+1}(F)$, which is given by $g\mapsto(g,\mathfrak{s}_r(g))$, restricts to the canonical section $\mathrm{SL}_{r_i+1}(\mathcal{O}_F)\to\widetilde{\mathrm{SL}}_{r_i+1}(F)$, which is given by $g_i\mapsto(g_i,\mathfrak{s}_{r_i}(g_i))$. Namely, we have the commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{SL}}_{r_i+1}(\mathcal{O}_F) \arrow[r, "{(g,\xi)\to(F(g),\xi)}"] & \widetilde{\mathrm{SL}}_{r+1}(\mathcal{O}_F) \\ \mathrm{SL}_{r_i+1}(\mathcal{O}_F) \arrow[u, "{g_i\mapsto(g_i,\,\mathfrak{s}_{r_i}(g_i))}"] \arrow[r, "F"] & \mathrm{SL}_{r+1}(\mathcal{O}_F), \arrow[u, "{g\mapsto(g,\,\mathfrak{s}_r(g))}"'] \end{tikzcd}$$

where all the maps are homomorphisms. In particular, we have

$$\mathfrak{s}_r(F(g_i)) = \mathfrak{s}_{r_i}(g_i), \tag{3.4}$$

for all $g_i\in\mathrm{SL}_{r_i+1}(\mathcal{O}_F)$. Thus,

$$\begin{aligned} s_r(k) &= \mathfrak{s}_r(l(k)) && \text{by (2.4)}\\ &= \mathfrak{s}_r(F(l_i(k_i))) && \text{by (3.3)}\\ &= \mathfrak{s}_{r_i}(l_i(k_i)) && \text{by (3.4)}\\ &= s_{r_i}(k_i) && \text{by (2.4) with } r \text{ replaced by } r_i. \end{aligned}$$

The lemma has been proved. ■

Global case: Assume F is a number field. We define $\widehat{s}_M\colon M(\mathbb{A})\to\mu_n$ by

$$\widehat{s}_M\big(\prod_v m_v\big) := \prod_v \widehat{s}_{M_v}(m_v)$$

for $\prod_v m_v\in M(\mathbb{A})$. The product is finite thanks to Lemma 3.3. Since both of the cocycles τ_r and τ_M are the products of the corresponding local ones, one can see that relation (3.2) holds globally as well.

Thus, analogously to the local case, we have the isomorphism

$$\alpha_M\colon {}^c\widetilde{M}(\mathbb{A}) \longrightarrow \widetilde{M}(\mathbb{A}), \quad (m,\xi) \longmapsto (m, \widehat{s}_M(m)\xi).$$

Lemma 3.4 *The splitting of $M(F)$ into ${}^c\widetilde{M}(\mathbb{A})$ is given by*

$$\mathbf{s}_M\colon M(F) \longrightarrow {}^c\widetilde{M}(\mathbb{A}), \quad \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix} \longmapsto \left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \prod_{i=1}^{k} s_i(g_i)^{-1} \right).$$

Proof For each i, the splitting $\mathbf{s}_{r_i}\colon \mathrm{GL}_{r_i}(F) \to \widetilde{\mathrm{GL}}_{r_i}(\mathbb{A})$ is given by

$$g_i \mapsto (g_i, s_{r_i}(g_i)^{-1}),$$

where $\widetilde{\mathrm{GL}}_{r_i}(\mathbb{A})$ is defined via the cocycle τ_{r_i}. The lemma follows by the block-compatibility of τ_M and the product formula for the Hilbert symbol. ■

As in the case of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$, the section $\mathbf{s}_M$ cannot be defined on all of $M(\mathbb{A})$ even set theoretically, because the expression $\prod_i s_{r_i}(g_i)$ does not make sense for all $\mathrm{diag}(g_1,\dots,g_k) \in M(\mathbb{A})$. So we only have a partial set theoretic section

$$\mathbf{s}_M\colon M(\mathbb{A}) \to {}^c\widetilde{M}(\mathbb{A}).$$

Analogously to Proposition 2.8, we have the following proposition.

Proposition 3.5 *The partial section $\mathbf{s}_M$ is defined on both $M(F)$ and $N_M(\mathbb{A})$, where $N_M(\mathbb{A})$ is the unipotent radical of the Borel subgroup of M, and, moreover, it gives rise to a group homomorphism on each of these subgroups. Also for $m \in M(F)$ and $n \in N_M(\mathbb{A})$, both $\mathbf{s}_M(mn)$ and $\mathbf{s}_M(nm)$ are defined and further $\mathbf{s}_M(mn) = \mathbf{s}_M(m)\mathbf{s}_M(n)$ and $\mathbf{s}_M(nm) = \mathbf{s}_M(n)\mathbf{s}_M(m)$.*

Proof This follows from Proposition 2.8 applied to each $\widetilde{\mathrm{GL}}_{r_i}(\mathbb{A})$ together with the block-compatibility of the cocycle τ_M. (Note that one also needs to use the fact that for all g, g' in the subgroup generated by $M(F)$ and $N_M(\mathbb{A})$, we have $(\det(g), \det(g'))_{\mathbb{A}} = 1$.) ■

This splitting is related to the splitting $\mathbf{s}_r\colon \mathrm{GL}_r(F) \to \mathrm{GL}_r(\mathbb{A})$ by the following proposition.

Proposition 3.6 *We have the following commutative diagram:*

$$\begin{tikzcd} {}^c\widetilde{M}(\mathbb{A}) \arrow[r, hook, "\alpha_M"] & \widetilde{\mathrm{GL}}_r(\mathbb{A}) \\ M(F) \arrow[u, "\mathbf{s}_M"] \arrow[r, hook] & \mathrm{GL}_r(F). \arrow[u, "\mathbf{s}_r"'] \end{tikzcd}$$

Proof For

$$m = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix} \in M(F),$$

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we have

$$\alpha_M(\mathbf{s}_M(m)) = \alpha_M\Big(m, \prod_{i=1}^{k} s_{r_i}(g_i)^{-1}\Big) = \Big(m, \widehat{s}_M(m)\prod_{i=1}^{k} s_{r_i}(g_i)^{-1}\Big)$$
$$= \big(m, s_r(m)^{-1}\big) = \mathbf{s}_r(m),$$

where for the elements in $M(F)$, all of s_{r_i} and s_r are defined globally, and the second equality follows from the definition of $\widehat{s}_M$ as in (3.1). ■

This proposition implies the following corollary.

Corollary 3.7 *Assume that π is an automorphic subrepresentation of ${}^c\widetilde{M}(\mathbb{A})$. The representation of $\widetilde{M}(\mathbb{A})$ defined by $\pi \circ \alpha_M^{-1}$ is also automorphic.*

Proof If π is realized in a space V of automorphic forms on ${}^c\widetilde{M}(\mathbb{A})$, then $\pi \circ \alpha_M^{-1}$ is realized in the space of functions of the form $f \circ \alpha_M^{-1}$ for $f \in V$. The automorphy follows from the commutativity of the diagram in the above lemma. ■

The following remark should be kept in mind for the rest of the paper.

Remark 3.8 The results of this subsection essentially show that we can identify ${}^c\widetilde{M}$ (locally or globally) with $\widetilde{M}$. We can even "pretend" that the cocycle τ_r has the block-compatibility property. We need to make the distinction between ${}^c\widetilde{M}$ and $\widetilde{M}$ only when we would like to view the group $\widetilde{M}$ as a subgroup of $\widetilde{\mathrm{GL}}_r$. For most part, however, we will not have to view $\widetilde{M}$ as a subgroup of $\widetilde{\mathrm{GL}}_r$. Hence, we suppress the superscript ^{c} from the notation and always denote ${}^c\widetilde{M}$ simply by $\widetilde{M}$, when there is no danger of confusion. Accordingly, we denote the partial section $\mathbf{s}_M$ simply by $\mathbf{s}$.

3.3 The Center $Z_{\widetilde{M}}$ of $\widetilde{M}$

In this subsection F is either local or global, and, accordingly, we let $R = F$ or $\mathbb{A}$ as in the notation section. All the groups are over R.

For any group H (metaplectic or not), we denote its center by Z_H. In particular for each group $\widetilde{H} \subseteq \widetilde{\mathrm{GL}}_r$, we let $Z_{\widetilde{H}} =$ center of $\widetilde{H}$.

For the Levi part $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_2} \subseteq \mathrm{GL}_r$, we have

$$Z_M = \left\{ \begin{pmatrix} a_1 I_{r_1} & & \\ & \ddots & \\ & & a_k I_{r_k} \end{pmatrix} : a_i \in R^\times \right\}.$$

But for the center $Z_{\widetilde{M}}$ of $\widetilde{M}$, we have $Z_{\widetilde{M}} \subsetneq \widetilde{Z_M}$ in general, and indeed $\widetilde{Z_M}$ might not even be commutative.

In what follows, we will describe $Z_{\widetilde{M}}$ in detail. For this purpose, let us start with the following lemma.

Lemma 3.9 *Assume F is local. Then for each $g \in \mathrm{GL}_r(F)$ and $a \in F^\times$, we have*

$$\sigma_r(g, aI_r)\sigma_r(aI_r, g)^{-1} = \big(\det(g), a^{r-1+2cr}\big).$$

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Proof First let us note that if we write $\sigma_r = \sigma_r^{(c)}$ to emphasize the parameter c, then

$$\sigma_r^{(c)}(g, aI_r)\sigma_r^{(c)}(aI_r, g)^{-1} = \sigma_r^{(0)}(g, aI_r)\sigma_r^{(0)}(aI_r, g)^{-1}\big(\det(g), a^r\big)^{2c},$$

because $(a^r, \det(g))^{-1} = (\det(g), a^r)$. Hence it suffices to show the lemma for the case $c = 0$.

But this can be done by using the recipe provided by [BLS]. Namely, let $g = nt\eta n'$ for $n, n' \in N_B$, $t \in T$ and $\eta \in \mathfrak{M}$. Then

$$\begin{aligned}
\sigma_r(g, aI_r) &= \sigma_r(nt\eta n', aI_r)\\
&= \sigma_r(t\eta, n'aI_r) \quad \text{by Proposition 2.1(ii) and (2)}\\
&= \sigma_r(t\eta, aI_r) \quad \text{by } n'aI_r = aI_rn' \text{ and Proposition 2.1(ii)}\\
&= \sigma_r(t, \eta aI_r)\sigma_r(\eta, aI_r)\sigma_r(t, \eta)^{-1} \quad \text{by Proposition 2.1(i)}\\
&= \sigma_r(t, aI_r\eta)\sigma_r(\eta, aI_r) \quad \text{by Proposition 2.1(vi)}\\
&= \sigma_r(taI_r, \eta)\sigma_r(t, aI_r)\sigma_r(aI_r, \eta)^{-1}\sigma_r(\eta, aI_r) \quad \text{by Proposition 2.1(i)}\\
&= \sigma_r(t, aI_r)\sigma_r(\eta, aI_r) \quad \text{by Proposition 2.1(vi).}
\end{aligned}$$

Now by Proposition 2.1(iv), $\sigma(\eta, aI_r)$ is a product of $(-a, a)$'s, which is 1. Hence, by using Proposition 2.1(v), we have

$$\sigma_r(g, aI_r) = \sigma_r(t, aI_r) = \prod_{i=1}^{r}(t_i, a)^{r-i}.$$

By an analogous computation, one can see

$$\sigma_r(aI_r, g) = \sigma_r(aI_r, t) = \prod_{i=1}^{r}(a, t_i)^{i-1}.$$

Using $(a, t_i)^{-1} = (t_i, a)$, one can see

$$\sigma_r(g, aI_r)\sigma_r(aI_r, g)^{-1} = \prod_{i=1}^{r}(t_i, a)^{r-1}.$$

But this is equal to $(\det(g), a^{r-1})$, because $\det(g) = \prod_{i=1}^{r} t_i$. ■

Note that this lemma immediately implies that the center $Z_{\widetilde{\mathrm{GL}}_r}$ of $\widetilde{\mathrm{GL}}_r$ is indeed as in (2.10), though a different proof is provided in [KP].

With this lemma, we can also prove the following proposition.

Proposition 3.10 *Both locally and globally, the center $Z_{\widetilde{M}}$ is described as*

$$Z_{\widetilde{M}} = \left\{ \begin{pmatrix} a_1I_{r_1} & & \\ & \ddots & \\ & & a_kI_{r_k} \end{pmatrix} : a_i^{r-1+2cr} \in R^{\times n} \ \textit{and} \ a_1 \equiv \cdots \equiv a_r \mod R^{\times n} \right\}.$$

Proof First assume F is local. Let

$$m = \operatorname{diag}(g_1, \ldots, g_k) \in M \quad \text{and} \quad a = \operatorname{diag}(a_1I_{r_1}, \ldots, a_kI_{r_k}).$$

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It suffices to show $\sigma_r(m,a)\sigma_r(a,m)^{-1}=1$ if and only if all a_i are as in the proposition. But

$$\begin{aligned}
&\sigma_r(m,a)\sigma_r(a,m)^{-1}\\
&\quad=\prod_{i=1}^{r}\sigma_{r_i}(g_i,a_iI_{r_i})\sigma_{r_i}(a_iI_{r_i},g_i)^{-1}\prod_{1\le i<j\le r}(\det(g_i),a_j^{r_j})\prod_{i\neq j}\big(\det(g_i),a_j^{r_j}\big)^{c}\\
&\qquad\times\prod_{1\le i<j\le r}\big(a_i^{r_i},\det(g_j)\big)^{-1}\prod_{i\neq j}\big(a_i^{r_i},\det(g_j)\big)^{-c}\\
&\quad=\prod_{i=1}^{r}\sigma_{r_i}(g_i,a_iI_{r_i})\sigma_{r_i}(a_iI_{r_i},g_i)^{-1}\prod_{i\neq j}(\det(g_i),a_j^{r_j})^{1+2c}\\
&\quad=\prod_{i=1}^{r}\big(\det(g_i),a_i^{r_i-1+2cr_i}\big)\prod_{i\neq j}\big(\det(g_i),a_j^{r_j+2cr_j}\big)\\
&\quad=\prod_{i=1}^{r}\Big(\det(g_i),\ a_i^{-1}\prod_{j=1}^{r}a_j^{r_j+2cr_j}\Big),
\end{aligned}$$

where for the third equality we used the above lemma with r replaced by r_i.

Now assume a is such that $(a,1)\in Z_{\widetilde{M}}$. Then the above product must be 1 for any m. In particular, choose m so that $g_j=1$ for all $i\neq j$. Then we must have $(\det(g_i),\ a_i^{-1}\prod_{j=1}^{r}a_j^{r_j+2cr_j})=1$ for all $g_i\in\mathrm{GL}_{r_i}$. This implies

$$a_i^{-1}\prod_{j=1}^{r}a_j^{r_j+2cr_j}\in F^{\times n}$$

for all i. Since this holds for all i, one can see $a_i^{-1}a_j\in F^{\times n}$ for all $i\neq j$, which implies $a_1\equiv\cdots\equiv a_r \mod F^{\times n}$. But if $a_1\equiv\cdots\equiv a_r \mod F^{\times n}$, then

$$\begin{aligned}
\prod_{i=1}^{r}\Big(\det(g_i),a_i^{-1}\prod_{j=1}^{r}a_j^{r_j+2cr_j}\Big)&=\prod_{i=1}^{r}\Big(\det(g_i),a_i^{-1}\prod_{j=1}^{r}a_i^{r_j+2cr_j}\Big)\\
&=\prod_{i=1}^{r}\big(\det(g_i),a_i^{r-1+2cr}\big).
\end{aligned}$$

This must be equal to 1 for any choice of g_i, which gives $a_i^{r-1+2cr}\in F^{\times n}$.

Conversely, if a is of the form as in the proposition, one can see that

$$\sigma_r(m,a)\sigma_r(a,m)^{-1}=\prod_{i=1}^{r}\Big(\det(g_i),a_i^{-1}\prod_{j=1}^{r}a_j^{r_j+2cr_j}\Big)=1$$

for any m.

The global case follows from the local one, because locally by using (2.3) and $am=ma$, one can see $\sigma_r(m,a)\sigma_r(a,m)^{-1}=1$ if and only if $\tau_r(m,a)\tau_r(a,m)^{-1}=1$, and the global τ_r is the product of local ones. ∎

Lemma 3.9 also implies the following lemma.

Lemma 3.11 *Both locally and globally, $\widetilde{Z_{\mathrm{GL}_r}}$ commutes with $\widetilde{\mathrm{GL}}_r^{(n)}$ pointwise.*

Proof The local case is an immediate corollary of Lemma 3.9, because if $g\in\mathrm{GL}_r^{(n)}$, the lemma implies $\sigma_r(g,aI_r)=\sigma_r(aI_r,g)$. Hence, by (2.3), locally $\tau_r(g,aI_r)=$

$\tau_r(aI_r, g)$ for all $g \in \mathrm{GL}_r^{(n)}$ and $a \in F^\times$. Since the global τ_r is the product of the local ones, the global case also follows. ■

Let us mention that in particular, if $n = 2$ and r is even, then $\widetilde{Z_{\mathrm{GL}_r}} \subseteq \widetilde{\mathrm{GL}}_r^{(n)}$ and $\widetilde{Z_{\mathrm{GL}_r}}$ is the center of $\widetilde{\mathrm{GL}}_r^{(n)}$. This fact is used crucially in [T1].

It should be mentioned that this description of the center $Z_{\widetilde{M}}$ easily implies that

$$Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)} = Z_{\widetilde{M}}\widetilde{M}^{(n)}. \tag{3.5}$$

Proposition 3.12 *Both locally and globally, the groups $\widetilde{Z_M}$ and $\widetilde{M}^{(n)}$ commute pointwise, which gives*

$$Z_{\widetilde{M}^{(n)}} = \widetilde{Z_M} \cap \widetilde{M}^{(n)}, \tag{3.6}$$

and hence

$$Z_{\widetilde{\mathrm{GL}}_r}Z_{\widetilde{M}^{(n)}} = Z_{\widetilde{\mathrm{GL}}_r}(\widetilde{Z_M} \cap \widetilde{M}^{(n)}) = \widetilde{Z_M} \cap (Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}).$$

Proof By the block compatibility of the cocycle τ_M, one can see that an element of the form

$$\left(\begin{pmatrix} a_1 I_{r_1} & & \\ & \ddots & \\ & & a_k I_{r_k} \end{pmatrix}, \xi\right)$$

commutes with all the elements in $\widetilde{M}^{(n)}$ if and only if each $(a_i I_{r_i}, \xi)$ commutes with all the elements in $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$. But this is always the case by the above lemma (with r replaced by r_i). This proves the proposition. ■

If F is global, we define

$$Z_{\widetilde{M}}(F) = Z_{\widetilde{M}}(\mathbb{A}) \cap \mathbf{s}\big(M(F)\big),$$

where recall that $\mathbf{s}\colon M(F) \to \widetilde{M}(\mathbb{A})$ is the section that splits $M(F)$. Similarly, we define groups like $Z_{\widetilde{\mathrm{GL}}_r}(F), \widetilde{M}^{(n)}(F)$, etc. Namely in general for any subgroup $\widetilde{H} \subseteq \widetilde{M}(\mathbb{A})$, we define the "$F$-rational points" $\widetilde{H}(F)$ of $\widetilde{H}$ by

$$\widetilde{H}(F) := \widetilde{H} \cap \mathbf{s}\big(M(F)\big). \tag{3.7}$$

3.4 The Abelian Subgroup $A_{\widetilde{M}}$

Again in this subsection, F is local or global, and $R = F$ or $\mathbb{A}$. As we mentioned above, the preimage $\widetilde{Z_M}$ of the center Z_M of the Levi subgroup M might not be even commutative. For later purposes, we let $A_{\widetilde{M}}$ be a closed abelian subgroup of $\widetilde{Z_M}$ containing the center $Z_{\widetilde{\mathrm{GL}}_r}$. Namely, $A_{\widetilde{M}}$ is a closed abelian subgroup such that $Z_{\widetilde{\mathrm{GL}}_r} \subseteq A_{\widetilde{M}} \subseteq \widetilde{Z_M}$. We let $A_M := p(A_{\widetilde{M}})$, where p is the canonical projection. If F is global, we always assume that $A_{\widetilde{M}}(\mathbb{A})$ is chosen compatibly with the local $A_{\widetilde{M}}(F_v)$ in the sense that we have

$$A_M(\mathbb{A}) = \prod_v{}' A_M(F_v).$$

Note that if $A_M(F_v)$ (hence $A_{\widetilde{M}}(F_v)$) is closed, then $A_M(\mathbb{A})$ (hence $A_{\widetilde{M}}(\mathbb{A})$) is closed by Lemma 2.10.

Of course there are many different choices for $A_{\widetilde{M}}$. But we would like to choose $A_{\widetilde{M}}$ so that the following hypothesis is satisfied.

Hypothesis $(*)$ Assume that F is global. The image of $M(F)$ in the quotient $A_M(\mathbb{A})M^{(n)}(\mathbb{A})\backslash M(\mathbb{A})$ is discrete in the quotient topology.

The author does not know if one can always find such $A_{\widetilde{M}}$ for general n, but at least we have the following proposition.

Proposition 3.13 *If $n = 2$, the above hypothesis is satisfied for a suitable choice of $A_{\widetilde{M}}$. For $n > 2$, if $d = \gcd(n, r - 1 + 2cr)$ is such that n divides nr_i/d for all $i = 1, \dots, k$ (which is the case, for example, if $d = 1$), then the above hypothesis is satisfied with $A_{\widetilde{M}} = Z_{\widetilde{M}}$.*

Proof This is proved in Appendix A. ∎

We believe that for any reasonable choice of $A_{\widetilde{M}}$ the above hypothesis is always satisfied, but the author does not at present know how to prove. This is unfortunate in that this subtle technical issue makes the main theorem of the paper conditional when $n > 2$. However, if $n = 2$, our main results are complete, and this is the only case we need for our applications to symmetric square L-functions in [T1,T2], which is the main motivation for this work.

Let us mention that the group $A_M(\mathbb{A})M^{(n)}(\mathbb{A})$ (for any choice of A_M) is a normal subgroup of $M(\mathbb{A})$, and hence the quotient $A_M(\mathbb{A})M^{(n)}(\mathbb{A})\backslash M(\mathbb{A})$ is a group. Accordingly, if the hypothesis is satisfied, the image of $M(F)$ in the quotient is a discrete subgroup and hence closed.

Also, we have

$$A_{\widetilde{M}}(F) = A_{\widetilde{M}}(\mathbb{A}) \cap \mathbf{s}\big(M(F)\big).$$

following the convention as in (3.7), and we set $A_M(F) = p(A_{\widetilde{M}}(F))$.

4 On the Local Metaplectic Tensor Product

In this section we first review the local metaplectic tensor product of Mezo [Me] and then extend his theory further, first by proving that the metaplectic tensor product behaves in the expected way under the Weyl group action, and second by establishing the compatibility of the metaplectic tensor product with parabolic inductions. Hence, in this section, all the groups are over a local (not necessarily nonarchimedean) field F unless otherwise stated. Accordingly, we assume that our metaplectic group is defined by the block-compatible cocycle σ_r of [BLS], and hence by $\widetilde{\mathrm{GL}}_r$, we actually mean ${}^{\sigma}\widetilde{\mathrm{GL}}_r$.

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4.1 Mezo's Metaplectic Tensor Product

Let $\pi_1, \dots, \pi_k$ be irreducible genuine representations of $\widetilde{\mathrm{GL}}_{r_1}, \dots, \widetilde{\mathrm{GL}}_{r_k}$, respectively. The construction of the metaplectic tensor product takes several steps. First, for each i, fix an irreducible constituent $\pi_i^{(n)}$ of the restriction $\pi_i|_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}}$ of π_i to $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$. Then we have

$$\pi_i|_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}} = \sum_g m_i\, {}^g(\pi_i^{(n)}),$$

where g runs through a finite subset of $\widetilde{\mathrm{GL}}_{r_i}$, m_i is a positive multiplicity, and ${}^g(\pi_i^{(n)})$ is the representation twisted by g. Then we construct the tenor product representation

$$\pi_1^{(n)} \otimes \cdots \otimes \pi_k^{(n)}$$

of the group $\widetilde{\mathrm{GL}}_{r_1}^{(n)} \times \cdots \times \widetilde{\mathrm{GL}}_{r_k}^{(n)}$. Note that this group is merely the direct product of the groups $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$. The genuineness of the representations $\pi_1^{(n)}, \dots, \pi_k^{(n)}$ implies that this tensor product representation descends to a representation of the group $\widetilde{\mathrm{GL}}_{r_1}^{(n)} \widetilde{\times} \cdots \widetilde{\times} \widetilde{\mathrm{GL}}_{r_k}^{(n)}$, *i.e.*, the representation factors through the natural surjection

$$\widetilde{\mathrm{GL}}_{r_1}^{(n)} \times \cdots \times \widetilde{\mathrm{GL}}_{r_k}^{(n)} \twoheadrightarrow \widetilde{\mathrm{GL}}_{r_1}^{(n)} \widetilde{\times} \cdots \widetilde{\times} \widetilde{\mathrm{GL}}_{r_k}^{(n)} = \widetilde{M}^{(n)}.$$

We denote this representation of $\widetilde{M}^{(n)}$ by

$$\pi^{(n)} := \pi_1^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k^{(n)},$$

and call it the metaplectic tensor product of $\pi_1^{(n)}, \dots, \pi_k^{(n)}$. Let us note that the space $V_{\pi^{(n)}}$ of $\pi^{(n)}$ is simply the tensor product $V_{\pi_1^{(n)}} \otimes \cdots \otimes V_{\pi_k^{(n)}}$ of the spaces of $\pi_i^{(n)}$. Let ω be a character on $Z_{\widetilde{\mathrm{GL}}_r}$ such that for all $(aI_r, \xi) \in Z_{\widetilde{\mathrm{GL}}_r} \cap \widetilde{M}^{(n)}$, where $a \in F^\times$ we have

$$\omega(aI_r, \xi) = \pi^{(n)}(aI_r, \xi) = \xi \pi_1^{(n)}(aI_{r_1}, 1) \cdots \pi_k^{(n)}(aI_{r_k}, 1).$$

Namely, ω agrees with $\pi^{(n)}$ on the intersection $Z_{\widetilde{\mathrm{GL}}_r} \cap \widetilde{M}^{(n)}$. We can extend $\pi^{(n)}$ to the representation $\pi_\omega^{(n)} := \omega\pi^{(n)}$ of $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$ by letting $Z_{\widetilde{\mathrm{GL}}_r}$ act by ω. Now extend the representation $\pi_\omega^{(n)}$ to a representation ρ_ω of a subgroup $\widetilde{H}$ of $\widetilde{M}$ so that ρ_ω satisfies Mackey's irreducibility criterion, and so the induced representation

$$\pi_\omega := \mathrm{Ind}_{\widetilde{H}}^{\widetilde{M}} \rho_\omega \tag{4.1}$$

is irreducible. It is always possible to find such $\widetilde{H}$, and, moreover, $\widetilde{H}$ can be chosen to be normal. Mezo shows in [Me] that π_ω is dependent only on ω and is independent of the other choices made throughout, namely, the choices of $\pi_i^{(n)}$, $\widetilde{H}$, and ρ_ω. We write $\pi_\omega = (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$ and call it the metaplectic tensor product of $\pi_1, \dots, \pi_k$ with the character ω.

Mezo also shows that the metaplectic tensor product π_ω is unique up to twist.

Proposition 4.1 *Let $\pi_1, \dots, \pi_k$ and $\pi'_1, \dots, \pi'_k$ be representations of $\widetilde{\mathrm{GL}}_{r_1}, \dots, \widetilde{\mathrm{GL}}_{r_k}$. They give rise to isomorphic metaplectic tensor products with a character ω,* i.e.,

$$(\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega \cong (\pi'_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi'_k)_\omega,$$

if and only if for each i there exists a character ω_i of $\widetilde{\mathrm{GL}}_{r_i}$ trivial on $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$ such that $\pi_i \cong \omega_i \otimes \pi_i'$.

Proof This is [Me, Lemma 5.1]. ■

Remark 4.2 Though the metaplectic tensor product generally depends on the choice of ω, if the center $Z_{\widetilde{\mathrm{GL}}_r}$ is already contained in $\widetilde{M}^{(n)}$, we have $\pi_\omega^{(n)} = \pi^{(n)}$ and hence there is no actual choice for ω and the metaplectic tensor product is canonical. This is the case, for example, when $n = 2$ and r is even, which is one of the important cases we consider in our applications in [T1, T2].

Remark 4.3 Equality (3.5) implies that extending a representation $\pi^{(n)}$ of $\widetilde{M}^{(n)}$ to $\pi_\omega^{(n)}$ multiplying the character ω on $Z_{\widetilde{\mathrm{GL}}_r}$ is the same as extending it by multiplying an appropriate character on $Z_{\widetilde{M}}$.

Let us mention the following, which is not made explicit in [Me].

Lemma 4.4 *Let π_ω be an irreducible admissible representation of $\widetilde{M}$ where ω is the character on $Z_{\widetilde{\mathrm{GL}}_r}$ defined by $\omega = \pi_\omega|_{Z_{\widetilde{\mathrm{GL}}_r}}$. Then there exist irreducible admissible representations $\pi_1, \dots \pi_k$ of $\widetilde{\mathrm{GL}}_{r_1}, \dots, \widetilde{\mathrm{GL}}_{r_k}$, respectively, such that*

$$\pi_\omega = (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_k)_\omega.$$

Namely, a representation of $\widetilde{M}$ is always a metaplectic tensor product.

Proof The restriction $\pi_\omega|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}$ contains a representation of the form

$$\omega\big(\pi_1^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k^{(n)}\big)$$

for some representations $\pi_i^{(n)}$ of $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$. Let π_i be an irreducible constituent of

$$\mathrm{Ind}_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}}^{\widetilde{\mathrm{GL}}_{r_i}} \pi_i^{(n)}.$$

Then one can see that π_ω is $(\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_k)_\omega$. ■

From Mezo's construction, one can tell that essentially the representation theory of the group $\widetilde{M}$ is determined by that of $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$. Let us briefly explain why this is so. Let π be an irreducible admissible representation of $\widetilde{M}$, and let $\chi_\pi \colon \widetilde{M} \to \mathbb{C}$ be the distribution character. If π is genuine, so is χ_π. Namely, $\chi_\pi((1,\xi)\widetilde{m}) = \xi\chi_\pi(\widetilde{m})$ for all $\xi \in \mu_n$ and $\widetilde{m} \in \widetilde{M}$. But if $\widetilde{m} \in \widetilde{M}$ is a regular element but not in $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$, then one can find $\xi \in \mu_n$ with $\xi \neq 1$ such that $(1,\xi)\widetilde{m}$ is conjugate to $\widetilde{m}$. This is proved in the same way as [KP, Proposition 0.1.4]. (The only modification one needs is to choose $A \subset M_r(F)$ in their proof so that $A \subset M_{r_1}(F) \times \cdots \times M_{r_k}(F)$.) Therefore, for such $\widetilde{m}$, one has $\chi_\pi(\widetilde{m}) = 0$. Namely, the support of χ_π is contained in $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$. (Indeed, this argument by the distribution character is crucially used in [Me, Lemma 4.2].) This explains why π is essentially determined by the restriction $\pi|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}$.

This idea can be observed in the following lemma.

Lemma 4.5 *Let π and π' be irreducible admissible representations of $\widetilde{M}$. Then π and π' are equivalent if and only if $\pi|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}$ and $\pi'|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}$ have an equivalent constituent.*

Proof This follows from Proposition 4.1 and Lemma 4.4. ■

Proposition 4.6 *We have*

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{M}}\pi_\omega^{(n)} = m\pi_\omega$$

for some finite multiplicity m, so every constituent of $\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{M}}\pi_\omega^{(n)}$ is isomorphic to π_ω.

Proof By inducting in stages, we have

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{M}}\pi_\omega^{(n)} = \mathrm{Ind}_{\widetilde{H}}^{\widetilde{M}}\,\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{H}}\pi_\omega^{(n)},$$

where $\widetilde{H}$ is as in (4.1), and by [Me, Lemma 4.1] we have

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{H}}\pi_\omega^{(n)} = \bigoplus_\chi \chi\otimes\rho_\omega,$$

where χ runs over the finite set of characters of $\widetilde{H}$ that are trivial on $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$. Moreover, it is shown in [Me, Lemma 4.1] that any extension of $\pi_\omega^{(n)}$ to $\widetilde{H}$ is of the form $\chi\otimes\rho_\omega$ and $\mathrm{Ind}_{\widetilde{H}}^{\widetilde{M}}\chi\otimes\rho_\omega = \pi_\omega$ for all χ by [Me, Lemma 4.2]. Hence, we have

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{M}}\pi_\omega^{(n)} = \bigoplus_\chi \mathrm{Ind}_{\widetilde{H}}^{\widetilde{M}}\chi\otimes\rho_\omega = m\pi_\omega.$$ ■

Let ω be as above and $A_{\widetilde{M}}$ as in Section 3.4. The restriction $\pi^{(n)}|_{A_{\widetilde{M}}\cap\widetilde{M}^{(n)}}$ gives a character on $A_{\widetilde{M}}\cap\widetilde{M}^{(n)}$, because $A_{\widetilde{M}}\cap\widetilde{M}^{(n)}$ is contained in the center of $\widetilde{M}^{(n)}$ by (3.6). The product $\omega(\pi^{(n)}|_{A_{\widetilde{M}}\cap\widetilde{M}^{(n)}})$ of ω and $\pi^{(n)}|_{A_{\widetilde{M}}\cap\widetilde{M}^{(n)}}$ defines a character on $Z_{\widetilde{\mathrm{GL}}_r}(A_{\widetilde{M}}\cap\widetilde{M}^{(n)})$, because the two characters agree on $Z_{\widetilde{\mathrm{GL}}_r}\cap(A_{\widetilde{M}}\cap\widetilde{M}^{(n)})$. Since the Pontryagin dual is an exact functor, one can extend it to a character on $A_{\widetilde{M}}$, which we denote again by ω. Namely, ω is a character on $A_{\widetilde{M}}$ extending ω such that $\omega(a) = \pi^{(n)}(a)$ for all $a\in A_{\widetilde{M}}\cap\widetilde{M}^{(n)}$. With this said, we have the following corollary.

Corollary 4.7 *Let ω be the character on $A_{\widetilde{M}}$ described above, and let $\pi_\omega^{(n)} := \omega\pi^{(n)}$ be the representation of $A_{\widetilde{M}}\widetilde{M}^{(n)}$ extending $\pi^{(n)}$ by letting $A_{\widetilde{M}}$ act as ω. Then*

$$\mathrm{Ind}_{A_{\widetilde{M}}\widetilde{M}^{(n)}}^{\widetilde{M}}\pi_\omega^{(n)} = m'\pi_\omega,$$

where m' is some finite multiplicity.

Proof This follows from the previous proposition, because we have the inclusion $\mathrm{Ind}_{A_{\widetilde{M}}\widetilde{M}^{(n)}}^{\widetilde{M}}\pi_\omega^{(n)} \hookrightarrow \mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{M}}\pi_\omega^{(n)}$. ■

4.2 The Archimedean Case

Strictly speaking, Mezo assumes that the field F is non-archimedean. If $F = \mathbb{C}$, then $\widetilde{M}^{(n)} = \widetilde{M}$. Indeed, $\widetilde{M}(\mathbb{C}) = M(\mathbb{C}) \times \mu_n$ (direct product), and the metaplectic tensor product is obtained simply by taking the tensor product $\pi_1 \otimes \cdots \otimes \pi_k$ and descending it to $\widetilde{M}(\mathbb{C})$. Hence, there is essentially no discrepancy between the metaplectic case and the non-metaplectic one.

If $F = \mathbb{R}$ (so necessarily $n = 2$), one can trace the argument of Mezo and make sure the construction works for this case as well, with the proviso that equivalence has to be considered as infinitesimal equivalence. However, it has been communicated to the author by J. Adams that for this case, the induced representation $\operatorname{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{M}} \pi_\omega^{(n)}$ is always irreducible. Hence one can simply define the metaplectic tensor product to be this induced representation.

4.3 Twists by Weyl Group Elements

As in the notation section, we let W_M be the subset of the Weyl group W_{GL_r} consisting of only those elements that permute the GL_{r_i}-factors of $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$. Though W_M is not a group in general, it is identified with the group S_k of permutations of k letters. Assume $w \in W_M$ is such that

$$M' := wMw^{-1} = \mathrm{GL}_{r_{\sigma(1)}} \times \cdots \times \mathrm{GL}_{r_{\sigma(k)}}$$

for a permutation $\sigma \in S_k$, and so $w(g_1, \dots, g_k)w^{-1} = (g_{\sigma(1)}, \dots, g_{\sigma(k)})$ for each $(g_1, \dots, g_k) \in M$. Namely, w corresponds to the permutation σ^{-1}. Then we have

$$\widetilde{M'} = \mathbf{s}(w)\widetilde{M}\mathbf{s}(w)^{-1}.$$

Let $\pi = (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$ be an irreducible admissible representation of $\widetilde{M}$. As in the notation section, one can define the twist ${}^{\mathbf{s}(w)}\pi$ of π by $\mathbf{s}(w)$ to be the representation of $\widetilde{M'}$ on the space V_π given by ${}^{\mathbf{s}(w)}\pi(\widetilde{m}') = \pi(\mathbf{s}(w)^{-1}\widetilde{m}'\mathbf{s}(w))$ for $\widetilde{m}' \in \widetilde{M'}$. To ease the notation we simply write ${}^{w}\pi := {}^{\mathbf{s}(w)}\pi$. Actually, since $\mu_n \subseteq \widetilde{M}$ is in the center, for any preimage $\widetilde{w}$ of w, we have ${}^{\mathbf{s}(w)}\pi = {}^{\widetilde{w}}\pi$, and hence the notation ${}^{w}\pi$ is not ambiguous.

The goal of this subsection is to show that the metaplectic tensor product behaves in the expected way under the Weyl group action. Namely, we will prove the following theorem.

Theorem 4.8 *With the above notations, we have*

$$ {}^{w}(\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega \cong (\pi_{\sigma(1)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{\sigma(k)})_\omega. \tag{4.2}$$

To prove this, we first need the following lemma.

Lemma 4.9 *For each $(m, 1) \in \widetilde{M}^{(n)}$ and $w \in W_M$, where $m \in M^{(n)}$, we have*

$$\mathbf{s}(w)(m, 1)\mathbf{s}(w)^{-1} = (wmw^{-1}, 1),$$

namely, $\mathbf{s}(w)\mathbf{s}(m)\mathbf{s}(w)^{-1} = \mathbf{s}(wmw^{-1})$.

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Proof Note that $\mathbf{s}(w) = (w, 1)$ and $\mathbf{s}(w)^{-1} = (w^{-1}, \sigma_r(w, w^{-1})^{-1})$ because we are using ${}^{\sigma}\widetilde{\mathrm{GL}}_r$, and hence

$$\mathbf{s}(w)(m,1)\mathbf{s}(w)^{-1} = \left(wmw^{-1}, \sigma_r(w, mw^{-1})\sigma_r(m, w^{-1})\sigma_r(w, w^{-1})^{-1}\right).$$

Let

$$\varphi_w(m) := \sigma_r(w, mw^{-1})\sigma_r(m, w^{-1})\sigma_r(w, w^{-1})^{-1}.$$

We need to show that $\varphi_w(m) = 1$ for all $m \in M^{(n)}$. Let us first show that the map $m \mapsto \varphi_w(m)$ is a homomorphism on $M^{(n)}$. To see it, for $m, m' \in M^{(n)}$, we have

$$\begin{aligned}\mathbf{s}(w)(m,1)(m',1)\mathbf{s}(w)^{-1} &= \mathbf{s}(w)\left(mm', \sigma_r(m,m')\right)\mathbf{s}(w)^{-1}\\ &= \left(wmm'w^{-1}, \sigma_r(m,m')\varphi_w(mm')\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathbf{s}(w)(m,1)(m',1)\mathbf{s}(w)^{-1} &= \mathbf{s}(w)(m,1)\mathbf{s}(w)^{-1}\mathbf{s}(w)(m',1)\mathbf{s}(w)^{-1}\\ &= \left(wmw^{-1}, \varphi_w(m)\right)\left(wm'w^{-1}, \varphi_w(m')\right)\\ &= \left(wmm'w^{-1}, \sigma_r(wmw^{-1}, wm'w^{-1})\varphi_w(m)\varphi_w(m')\right)\\ &= \left(wmm'w^{-1}, \sigma_r(m,m')\varphi_w(m)\varphi_w(m')\right),\end{aligned}$$

where the last equality follows because $\sigma_r(wmw^{-1}, wm'w^{-1}) = \sigma_r(m,m')$ by the block-compatibility of σ_r. Hence, by comparing those two, one obtains $\varphi_w(mm') = \varphi_w(m)\varphi_w(m')$. Therefore, to show $\varphi_w(m) = 1$, it suffices to show it for the elements of the form

$$m = \mathrm{diag}(I_{r_1}, \dots, I_{r_{i-1}}, g_i, I_{r_{i+1}}, \dots, I_{r_k}) \tag{4.3}$$

for $g_i \in \mathrm{GL}_{r_i}^{(n)}$.

Then one can rewrite $\varphi_w(m)$ as follows:

$$\begin{aligned}&\varphi_w(m)\\ &\quad= \sigma_r(w, mw^{-1})\sigma_r(m, w^{-1})\sigma_r(w, w^{-1})^{-1}\\ &\quad= \sigma_r(w, w^{-1}wmw^{-1})\sigma_r(m, w^{-1})\sigma_r(w, w^{-1})^{-1}\\ &\quad= \sigma_r(ww^{-1}, wmw^{-1})\sigma_r(w, w^{-1})\sigma_r(w^{-1}, wmw^{-1})^{-1}\sigma_r(m, w^{-1})\sigma_r(w, w^{-1})^{-1}\\ &\quad= \sigma_r(w^{-1}, wmw^{-1})^{-1}\sigma_r(m, w^{-1}),\end{aligned}$$

where for the third equality we used Proposition 2.1(i). So we only have to show

$$\sigma_r(w^{-1}, wmw^{-1})^{-1}\sigma_r(m, w^{-1}) = 1. \tag{4.4}$$

This can be shown by using the algorithm computing the cocycle σ_r given by [BLS]. To use the results of [BLS], it should be mentioned that one needs to use the set $\mathfrak{M}$ for a set of representatives of the Weyl group of GL_r as defined in the notation section. Also, let us recall the following notation from [BLS]. For each $g \in \mathrm{GL}_r$, the "torus part function" $\mathbf{t}\colon \mathrm{GL}_r \to T$ is the unique map such that $\mathbf{t}(nt\eta n') = t$, where $n, n' \in N_B$, $t \in T$ and $\eta \in \mathfrak{M}$ when GL_r is written as

$$\mathrm{GL}_r = \coprod_{\eta \in \mathfrak{M}} N_B T \eta N_B$$

by the Bruhat decomposition. Namely, $\mathbf{t}(g)$ is the "torus part" of g.

Using this language, each $w \in W_M$ is written as $w = \mathbf{t}(w)\eta_w$, where $\eta_w \in \mathfrak{M}$, and $\mathbf{t}(w) \in \mathrm{GL}_{r_{\sigma(1)}} \times \cdots \times \mathrm{GL}_{r_{\sigma(k)}}$ is of the form

$$\mathbf{t}(w) = \left(\varepsilon_{\sigma(1)} I_{\sigma(1)}, \ldots, \varepsilon_{\sigma(k)} I_{\sigma(k)}\right),$$

where $\varepsilon_i \in \{\pm 1\}$.

We are now ready to carry out our cocycle computations for (4.4). Let us deal with $\sigma_r(m, w^{-1})$ first. Write $m = nt\eta n'$ by the Bruhat decomposition, so $\mathbf{t}(m) = t$. But recall that we are assuming m is of the form as in (4.3), so the decomposition $nt\eta n'$ takes place essentially inside the GL_{r_i}-block. In particular, we can write

$$m = \mathrm{diag}(I_{r_1}, \ldots, I_{r_{i-1}}, n_i t_i \eta_i n_i', I_{r_{i+1}} \ldots, I_{r_k}),$$

where $t_i \in \mathrm{GL}_{r_i}^{(n)}$. (Note that $\det(t_i) \in F^{\times n}$.) Then one can compute $\sigma_r(m, w^{-1})$ as follows:

$$\begin{aligned}
\sigma_r(m, w^{-1}) &= \sigma_r(nt\eta n', w^{-1}) \\
&= \sigma_r(t\eta, n'w^{-1}) \quad \text{by Proposition 2.1(ii), (iii)} \\
&= \sigma_r(t\eta, w^{-1}wn'w^{-1}) \\
&= \sigma_r(t\eta, w^{-1}) \quad \text{because } wn'w^{-1} \in N_B \text{ and by Proposition 2.1(ii)} \\
&= \sigma_r(t\eta, \mathbf{t}(w^{-1})\eta_{w^{-1}}).
\end{aligned}$$

Now since η is essentially inside the GL_{r_i}-factor of M and $\eta_{w^{-1}}$ only permutes the GL_{r_j}-factors of M, we have $l(\eta\eta_{w^{-1}}) = l(\eta) + l(\eta_{w^{-1}})$, where l is the length function. Hence, by applying [BLS, Lemma 10, p. 155], we have

$$\sigma_r\left(t\eta, \mathbf{t}(w^{-1})\eta_{w^{-1}}\right) = \sigma_r\left(t, \eta\mathbf{t}(w^{-1})\eta^{-1}\right)\sigma_r\left(\eta, \mathbf{t}(w^{-1})\right). \tag{4.5}$$

Here note that $\mathbf{t}(w^{-1}) \in M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$ is of the form $(\varepsilon_1 I_{r_1}, \ldots, \varepsilon_k I_{r_k})$ and η is in the GL_{r_i}-block. Hence, $\eta\mathbf{t}(w^{-1})\eta^{-1} = \mathbf{t}(w^{-1})$. Thus, by the block-compatibility of σ_r, (4.5) is written as

$$\sigma_{r_i}(t_i, \varepsilon_i I_{r_i})\sigma_{r_i}(\eta_i, \varepsilon_i I_{r_i}).$$

Clearly, if $\varepsilon_i = 1$, then both $\sigma_{r_i}(t_i, \varepsilon_i I_{r_i})$ and $\sigma_{r_i}(\eta_i, \varepsilon_i I_{r_i})$ are 1. If $\varepsilon_i = -1$, then by Proposition 2.1(iv), one can see that $\sigma_{r_i}(\eta_i, \varepsilon_i I_{r_i}) = 1$. Hence, in either case, one has

$$\sigma_r(m, w^{-1}) = \sigma_{r_i}(t_i, \varepsilon_i I_{r_i}).$$

Next let us deal with $\sigma_r(w^{-1}, wmw^{-1})$ in (4.4). First, by the analogous computation to what we did for $\sigma(m, w^{-1})$, one can write

$$\sigma_r(w^{-1}, wmw^{-1}) = \sigma_r(w^{-1}, wt\eta w^{-1}) = \sigma_r\left(\mathbf{t}(w^{-1})\eta_{w^{-1}}, wt\eta w^{-1}\right). \tag{4.6}$$

Since w corresponds to the permutation σ^{-1}, if we let

$$\tau_i\colon \mathrm{GL}_{r_i} \longrightarrow wMw^{-1} = \mathrm{GL}_{r_{\sigma(1)}} \times \cdots \times \mathrm{GL}_{r_{\sigma(k)}}$$

be the embedding of GL_{r_i} into the corresponding GL_{r_i}-factor of wMw^{-1}, then (4.6) can be written as

$$\sigma_r\left(\mathbf{t}(w^{-1})\eta_{w^{-1}}, \tau_i(t_i)\tau_i(\eta_i)\right).$$

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Note that $\tau_i(\eta_i) \in \mathfrak{M}$ and $l(\eta_{w^{-1}}\tau_i(\eta_i)) = l(\eta_{w^{-1}}) + l(\tau_i(\eta_i))$. Hence, by using [BLS, Lemma 10, p. 155], this can be written as

$$\sigma_r\Big(\mathbf{t}(w^{-1}), \eta_{w^{-1}}\tau_i(t_i)\eta_{w^{-1}}^{-1}\Big)\,\sigma_r\Big(\eta_{w^{-1}}, \tau_i(t_i)\Big)\,. \tag{4.7}$$

By the block compatibility of σ_r, one can see that

$$\sigma_r\Big(\mathbf{t}(w^{-1}), \eta_{w^{-1}}\tau_i(t_i)\eta_{w^{-1}}^{-1}\Big) = \sigma_{r_i}(\varepsilon_i I_{r_i}, t_i).$$

Also, to compute $\sigma_r(\eta_{w^{-1}}, \tau_i(t_i))$, one needs to use Proposition 2.1(iv). For this purpose, let us write

$$t_i = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{r_i} \end{pmatrix} \in \mathrm{GL}_{r_i}$$

where $\det(t_i) = a_1 \cdots a_{r_i} \in F^{\times n}$. By looking at the formula in Proposition 2.1(iv), one can see that $\sigma_r(\eta_{w^{-1}}, \tau_i(t_i))$ is a power of $(-1, a_1) \cdots (-1, a_{r_i})$, which is equal to $(-1, a_1 \cdots a_{r_i}) = 1$, because $\det(t_i) = a_1 \cdots a_{r_i} \in F^{\times n}$. Hence (4.7), which is the same as (4.6), becomes $\sigma_{r_i}(\varepsilon_i I_{r_i}, t_i)$. Hence the left-hand side of (4.4) can be written as

$$\sigma_{r_i}(\varepsilon_i I_{r_i}, t_i)^{-1}\sigma_{r_i}(t_i, \varepsilon_i I_{r_i}).$$

We need to show that this is 1. But clearly this is the case if $\varepsilon_i = 1$. So let us assume $\varepsilon_i = -1$. Namely, we will show $\sigma_{r_i}(-I_{r_i}, t_i)^{-1}\sigma_{r_i}(t_i, -I_{r_i}) = 1$. But by Proposition 2.1(v), one can compute

$$\begin{aligned}
\sigma_{r_i}(-I_{r_i}, t_i) &= (-1, a_2)(-1, a_3)^2(-1, a_4)^3 \cdots (-1, a_r)^{r-1+2c},\\
\sigma_{r_i}(t_i, -I_{r_i}) &= (a_1, -1)^{r-1}(a_2, -1)^{r-2}(a_3, -1)^{r-3} \cdots (-1, a_{r-1}).
\end{aligned}$$

Noting that $(-1, a_i)^{-1} = (a_i, -1)$, we have

$$\sigma_{r_i}(-I_{r_i}, t_i)^{-1}\sigma_{r_i}(t_i, -I_{r_i}) = \prod_{i=1}^{r}(a_i, -1)^{r-1+2c} = \Big(\prod_{i=1}^{r} a_i, -1\Big)^{r-1+2c} = 1,$$

where the last equality follows, because $\det(t_i) = \prod_{i=1}^{r} a_i \in F^{\times n}$. This completes the proof. ∎

We are now ready to prove Theorem 4.8.

Proof of Theorem 4.8 By restricting to $\widetilde{M'}^{(n)}$, one can see that the left-hand side of (4.2) contains the representation ${}^{w}(\pi_1^{(n)}\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k^{(n)})$ and the right-hand side of (4.2) contains $\pi_{\sigma(1)}^{(n)}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\pi_{\sigma(k)}^{(n)}$, where ${}^{w}(\pi_1^{(n)}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\pi_k^{(n)})$ is the representation of

$$\widetilde{M'}^{(n)} = \mathbf{s}(w)\widetilde{M}^{(n)}\mathbf{s}(w)^{-1}$$

whose space is the space of $\pi_1^{(n)}\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k^{(n)}$. Hence, by Lemma 4.5, it suffices to show that

$${}^{w}(\pi_1^{(n)}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\pi_k^{(n)}) \cong \pi_{\sigma(1)}^{(n)}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\pi_{\sigma(k)}^{(n)}.$$

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But this can be seen from the commutative diagram

$$\begin{tikzcd}
& \widetilde{\mathrm{GL}}^{(n)}_{r_{\sigma(1)}} \times \cdots \times \widetilde{\mathrm{GL}}^{(n)}_{r_{\sigma(k)}} \arrow[d] \arrow[ddl] \arrow[ddr] & \\
& \widetilde{M'}^{(n)} \arrow[dl] \arrow[dr] & \\
\mathrm{Aut}(V_{\pi_1^{(n)}} \otimes \cdots \otimes V_{\pi_k^{(n)}}) \arrow[rr, "\sim"] & & \mathrm{Aut}(V_{\pi^{(n)}_{\sigma(1)}} \otimes \cdots \otimes V_{\pi^{(n)}_{\sigma(k)}}),
\end{tikzcd}$$

where the leftmost arrow is the representation of $\widetilde{\mathrm{GL}}^{(n)}_{r_{\sigma(1)}} \times \cdots \times \widetilde{\mathrm{GL}}^{(n)}_{r_{\sigma(k)}}$ (direct product) acting on the space of $\pi_1^{(n)} \otimes \cdots \otimes \pi_k^{(n)}$ by permuting each factor by σ^{-1}, which descends to the representation ${}^{w}(\pi_1^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k^{(n)})$ of $\widetilde{M}^{(n)}$. To see that this indeed descends to ${}^{w}(\pi_1^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k^{(n)})$, one uses the above lemma. ■

4.4 Compatibility with Parabolic Induction

We will show the compatibility of the metaplectic tensor product with parabolic induction. Hence, we consider the standard parabolic subgroup $P = MN \subseteq \mathrm{GL}_r$ where M is the Levi part and N the unipotent radical.

Lemma 4.10 *The image N^* of the unipotent radical N via the section $\mathbf{s}\colon \mathrm{GL}_r \to \widetilde{\mathrm{GL}}_r$ is normalized by the metaplectic preimage $\widetilde{M}$ of the Levi part M.*

Proof This is known not only for $\widetilde{\mathrm{GL}}_r$ but for any covering group (see [MW, Appendix I]), but we will give a simple proof for $\widetilde{\mathrm{GL}}_r$. Let $\widetilde{m} \in \widetilde{M}$ and $(n,1) \in N^*$, where $n \in N$. (Note that since we are assuming the group $\widetilde{\mathrm{GL}}_r$ is defined by σ_r, each element in N^* is written as $(n,1)$.) We may assume $\widetilde{m} = (m,1)$ for $m \in M$. Noting that $\widetilde{m}^{-1} = (m^{-1}, \sigma_r(m,m^{-1})^{-1})$, we compute

$$\begin{aligned}
\widetilde{m}(n,1)\widetilde{m}^{-1} &= (m,1)(n,1)\big(m^{-1}, \sigma_r(m,m^{-1})^{-1}\big) \\
&= \big(mn, \sigma_r(m,n)\big)\,\big(m^{-1}, \sigma_r(m,m^{-1})^{-1}\big) \\
&= \big(mnm^{-1}, \sigma_r(mn,m^{-1})\sigma_r(m,n)\sigma_r(m,m^{-1})^{-1}\big).
\end{aligned}$$

By Proposition 2.1(ii), $\sigma_r(m,n) = 1$. Also, since $mnm^{-1} \in N$, we have

$$\sigma_r(mn,m^{-1}) = \sigma_r(mnm^{-1}m, m^{-1}) = \sigma_r(m,m^{-1})$$

again by Proposition 2.1(ii). Thus, we have $\widetilde{m}(n,1)\widetilde{m}^{-1} = (mnm^{-1},1) \in N^*$. ■

By this lemma, we can write $\widetilde{P} = \widetilde{M}N^*$, where $\widetilde{M}$ normalizes N^* and hence for a representation π of $\widetilde{M}$ one can form the induced representation $\mathrm{Ind}_{\widetilde{M}N^*}^{\widetilde{\mathrm{GL}}_r} \pi$ by letting N^* act trivially.

Theorem 4.11 *Let $P = MN \subseteq \mathrm{GL}_r$ be the standard parabolic subgroup whose Levi part is $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$. Further, for each $i = 1, \dots, k$, let $P_i = M_iN_i \subseteq \mathrm{GL}_{r_i}$*

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be the standard parabolic of GL_{r_i} *whose Levi part is* $M_i = \mathrm{GL}_{r_{i,1}} \times \cdots \times \mathrm{GL}_{r_{i,l_i}}$. *For each i, we are given a representation*

$$\sigma_i := (\tau_{i,1} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{i,l_i})_{\omega_i}$$

of $\widetilde{M}_i$, *which is given as the metaplectic tensor product of the representations* $\tau_{i,1}, \ldots, \tau_{i,l_i}$ *of* $\widetilde{\mathrm{GL}}_{r_{i,1}}, \ldots, \widetilde{\mathrm{GL}}_{r_{i,l_i}}$. *Assume that* π_i *is an irreducible constituent of the induced representation* $\mathrm{Ind}_{\widetilde{P}_i}^{\widetilde{\mathrm{GL}}_{r_i}} \sigma_i$. *Then the metaplectic tensor product*

$$\pi_\omega := (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$$

is an irreducible constituent of the induced representation

$$\mathrm{Ind}_{\widetilde{Q}}^{\widetilde{M}} (\tau_{1,1} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{1,l_1} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{k,1} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{k,l_k})_\omega,$$

where Q is the standard parabolic of M whose Levi part is $M_1 \times \cdots \times M_k$.

First we need the following lemma.

Lemma 4.12 *For a genuine representation* π *of a Levi part* $\widetilde{M}$, *the map*

$$\mathrm{Ind}_{\widetilde{M}N^*}^{\widetilde{\mathrm{GL}}_r} \pi \to \mathrm{Ind}_{(\widetilde{M})^{(n)}N^*}^{\widetilde{\mathrm{GL}}_r^{(n)}} \pi|_{(\widetilde{M})^{(n)}}$$

given by the restriction $\varphi \mapsto \varphi|_{\widetilde{\mathrm{GL}}_r^{(n)}}$ *for* $\varphi \in \mathrm{Ind}_{\widetilde{M}N^*}^{\widetilde{\mathrm{GL}}_r} \pi$ *is an isomorphism, where*

$$(\widetilde{M})^{(n)} = \widetilde{M} \cap \widetilde{\mathrm{GL}}_r^{(n)}.$$

Hence, in particular,

$$\Big(\mathrm{Ind}_{\widetilde{M}N^*}^{\widetilde{\mathrm{GL}}_r} \pi\Big)|_{\widetilde{\mathrm{GL}}_r^{(n)}} \cong \Big(\mathrm{Ind}_{\widetilde{M}N^*}^{\widetilde{\mathrm{GL}}_r} \pi\Big)\|_{\widetilde{\mathrm{GL}}_r^{(n)}} \cong \mathrm{Ind}_{(\widetilde{M})^{(n)}N^*}^{\widetilde{\mathrm{GL}}_r^{(n)}} \pi|_{(\widetilde{M})^{(n)}}$$

as representations of $\widetilde{\mathrm{GL}}_r^{(n)}$.

Proof To show it is one-to-one, assume that $\varphi|_{\widetilde{\mathrm{GL}}_r^{(n)}} = 0$. We need to show $\varphi = 0$, but for any $g \in \mathrm{GL}_r$, one can write

$$g = \begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix} \begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g,$$

where

$$\begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix} \in M \quad \text{and} \quad \begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g \in \mathrm{GL}_r^{(n)}.$$

Hence, any $\widetilde{g} \in \widetilde{\mathrm{GL}}_r$ is written as $\widetilde{g} = \widetilde{m}\widetilde{g}'$ for some $\widetilde{m} \in \widetilde{M}$ and $\widetilde{g}' \in \widetilde{\mathrm{GL}}_r^{(n)}$. Hence, $\varphi(\widetilde{g}) = \pi(\widetilde{m})\varphi(\widetilde{g}')$, but $\varphi(\widetilde{g}') = 0$. Hence, $\varphi(\widetilde{g}) = 0$.

To show it is onto, let $\varphi \in \mathrm{Ind}_{(\widetilde{M})^{(n)}N^*}^{\widetilde{\mathrm{GL}}_r^{(n)}} \pi|_{(\widetilde{M})^{(n)}}$. Define $\widetilde{\varphi} \colon \widetilde{\mathrm{GL}}_r \to \pi$ by

$$\widetilde{\varphi}(g, \xi) = \xi\pi\Big(\begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \eta\Big)\varphi\Big(\begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g, 1\Big),$$

where η is chosen to be such that $\Big(\begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \eta\Big)\Big(\begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g, 1\Big) = (g, 1)$. Namely, η is given by the cocycle as

$$\eta = \sigma_r\Big(\begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g\Big)^{-1}.$$

That $\widetilde{\varphi}|_{\widetilde{\mathrm{GL}}_r^{(n)}} = \varphi$ follows because if $g \in \mathrm{GL}_r^{(n)}$, then

$$\Big(\begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \eta \Big) \in (\widetilde{M})^{(n)}.$$

Also one can check $\widetilde{\varphi} \in \mathrm{Ind}_{\widetilde{M}N^*}^{\widetilde{\mathrm{GL}}_r} \pi$ as follows. We need to check $\varphi(\widetilde{m}(g, \xi)) = \pi(\widetilde{m})\varphi(g, \xi)$ for all $\widetilde{m} \in \widetilde{M}$. But since π (and hence φ) is genuine, we may assume that $\widetilde{m}$ is of the form $(m, 1)$ for $m \in M$ and $\xi = 1$. Then

$$\begin{aligned} \widetilde{\varphi}\big((m,1)(g,1)\big) &= \widetilde{\varphi}\big(mg, \sigma_r(m,g)\big) \\ &= \sigma_r(m,g)\pi\Big(\begin{pmatrix} \det(mg)^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \eta_1 \Big) \\ &\qquad \varphi\Big(\begin{pmatrix} \det(mg)^{n-1} & \\ & I_{r-1} \end{pmatrix} mg, 1 \Big), \end{aligned} \tag{4.8}$$

where

$$\eta_1 = \sigma_r\Big(\begin{pmatrix} \det(mg)^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \begin{pmatrix} \det(mg)^{n-1} & \\ & I_{r-1} \end{pmatrix} mg \Big)^{-1}.$$

Now

$$\begin{aligned} &\Big(\begin{pmatrix} \det(mg)^{n-1} & \\ & I_{r-1} \end{pmatrix} mg, 1 \Big) = \\ &\qquad \Big(\begin{pmatrix} \det(mg)^{n-1} & \\ & I_{r-1} \end{pmatrix} m \begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \eta_2 \Big) \Big(\begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g, 1 \Big), \end{aligned}$$

where

$$\eta_2 = \sigma_r\Big(\begin{pmatrix} \det(mg)^{n-1} & \\ & I_{r-1} \end{pmatrix} m \begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g \Big)^{-1}.$$

Since

$$\Big(\begin{pmatrix} \det(mg)^{n-1} & \\ & I_{r-1} \end{pmatrix} m \begin{pmatrix} \det g^{-n-1} & \\ & I_{r-1} \end{pmatrix}, \eta_2 \Big) \in (\widetilde{M})^{(n)},$$

the right-hand side of (4.8) becomes

$$\begin{aligned} &\sigma_r(m,g)\pi\Big(\Big(\begin{pmatrix} \det(mg)^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \eta_1 \Big) \Big(\begin{pmatrix} \det(mg)^{n-1} & \\ & I_{r-1} \end{pmatrix} m \begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \eta_2 \Big) \Big) \\ &\qquad \varphi\Big(\begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g, 1 \Big) \\ &= \sigma_r(m,g)\pi\Big(m \begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \eta_1\eta_2\eta_3 \Big) \varphi\Big(\begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g, 1 \Big) \\ &= \sigma_r(m,g)\pi(m, \eta_1\eta_2\eta_3\eta_4)\pi\Big(\begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix}, 1 \Big) \varphi\Big(\begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g, 1 \Big), \end{aligned}$$

where

$$\eta_3 = \sigma_r\Big(\begin{pmatrix} \det(mg)^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \begin{pmatrix} \det(mg)^{n-1} & \\ & I_{r-1} \end{pmatrix} m \begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix} \Big)$$

and

$$\eta_4 = \sigma_r\Big(m, \begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix} \Big)^{-1}.$$

Then one can compute

$$\sigma_r(m,g)\eta_1\eta_2\eta_3\eta_4 = \eta$$

by using Proposition 2.1(i). Hence (4.8) is written as

$$\pi(m,1)\pi\Big(\begin{pmatrix} \det g^{-n+1} & \\ & I_{r-1} \end{pmatrix}, \eta \Big) \varphi\Big(\begin{pmatrix} \det g^{n-1} & \\ & I_{r-1} \end{pmatrix} g, 1 \Big) = \pi(m,1)\widetilde{\varphi}(g,1).$$

This completes the proof. ∎

With this lemma, one can prove the theorem.

Proof of Theorem 4.11 Let $\pi_i^{(n)}$ be an irreducible constituent of the restriction $\pi_i|_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}}$. By Lemma 4.12, it is an irreducible constituent of

$$\mathrm{Ind}_{(\widetilde{M}_i)^{(n)}N_i^*}^{\widetilde{\mathrm{GL}}_{r_i}^{(n)}} \sigma_i|_{(\widetilde{M}_i)^{(n)}}.$$

Noting that $\widetilde{M}_i^{(n)} \subseteq (\widetilde{M}_i)^{(n)}$, we have the inclusion

$$\mathrm{Ind}_{(\widetilde{M}_i)^{(n)}N_i^*}^{\widetilde{\mathrm{GL}}_{r_i}^{(n)}} \sigma_i|_{(\widetilde{M}_i)^{(n)}} \hookrightarrow \mathrm{Ind}_{\widetilde{M}_i^{(n)}N_i^*}^{\widetilde{\mathrm{GL}}_{r_i}^{(n)}} \sigma_i|_{\widetilde{M}_i^{(n)}}.$$

But since σ_i is a metaplectic tensor product of $\tau_{i,1}, \dots, \tau_{i,l_i}$, the restriction $\sigma_i|_{\widetilde{M}_i^{(n)}}$ is a sum of representations of the form $\tau_{i,1}^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{i,l_i}^{(n)}$, where each $\tau_{i,t}^{(n)}$ is an irreducible constituent of the restriction

$$\tau_{i,t}|_{\widetilde{\mathrm{GL}}_{i_{r_t}}^{(n)}} \text{ of } \tau_{i,t} \text{ to } \widetilde{\mathrm{GL}}_{r_{i,t}}^{(n)}.$$

Note that this is a metaplectic tensor product representation of $\widetilde{M}_i^{(n)}$. Hence the metaplectic tensor product

$$\pi^{(n)} := \pi_1^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k^{(n)}$$

is an irreducible constituent of

$$\widetilde{\bigotimes}_{i=1}^{k} \mathrm{Ind}_{\widetilde{M}_i^{(n)}N_i^*}^{\widetilde{\mathrm{GL}}_{r_i}^{(n)}} \tau_{i,1}^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{i,l_i}^{(n)}. \tag{4.9}$$

Note that the metaplectic tensor product for the group $\widetilde{M}^{(n)}$ can be defined for reducible representations, and hence $\widetilde{\bigotimes}_{i=1}^{k}$ is defined and the space of the representation is the same as the one for the usual tensor product. In particular, the space of the representation (4.9) is the usual tensor product. Then one can see that (4.9) is equivalent to

$$\mathrm{Ind}_{\widetilde{M}_1^{(n)} \widetilde{\times} \cdots \widetilde{\times} \widetilde{M}_k^{(n)} (N_1 \times \cdots \times N_k)^*}^{\widetilde{M}^{(n)}} \widetilde{\bigotimes}_{i=1}^{k} \tau_{i,1}^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{i,l_i}^{(n)}. \tag{4.10}$$

(To see this one can define a map from (4.9) to (4.10) by $\varphi_1 \otimes \cdots \otimes \varphi_k \mapsto \varphi_1 \cdots \varphi_k$, where $\varphi_1 \cdots \varphi_k$ is the product of functions that can be naturally viewed as a function on $\widetilde{M}^{(n)}$.)

Now let ω be a character on $Z_{\widetilde{\mathrm{GL}}_r}$ that agrees with $\pi^{(n)}$ on $Z_{\widetilde{\mathrm{GL}}_r} \cap \widetilde{M}^{(n)}$, so that the product $\pi_\omega^{(n)} := \omega \cdot \pi_n^{(n)}$ is a well-defined representation of $Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}$. Now all the constituents of the representation (4.10) have the same central character, and hence ω agrees with (4.10) on $Z_{\widetilde{\mathrm{GL}}_r} \cap \widetilde{M}^{(n)}$, and hence $\pi_\omega^{(n)}$ is a constituent of

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}_1^{(n)} \widetilde{\times} \cdots \widetilde{\times} \widetilde{M}_k^{(n)} (N_1 \times \cdots \times N_k)^*}^{Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}} \omega \cdot \widetilde{\bigotimes}_{i=1}^{k} \tau_{i,1}^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_{i,l_i}^{(n)}.$$

Recall that the metaplectic tensor product π_ω is a constituent of

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}}^{\widetilde{M}} \pi_\omega^{(n)}$$

and hence a constituent of

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{M}} \mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}_1^{(n)}\widetilde{\times}\cdots\widetilde{\times}\widetilde{M}_k^{(n)}(N_1\times\cdots\times N_k)^*}^{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}} \omega\cdot\widetilde{\bigotimes}_{i=1}^k \tau_{i,1}^{(n)}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\tau_{i,l_i}^{(n)},$$

which is

$$\mathrm{Ind}_{\widetilde{Q}}^{\widetilde{M}} \mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}_1^{(n)}\widetilde{\times}\cdots\widetilde{\times}\widetilde{M}_k^{(n)}(N_1\times\cdots\times N_k)^*}^{\widetilde{Q}} \omega\cdot\widetilde{\bigotimes}_{i=1}^k \tau_{i,1}^{(n)}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\tau_{i,l_i}^{(n)} \tag{4.11}$$

by inducing in stages.

Now one can see that the inner induced representation in (4.11) is equal to

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M_Q}^{(n)}}^{\widetilde{M_Q}} \omega\cdot\widetilde{\bigotimes}_{i=1}^k \tau_{i,1}^{(n)}\widetilde{\otimes}\cdots\widetilde{\otimes}\tau_{i,l_i}^{(n)}, \tag{4.12}$$

where the unipotent group $(N_1\times\cdots\times N_k)^*$ acts trivially and $\widetilde{M_Q}$ is the Levi part of $\widetilde{Q}$, namely

$$\widetilde{M_Q} = \widetilde{M}_1\widetilde{\times}\cdots\widetilde{\times}\widetilde{M}_k.$$

By Proposition 4.6 applied to the Levi subgroup $\widetilde{M_Q}$, the representation (4.12) is a sum of the metaplectic tensor product

$$(\tau_{1,1}^{(n)}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\tau_{1,l_1}^{(n)}\,\widetilde{\otimes}\,\cdots\widetilde{\otimes}\,\tau_{k,1}^{(n)}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\tau_{k,l_k}^{(n)})_\omega.$$

Hence, π_ω is a constituent of

$$\mathrm{Ind}_{\widetilde{Q}}^{\widetilde{M}}\big(\tau_{1,1}^{(n)}\widetilde{\otimes}\cdots\widetilde{\otimes}\tau_{1,l_1}^{(n)}\widetilde{\otimes}\cdots\widetilde{\otimes}\tau_{k,1}^{(n)}\widetilde{\otimes}\cdots\widetilde{\otimes}\tau_{k,l_k}^{(n)}\big)_\omega,$$

as claimed. ■

Remark 4.13 In the statement of Theorem 4.11, one can replace "constituent" by "irreducible subrepresentation" or "irreducible quotient", and the analogous statement is still true. Namely, if each π_i is an irreducible subrepresentation (resp. quotient) of the induced representation in the theorem, then the metaplectic tensor product $(\pi_1\ \widetilde{\otimes}\cdots\widetilde{\otimes}\ \pi_k)_\omega$ is also an irreducible subrepresentation (resp. quotient) of the corresponding induced representation. To prove it, one can simply replace all the occurrences of "constituent" by "irreducible subrepresentation" or "irreducible quotient" in the above proof.

5 The Global Metaplectic Tensor Product

Starting from this section, we will show how to construct the metaplectic tensor product of unitary automorphic subrepresentations. Hence all the groups are over the ring of adeles unless otherwise stated, and it should be recalled here that as in (2.11) the group $\mathrm{GL}_r(F)^*$ is the image of $\mathrm{GL}_r(F)$ under the partial map $\mathbf{s}\colon \mathrm{GL}_r(\mathbb{A})\to\widetilde{\mathrm{GL}}_r(\mathbb{A})$, and we simply write $\mathrm{GL}_r(F)$ for $\mathrm{GL}_r(F)^*$, when there is no danger of confusion. Also, throughout the section the group $A_{\widetilde{M}}(\mathbb{A})$ is an abelian group that satisfies Hypothesis $(*)$.

5.1 The Construction

The construction is similar to the local case in that first we consider the restriction to $\widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A})$, though we need an extra care to ensure the automorphy.

Lemma 5.1 *Let π be a genuine irreducible automorphic unitary subrepresentation of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. Then the restriction $\pi|_{\widetilde{\mathrm{GL}}_r^{(n)}(\mathbb{A})}$ is completely reducible, namely,*

$$\pi|_{\widetilde{\mathrm{GL}}_r^{(n)}\mathbb{A}} = \bigoplus \pi_i^{(n)},$$

where π_i is an irreducible unitary representation of $\widetilde{\mathrm{GL}}_r^{(n)}(\mathbb{A})$.

Proof This follows from the admissibility and unitarity of $\pi|_{\widetilde{\mathrm{GL}}_r^{(n)}\mathbb{A}}$. ∎

The lemma implies that the restriction $\pi_i\|_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A})}$ is also completely reducible. (See the notation section for the notation $\|$.) Hence each irreducible constituent of $\pi_i\|_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A})}$ is a subrepresentation. Let $\pi_i^{(n)} \subseteq \pi_i$ be an irreducible subrepresentation. Then each vector $f \in \pi_i^{(n)}$ is the restriction to $\widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A})$ of an automorphic form on $\widetilde{\mathrm{GL}}_{r_i}(\mathbb{A})$. Hence one can naturally view each vector $f \in \pi_i^{(n)}$ as a function on the group

$$H_i := \mathrm{GL}_{r_i}(F)\widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A}).$$

Namely the representation $\pi_i^{(n)}$ is an irreducible representation of the group $\widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A})$ realized in a space of "automorphic forms on H_i".

Note that H_i is indeed a group, and moreover it is closed in $\widetilde{\mathrm{GL}}_r(\mathbb{A})$, which can be shown by using Lemma A.5. Also note that each element in H_i is of the form (h_i, ξ_i) for $h_i \in \mathrm{GL}_{r_i}(F)\,\mathrm{GL}_{r_i}(\mathbb{A})$ and $\xi_i \in \mu_n$. By the product formula for the Hilbert symbol and the block-compatibility of the cocycle τ_M, we have the natural surjection

$$H_1 \times \cdots \times H_k \to M(F)\widetilde{M}^{(n)}(\mathbb{A}) \tag{5.1}$$

given by the map

$$\big((h_1, \xi_1), \dots, (h_k, \xi_k)\big) \longmapsto (h_1 \cdots h_k, \xi_1 \cdots \xi_k),$$

because $(\det(h_i), \det(h_j))_{\mathbb{A}} = 1$ for all $i, j = 1, \dots, k$.

Now we can construct a metaplectic tensor product of $\pi_1^{(n)}, \dots, \pi_k^{(n)}$, which is an "automorphic representation" of $\widetilde{M}^{(n)}(\mathbb{A})$ realized in a space of "automorphic forms on $M(F)\widetilde{M}^{(n)}(\mathbb{A})$" as follows.

Proposition 5.2 *Let $V_{\pi_1^{(n)}} \otimes \cdots \otimes V_{\pi_k^{(n)}}$ be the space of functions on the direct product $H_1 \times \cdots \times H_k$, which gives rise to an irreducible representation of*

$$\widetilde{\mathrm{GL}}_{r_1}^{(n)}(\mathbb{A}) \times \cdots \times \widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A}),$$

which acts by right translation. Then each function in this space can be viewed as a function on the group $M(F)\widetilde{M}^{(n)}(\mathbb{A})$; namely, it factors through the surjection as in (5.1) *and thus gives rise to a representation of $\widetilde{M}^{(n)}(\mathbb{A})$, which we denote by*

$$\pi^{(n)} := \pi_1^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k^{(n)}.$$

Moreover, each function in $V_{\pi^{(n)}} = V_{\pi_1^{(n)}} \otimes \cdots \otimes V_{\pi_k^{(n)}}$ is "automorphic" in the sense that it is left invariant on $M(F)$.

Proof Since π_i is genuine, for each $f_i \in V_{\pi_i^{(n)}}$ and $g \in H_i$, we have $f_i(g(1,\xi)) = f_i((1,\xi)g) = \xi f_i(g)$ for all $\xi \in \mu_n$. Now the kernel of the map (5.1) consists of the elements of the form $((I_{r_1},\xi_1),\dots,(I_{r_k},\xi_k))$ with $\xi_1\cdots\xi_k = 1$. Hence each $f_1 \otimes \cdots \otimes f_k \in V_{\pi_1^{(n)}} \otimes \cdots \otimes V_{\pi_k^{(n)}}$, viewed as a function on the direct product $H_1 \times \cdots \times H_k$, factors through the map (5.1), which we denote by $f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k$. Namely we can naturally define a function $f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k$ on $M(F)\widetilde{M}^{(n)}(\mathbb{A})$ by

$$(f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k)\left(\begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_k \end{pmatrix}, \xi\right) = \xi f_1(h_1,1)\cdots f_k(h_k,1).$$

One can see each function $f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k$ is "automorphic" as follows. For

$$\begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_k \end{pmatrix} \in M(F) \quad \text{and} \quad \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix} \in M(F)M^{(n)}(\mathbb{A}),$$

we have

$$\begin{aligned}
&(f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k)\left(\mathbf{s}\begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_k \end{pmatrix}\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \xi\right)\right)\\
&= (f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k)\left(\left(\begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_k \end{pmatrix}, \prod_{i=1}^k s_{r_i}(\gamma_i)^{-1}\right)\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \xi\right)\right)\\
&\hspace{20em}\text{by definition of } \mathbf{s}\\
&= (f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k)\left(\left(\begin{pmatrix} \gamma_1 g_1 & & \\ & \ddots & \\ & & \gamma_k g_k \end{pmatrix}, \xi \prod_{i=1}^k s_{r_i}(\gamma_i)^{-1} \tau_M\left(\begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_k \end{pmatrix}, \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}\right)\right)\right)\\
&= (f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k)\left(\begin{pmatrix} \gamma_1 g_1 & & \\ & \ddots & \\ & & \gamma_k g_k \end{pmatrix}, \xi \prod_{i=1}^k s_{r_i}(\gamma_i)^{-1} \tau_{r_i}(\gamma_i, g_i)\right)\\
&\hspace{18em}\text{by block-compatibility of } \tau_M\\
&= \left(\xi \prod_{i=1}^k s_{r_i}(\gamma_i)^{-1} \tau_{r_i}(\gamma_i, g_i)\right)\left(\prod_{i=1}^k f_i(\gamma_i g_i, 1)\right) \quad \text{by definition of } f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k\\
&= \xi \prod_{i=1}^k f_i\big(\gamma_i g_i, s_{r_i}(\gamma_i)^{-1} \tau_{r_i}(\gamma_i, g_i)\big) \quad \text{because each } f_i \text{ is genuine}\\
&= \xi \prod_{i=1}^k f_i\Big((\gamma_i, s_{r_i}(\gamma_i)^{-1})(g_i, 1)\Big) \quad \text{by definition of } \tau_{r_i}\\
&= \xi \prod_{i=1}^k f_i(\mathbf{s}_{r_i}(\gamma_i)(g_i, 1)) \quad \text{by definition of } \mathbf{s}_{r_i}
\end{aligned}$$

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$$= \xi \prod_{i=1}^{k} f_i(g_i, 1) \quad \text{by automorphy of } f_i$$

$$= (f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k)\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \xi\right) \quad \text{by definition of } f_1 \widetilde{\otimes} \cdots \widetilde{\otimes} f_k. \qquad \blacksquare$$

As in the local case, we would like to extend the representation $\pi^{(n)}$ to a representation of $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$ by letting $A_{\widetilde{M}}(\mathbb{A})$ act as a character. This is certainly possible by choosing an appropriate character, because $A_{\widetilde{M}}(\mathbb{A}) \cap \widetilde{M}^{(n)}(\mathbb{A})$ is in the center of $\widetilde{M}^{(n)}(\mathbb{A})$. However, one needs extra steps to ensure the resulting representation is automorphic.

For this purpose, let us first define

$$A_{\widetilde{M}}\widetilde{M}^{(n)}(F) := A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A}) \cap \mathbf{s}(M(F)).$$

Note that this is not necessarily the same as $A_{\widetilde{M}}(F)\widetilde{M}^{(n)}(F)$. Also let

$$H := A_{\widetilde{M}}\widetilde{M}^{(n)}(F)\widetilde{M}^{(n)}(\mathbb{A}).$$

By Proposition A.2, the image of $\mathbf{s}(M(F))$ (and hence $A_{\widetilde{M}}\widetilde{M}^{(n)}(F)$) in the quotient $\widetilde{M}^{(n)}(\mathbb{A})\backslash\widetilde{M}(\mathbb{A})$ is discrete. Hence H is a closed (and hence locally compact) subgroup of $\widetilde{M}(\mathbb{A})$ by using Lemma A.5 with $G = \widetilde{M}(\mathbb{A})$, $Y = \widetilde{M}^{(n)}(\mathbb{A})$, and $\Gamma = A_{\widetilde{M}}\widetilde{M}^{(n)}(F)$. Also note that the group $A_{\widetilde{M}}(\mathbb{A})$ commutes pointwise with the group H by Proposition 3.12 and hence $A_{\widetilde{M}}(\mathbb{A}) \cap H$ is in the center of H.

We need the following subtle but important lemma.

Lemma 5.3 *There exists a character χ on the center Z_H of H such that $f(ah) = \chi(a)f(h)$ for $a \in Z_H$, $h \in H$ and $f \in \pi^{(n)}$. (Note that each $f \in \pi^{(n)}$ is a function on $M(F)\widetilde{M}^{(n)}(\mathbb{A})$ and hence can be viewed as a function on H.)*

Proof Let $\pi_{H_i}^{(n)}$ be an irreducible subrepresentation of $\pi_i\|_{H_i}$ such that

$$\pi_i^{(n)} \subseteq \pi_{H_i}^{(n)}\|_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A})}.$$

Analogously to the construction of $\pi^{(n)} = \pi_1^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k^{(n)}$, one can construct the representation $\pi_{H_1}^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{H_k}^{(n)}$ of $M(F)\widetilde{M}^{(n)}(\mathbb{A})$. (The space of this representation is again a space of "automorphic forms on $M(F)\widetilde{M}^{(n)}(\mathbb{A})$", but this time it is an irreducible representation of the group $M(F)\widetilde{M}^{(n)}(\mathbb{A})$, rather than just $\widetilde{M}^{(n)}(\mathbb{A})$. The construction is completely the same as $\pi^{(n)}$, and one can just modify the proof of Proposition 5.2.) Then one can see

$$V_{\pi^{(n)}} \subseteq V_{\pi_{H_1}^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{H_k}^{(n)}}$$

and

$$\left(\pi_{H_1}^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{H_k}^{(n)}\right) \|_{\widetilde{M}^{(n)}(\mathbb{A})} \cong \left(\pi_{H_1}^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{H_k}^{(n)}\right) |_{\widetilde{M}^{(n)}(\mathbb{A})}.$$

Let $\pi_H^{(n)}$ be an irreducible subrepresentation of $(\pi_{H_1}^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{H_k}^{(n)})\big|_H$ such that

$$V_{\pi^{(n)}} \subseteq V_{\pi_H^{(n)}},$$

where both sides are spaces of functions on $M(F)\widetilde{M}^{(n)}(\mathbb{A})$. Such $\pi_H^{(n)}$ certainly exists, since each π_i is unitary and the unitary structure descends to $\pi_{H_1}^{(n)}\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_{H_k}^{(n)}$, making it unitary. Now since $\pi_H^{(n)}$ is unitary and H is locally compact, $\pi_H^{(n)}$ admits a central character χ. Thus for each $f \in V_{\pi_H^{(n)}}$ and *a fortiori* each $f \in V_{\pi^{(n)}}$, we have $f(ah) = \chi(a)f(h)$ for $a \in Z_H$ and $h \in H$. ∎

In the above lemma, if $a \in Z_H \cap \mathbf{s}(M(F))$, we have $\chi(a) = 1$ by the automorphy of f, namely χ is a "Hecke character on Z_H".

Now define a character ω on $A_{\widetilde{M}}(\mathbb{A})$ such that ω is trivial on $A_{\widetilde{M}}(F)$ and

$$\omega|_{A_{\widetilde{M}}(\mathbb{A})\cap H} = \chi|_{A_{\widetilde{M}}(\mathbb{A})\cap H}.$$

Such ω certainly exists, because $\chi|_{A_{\widetilde{M}}(\mathbb{A})\cap H}$ is viewed as a character on the group $\mathbf{s}(M(F)) \cap (A_{\widetilde{M}}(\mathbb{A}) \cap H)\backslash A_{\widetilde{M}}(\mathbb{A}) \cap H$, which is a locally compact abelian group naturally viewed as a closed subgroup of the locally compact abelian group $A_{\widetilde{M}}(F)\backslash A_{\widetilde{M}}(\mathbb{A})$, and thus it can be extended to $A_{\widetilde{M}}(F)\backslash A_{\widetilde{M}}(\mathbb{A})$.

For each $f \in \pi^{(n)}$ viewed as a function on $H = A_{\widetilde{M}}\widetilde{M}^{(n)}(F)\widetilde{M}^{(n)}(\mathbb{A})$, we extend it to a function $f_\omega \colon A_{\widetilde{M}}(\mathbb{A})H \to \mathbb{C}$ by

$$f_\omega(ah) = \omega(a)f(h), \quad \text{for all } a \in A_{\widetilde{M}}(\mathbb{A}) \text{ and } h \in H.$$

This is well defined because of our choice of ω and the following lemma.

Lemma 5.4 *The function f_ω is a function on $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$ such that*

$$f_\omega(\gamma m) = f_\omega(m)$$

for all $\gamma \in A_{\widetilde{M}}\widetilde{M}^{(n)}(F)$ and $m \in \widetilde{M}^{(n)}(\mathbb{A})$. Namely, f_ω is an "automorphic form on $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$".

Proof The lemma follows from the definition of f_ω and the obvious equality

$$A_{\widetilde{M}}(\mathbb{A})H = A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A}).$$ ∎

The group $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$ acts on the space of functions of the form f_ω, giving rise to an "automorphic representation" $\pi_\omega^{(n)}$ of $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$, namely

$$V_{\pi_\omega^{(n)}} := \{f_\omega : f \in \pi^{(n)}\},$$

and $A_{\widetilde{M}}(\mathbb{A})$ acts as the character ω. As abstract representations, we have

$$\pi_\omega^{(n)} \cong \omega \cdot \pi^{(n)} \tag{5.2}$$

where by $\omega \cdot \pi^{(n)}$ is the representation of the group $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$ extended from $\pi^{(n)}$ by letting $A_{\widetilde{M}}(\mathbb{A})$ act via the character ω.

We need to establish the relation between $\pi_\omega^{(n)}$ and its local analogue we constructed in the previous section. For this, let us start with the following lemma.

Lemma 5.5 *Let $\pi \cong \widetilde{\bigotimes}'_v \pi_v$ be a genuine admissible representation of $\widetilde{M}(\mathbb{A})$. Let $\pi^{(n)}$ be an irreducible quotient of the restriction $\pi|_{\widetilde{M}^{(n)}(\mathbb{A})}$. If we write*

$$\pi^{(n)} \cong \widetilde{\bigotimes_v}' \pi_v^{(n)},$$

then each $\pi_v^{(n)}$ is an irreducible constituent of the restriction $\pi_v|_{\widetilde{M}_r^{(n)}(F_v)}$.

Proof Since $\pi^{(n)}$ is an irreducible quotient, there is a surjective $\widetilde{M}^{(n)}(\mathbb{A})$ map

$$T\colon \widetilde{\bigotimes_v}' \pi_v \longrightarrow \widetilde{\bigotimes_v}' \pi_v^{(n)}.$$

Fix a place v_0. Since $T \neq 0$, there exists a pure tensor $\bigotimes w_v \in \widetilde{\bigotimes}' \pi_v$ such that $T(\bigotimes w_v) \neq 0$. (Note that, as we have seen, the space of $\widetilde{\bigotimes}' \pi_v$ is the space of the usual restricted tensor product $\bigotimes_v' \pi_v$.) Define $i\colon \pi_{v_0} \to \widetilde{\bigotimes}' \pi_v$ by $i(w) = w \otimes (\otimes_{v\neq v_0} w_v)$ for $w \in V_{\pi_{v_0}}$. Then the composite $T \circ i\colon \pi_{v_0} \to \bigotimes_v' \pi_v^{(n)}$ is a non-zero $\widetilde{M}^{(n)}(F_{v_0})$ intertwining. Let $w \in \pi_{v_0}$ be such that $T \circ i(w) \neq 0$. Then $T \circ i(w)$ is a finite linear combination of pure tensors, and indeed it can be written as

$$T \circ i(w) = x_1 \otimes y_1 + \cdots + x_t \otimes y_t,$$

where $x_i \in \pi_{v_0}^{(n)}$ and $y_i \in \bigotimes_{v\neq v_0}' \pi_v^{(n)}$. Here one can assume that $y_1, \ldots, y_t$ are linearly independent. Let $\lambda\colon \otimes_{v\neq v_0} \pi_v^{(n)} \to \mathbb{C}$ be a linear functional such that $\lambda(y_1) \neq 0$ and $\lambda(y_2) = \cdots = \lambda(y_t) = 0$. (Such λ certainly exits, because $y_1, \ldots, y_t$ are linearly independent.) Consider the map

$$U\colon \widetilde{\bigotimes}' \pi_v^{(n)} \to \pi_{v_0}^{(n)}$$

defined on pure tensors by

$$U(\bigotimes x_v) = \lambda(\bigotimes_{v\neq v_0} x)x_{v_0}.$$

This is a non-zero $\widetilde{M}^{(n)}(F_v)$ intertwining map. Moreover, the composite $U\circ T\circ i$ gives a non-zero $\widetilde{M}^{(n)}(F_v)$ intertwining map from π_{v_0} to $\pi_{v_0}^{(n)}$. Hence $\pi_{v_0}^{(n)}$ is an irreducible constituent of the restriction $\pi_{v_0}|_{\widetilde{M}^{(n)}(F_{v_0})}$. ■

By taking $k = 1$ in the above lemma, one can see that if one writes

$$\pi_i^{(n)} \cong \widetilde{\bigotimes_v}' \pi_{i,v}^{(n)},$$

then each local component $\pi_{i,v}^{(n)}$ is an irreducible constituent of $\pi_{i,v}|_{\widetilde{\mathrm{GL}}_{r_i}(F_v)}$ where $\pi_{i,v}$ is the v-component of $\pi_i \cong \widetilde{\bigotimes}_v' \pi_{i,v}$. Then one can see that for $\pi^{(n)} = \pi_1^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k^{(n)}$, if we write $\pi^{(n)} \cong \widetilde{\bigotimes}_v' \pi_v^{(n)}$, we have

$$\pi_v^{(n)} \cong \pi_{1,v}^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{k,v}^{(n)},$$

where the right-hand side is the local metaplectic tensor product representation of $\widetilde{M}^{(n)}(F_v)$. Also one can see that the character ω decomposes as $\omega = \widetilde{\bigotimes}_v' \omega_v$, where ω_v is a character on $A_{\widetilde{M}(F_v)}$. Hence by (5.2) we have the following proposition.

Proposition 5.6 *As abstract representations of $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$, we have*

$$\pi_\omega^{(n)} \cong \widetilde{\bigotimes_v}' \pi_{\omega_v}^{(n)},$$

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where

$$\pi^{(n)}_{\omega_v} = \omega_v \cdot \pi^{(n)}_{1,v}\widetilde{\otimes}\cdots\widetilde{\otimes}\pi^{(n)}_{k,v}$$

is the representation of $A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)$ *as defined in the previous section.*

Now that we have constructed the representation $\pi^{(n)}_{\omega}$ of $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$, we can construct an automorphic representation of $\widetilde{M}(\mathbb{A})$ analogously to the local case by inducing it to $\widetilde{M}(\mathbb{A})$, though we need extra care for the global case. First consider the compactly induced representation

$$\text{c-Ind}^{\widetilde{M}(\mathbb{A})}_{A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})}\,\pi^{(n)}_{\omega} = \{\varphi\colon \widetilde{M}(\mathbb{A}) \to \pi^{(n)}_{\omega}\},$$

where φ is such that $\varphi(hm) = \pi^{(n)}_{\omega}(h)\varphi(m)$ for all $h \in A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$ and $m \in \widetilde{M}(\mathbb{A})$, and the map $m \mapsto \varphi(m;1)$ is a smooth function on $\widetilde{M}(\mathbb{A})$ whose support is compact modulo $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$. (Note here that for each

$$\varphi \in \mathrm{Ind}^{\widetilde{M}(\mathbb{A})}_{A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})}\,\pi^{(n)}_{\omega} \quad\text{and}\quad m \in \widetilde{M}(\mathbb{A}),$$

$\varphi(m) \in V_{\pi^{(n)}_{\omega}}$ is a function on $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$. For $m' \in A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$, we use the notation $\varphi(m;m')$ for the value of $\varphi(m)$ at m' instead of writing $\varphi(m)(m')$.) Also, consider the metaplectic restricted tensor product

$$\widetilde{\bigotimes_v}' \mathrm{Ind}^{\widetilde{M}(F_v)}_{A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)}\,\pi^{(n)}_{\omega_v},$$

where for almost all v at which all the data defining $\mathrm{Ind}^{\widetilde{M}(F_v)}_{A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)}\,\pi^{(n)}_{\omega_v}$ are unramified, we choose the spherical vector

$$\varphi^{\circ}_v \in \mathrm{Ind}^{\widetilde{M}(F_v)}_{A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)}\,\pi^{(n)}_{\omega_v}$$

to be the one defined by

$$\varphi^{\circ}_v(m) = \begin{cases} \pi^{(n)}_{\omega_v}(h)f^{\circ}_v & \text{if } m = h(k,1) \text{ for } h \in A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v) \text{ and } (k,1) \in \widetilde{M}(\mathcal{O}_{F_v}),\\ 0 & \text{otherwise,}\end{cases}$$

where $f^{\circ}_v \in \pi^{(n)}_{\omega_v}$ is the spherical vector defining the restricted metaplectic tensor product $\pi^{(n)}_{\omega} = \widetilde{\bigotimes}'_v \pi^{(n)}_{\omega_v}$. (We do not know if the dimension of the spherical vectors in $\mathrm{Ind}^{\widetilde{M}(F_v)}_{A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)}\,\pi^{(n)}_{\omega_v}$ is one or not.) One has the injection

$$T\colon \widetilde{\bigotimes_v}' \mathrm{Ind}^{\widetilde{M}(F_v)}_{A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)}\,\pi^{(n)}_{\omega_v} \longrightarrow \text{c-Ind}^{\widetilde{M}(\mathbb{A})}_{A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})}\,\pi^{(n)}_{\omega}$$

given by $T(\otimes_v\varphi_v)(m) = \otimes_v\varphi_v(m_v) \in \widetilde{\otimes}'_v\pi^{(n)}_{\omega_v}$. The image of T lies in the compactly induced space because for almost all v, the support of φ° is $A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)\widetilde{M}(\mathcal{O}_{F_v})$, and for all v the index of $A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)$ in $\widetilde{M}(F_v)$ is finite by (2.8). (Indeed, the support property and the finiteness of this index imply that T is actually onto as well, though we do not use this fact.)

Let

$$V(\pi^{(n)}_{\omega}) = T\Big(\widetilde{\bigotimes_v}' \mathrm{Ind}^{\widetilde{M}(F_v)}_{A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)}\,\pi^{(n)}_{\omega_v}\Big);$$

namely $V(\pi_\omega^{(n)})$ is the image of T. For each $\varphi \in V(\pi_\omega^{(n)})$, define $\widetilde{\varphi}\colon \widetilde{M}(\mathbb{A}) \to \mathbb{C}$ by

$$\widetilde{\varphi}(m) = \sum_{\gamma \in A_M M^{(n)}(F)\backslash M(F)} \varphi\big(\mathbf{s}(\gamma)m; 1\big). \tag{5.3}$$

Let us note that by $A_M M^{(n)}(F)$ we mean $p(A_{\widetilde{M}}\widetilde{M}^{(n)}(F))$, which is not necessarily the same as $A_M(F)M^{(n)}(F)$, and

$$\mathbf{s}\big(A_M M^{(n)}(F)\big) = A_{\widetilde{M}}\widetilde{M}^{(n)}(F) \subseteq A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A}).$$

By the automorphy of $\pi_\omega^{(n)}$, φ is left invariant on $\mathbf{s}(A_M M^{(n)}(F))$, and hence the sum is well defined. Also note that for each fixed $m \in \widetilde{M}(\mathbb{A})$, the map $m' \mapsto \varphi(m'm; 1)$ is compactly supported modulo $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})$. By our assumption on $A_{\widetilde{M}}$ (Hypothesis $(*)$), the image of $M(F)$ is discrete in $A_M(\mathbb{A})M^{(n)}(\mathbb{A})\backslash M(\mathbb{A})$, and hence the group $A_M M^{(n)}(F)\backslash M(F)$ naturally viewed as a subgroup of $A_M(\mathbb{A})M^{(n)}(\mathbb{A})\backslash M(\mathbb{A})$ is discrete. A discrete subgroup is always closed by [D-E, Lemma 9.1.3 (b)]. Thus, the above sum is a finite sum, and in particular the sum is convergent. Moreover, one can find φ with the property that the support of the map $m' \mapsto \varphi(m'; 1)$ is small enough so that if $\gamma \in A_M M^{(n)}(F)\backslash M(F)$, then $\varphi(\gamma; 1) \neq 0$ only at $\gamma = 1$. Thus, the map $\varphi \mapsto \widetilde{\varphi}$ is not identically zero.

Remark 5.7 It should be mentioned here that Hypothesis $(*)$ is needed to make sure that the sum in (5.3) is convergent and not identically zero. The author suspects that either one can always find $A_{\widetilde{M}}$ so that Hypothesis $(*)$ is satisfied (which is the case if $n = 2$), or even without Hypothesis $(*)$ one can show that the sum in (5.3) is convergent and not identically zero. But the thrust of this paper is our application to symmetric square L-functions ([T1, T2]) for which we only need the case for $n = 2$.

One can verify that $\widetilde{\varphi}$ is a smooth automorphic form on $\widetilde{M}(\mathbb{A})$. The automorphy is clear. The smoothness and K_f-finiteness follows from the fact that at each non-archimedean v, the induced representation $\mathrm{Ind}_{A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)}^{\widetilde{M}(F_v)} \pi_{\omega_v}^{(n)}$ is smooth and admissible. That $\widetilde{\varphi}$ is $\mathcal{Z}$-finite and of uniform moderate growth follows from the analogous property of $\varphi(\mathbf{s}(\gamma)m)$, because the Lie algebra of $\widetilde{M}(F_v)$ at archimedean v is the same as that of $\widetilde{M}^{(n)}(F_v)$.

As we mentioned, the sum in (5.3) is finite, but which γ contributes to the sum depends on m. Yet, we have the following lemma.

Lemma 5.8 *For each $\varphi \in V(\pi_\omega^{(n)})$, there exists a finite set S of places containing all the archimedean places such that those γ's that contribute to the sum in* (5.3) *depend only on the classes in $\widetilde{M}(\mathbb{A})/\widetilde{M}^{(n)}(\mathbb{A})\kappa(M(\mathcal{O}_S))$, where $\mathcal{O}_S = \prod_{v \notin S} \mathcal{O}_{F_v}$ and $\kappa\colon M(\mathbb{A}) \to \widetilde{M}(\mathbb{A})$ is the section $m \mapsto (m, 1)$.*

Proof By smoothness of φ at the non-archimedean places, there exists a finite set S of places such that for all $k \in \kappa(M(\mathcal{O}_S))$, we have $k \cdot \varphi = \varphi$. Hence one can see that $\mathrm{supp}(\varphi) = \mathrm{supp}(m \cdot \varphi)$ for all $m \in \widetilde{M}^{(n)}(\mathbb{A})\kappa(M(\mathcal{O}_S))$, because $\widetilde{M}^{(n)}(\mathbb{A})$ is a normal subgroup. This proves the lemma. ∎

Theorem 5.9 *Let $\widetilde{V}(\pi_\omega^{(n)}) = \{\widetilde{\varphi} : \varphi \in V(\pi_\omega^{(n)})\}$ and let π_ω be an irreducible constituent of $\widetilde{V}(\pi_\omega^{(n)})$. Then it is an irreducible automorphic representation of $\widetilde{M}(\mathbb{A})$ and $\pi_\omega \cong \widetilde{\bigotimes}'_v \pi_{\omega_v}$, where π_{ω_v} is the local metaplectic tensor product of Mezo. Also, the isomorphism class of π_ω depends only on the choice of the character $\omega|_{Z_{\widetilde{\mathrm{GL}}_r}(\mathbb{A})}$.*

Proof Since the map $\varphi \mapsto \widetilde{\varphi}$ is $\widetilde{M}(\mathbb{A})$-intertwining, the space $\widetilde{V}(\pi_\omega^{(n)})$ provides a space of (possibly reducible) automorphic representation of $\widetilde{M}(\mathbb{A})$. Hence π_ω is an automorphic representation of $\widetilde{M}(\mathbb{A})$.

Since each π_i is unitary, so is each $\pi_i^{(n)}$, from which one can see that $\pi_\omega^{(n)}$ is unitary. Since $V(\pi_\omega^{(n)})$ is a subrepresentation of the compactly induced representation induced from the unitary $\pi_\omega^{(n)}$, $V(\pi_\omega^{(n)})$ is unitary. Hence π_ω, which is a subquotient of

$$V(\pi_\omega^{(n)}) \cong \widetilde{\bigotimes_v}' \operatorname{Ind}_{A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)}^{\widetilde{M}(F_v)} \pi_{\omega_v}^{(n)},$$

is actually a quotient of

$$\widetilde{\bigotimes_v}' \operatorname{Ind}_{A_{\widetilde{M}}(F_v)\widetilde{M}^{(n)}(F_v)}^{\widetilde{M}(F_v)} \pi_{\omega_v}^{(n)}$$

by admissibility. With this said, one can derive the isomorphism $\pi_\omega \cong \widetilde{\bigotimes_v}' \pi_{\omega_v}$ from Lemma 5.5. Since the local π_{ω_v} depends only on the choice of $\omega_v|_{Z_{\widetilde{\mathrm{GL}}_r}(F_v)}$, the global π_ω depends only on $\omega|_{Z_{\widetilde{\mathrm{GL}}_r}(\mathbb{A})}$ up to equivalence. ∎

We call π_ω constructed above the global metaplectic tensor product of $\pi_1, \dots, \pi_k$ (with respect to ω) and write

$$\pi_\omega = (\pi_1 \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_k)_\omega.$$

Remark 5.10 We do not know if the multiplicity one theorem holds for the group $\widetilde{M}(\mathbb{A})$, and hence do not know if the space $\widetilde{V}(\pi_\omega^{(n)})$ has only one irreducible constituent. In this sense, the definition of π_ω depends on the choice of the irreducible constituent. For this reason, the metaplectic tensor product should be construed as an equivalence class of automorphic representations, although we know a more or less explicit ways of expressing automorphic forms in π_ω.

5.2 The Uniqueness

Just like the local case, the metaplectic tensor product of automorphic representations is unique up to twist.

Proposition 5.11 *Let $\pi_1, \dots, \pi_k$ and $\pi'_1, \dots, \pi'_k$ be unitary automorphic subrepresentations of $\widetilde{\mathrm{GL}}_{r_1}(\mathbb{A}), \dots, \widetilde{\mathrm{GL}}_{r_k}(\mathbb{A})$. They give rise to isomorphic metaplectic tensor products with a character ω, i.e., $(\pi_1 \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_k)_\omega \cong (\pi'_1 \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi'_k)_\omega$ if and only if for each i there exists an automorphic character ω_i of $\widetilde{\mathrm{GL}}_{r_i}(\mathbb{A})$ trivial on $\widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A})$ such that $\pi_i \cong \omega_i \otimes \pi'_i$.*

Proof By Theorem 5.9, the global metaplectic tensor product is written as the metaplectic restricted tensor product of the local metaplectic tensor products of Mezo. Hence by Proposition 4.1, for each i and each place v, there is a character $\omega_{i,v}$ on

$\widetilde{\mathrm{GL}}_{r_i}(F_v)$ trivial on $\widetilde{\mathrm{GL}}_{r_i}^{(n)}(F_v)$ such that $\pi_{i,v} \cong \omega_{i,v} \otimes \pi'_{i,v}$. Let $\omega_i = \widetilde{\otimes}'_v \omega_{i,v}$. Then $\pi_i \cong \omega_i \otimes \pi'_i$. The automorphy of ω follows from that of π_i and π'_i. This proves the only if part. The if part follows similarly. ∎

5.3 Cuspidality and Square-integrability

In this subsection, we will show that the cuspidality and square-integrability are preserved for the metaplectic tensor product.

Theorem 5.12 *Assume that $\pi_1, \ldots, \pi_k$ are all cuspidal. Then the metaplectic tensor product $\pi_\omega = (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$ is cuspidal.*

Proof Assume that $\pi_1, \ldots, \pi_k$ are all cuspidal. It suffices to show that for each $\varphi \in V(\pi_\omega^{(n)})$

$$\int_{U(F)\backslash U(\mathbb{A})} \widetilde{\varphi}\big(\mathbf{s}(u)\big)\, du = 0$$

for each unipotent radical U of the standard proper parabolic subgroup of M, where we recall from Proposition 3.5 that the partial set theoretic section $\mathbf{s}\colon M(\mathbb{A}) \to \widetilde{M}(\mathbb{A})$ is defined (and a group homomorphism) on the groups $M(F)$ and $U(\mathbb{A})$. Note that by definition of $\widetilde{\varphi}$, we have

$$\int_{U(F)\backslash U(\mathbb{A})} \widetilde{\varphi}(\mathbf{s}(u))\, du = \int_{U(F)\backslash U(\mathbb{A})} \sum_{\gamma \in A_M M^{(n)}(F)\backslash M(F)} \varphi(\mathbf{s}(\gamma)\mathbf{s}(u))\, du. \tag{5.4}$$

Here we may assume that $\gamma \in M(F)$ is a diagonal matrix, because for each $\gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_k)$ with $\gamma_i \in \mathrm{GL}_{r_i}(F)$, we have

$$\gamma_i = \gamma_i \begin{pmatrix} \det(\gamma_i)^{n-1} & \\ & I_{r_i-1} \end{pmatrix} \begin{pmatrix} \det(\gamma_i)^{-n+1} & \\ & I_{r_i-1} \end{pmatrix},$$

where $\gamma_i \left(\begin{smallmatrix} \det(\gamma_i)^{n-1} & \\ & I_{r_i-1} \end{smallmatrix}\right) \in \mathrm{GL}_r^{(n)}(F)$. So for each $u \in U(F)$, we have $\gamma u \gamma^{-1} \in U(F)$. Thus by the automophy of $\varphi(\mathbf{s}(u); -)$, for each $u \in U(\mathbb{A})$ one can see that the map $u \mapsto \varphi(\mathbf{s}(\gamma)\mathbf{s}(u); 1)$ is left invariant on $U(F)$. Hence for the right-hand side of (5.4), one can change the sum and integral. So it suffices to show

$$\int_{U(F)\backslash U(\mathbb{A})} \varphi\big(\mathbf{s}(\gamma)\mathbf{s}(u); 1\big)\, du = 0.$$

Since we are assuming γ is a diagonal matrix, we have $\gamma u \gamma^{-1} \in U(\mathbb{A})$ for all $u \in U(\mathbb{A})$. Then

$$\begin{aligned}
&\int_{U(F)\backslash U(\mathbb{A})} \varphi(\mathbf{s}(\gamma)\mathbf{s}(u); 1)\, du \\
&\quad = \int_{U(F)\backslash U(\mathbb{A})} \varphi(\mathbf{s}(\gamma)\mathbf{s}(u)\mathbf{s}(\gamma^{-1})\mathbf{s}(\gamma); 1)\, du \\
&\quad = \int_{U(F)\backslash U(\mathbb{A})} \varphi(\mathbf{s}(\gamma); \mathbf{s}(\gamma)\mathbf{s}(u)\mathbf{s}(\gamma^{-1}))\, du \quad \text{because } \mathbf{s}(\gamma)\mathbf{s}(u)\mathbf{s}(\gamma^{-1}) \in \widetilde{U}(\mathbb{A})
\end{aligned}$$

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$$
\begin{aligned}
&= \int_{U(F)\backslash U(\mathbb{A})} \varphi(\mathbf{s}(\gamma); \mathbf{s}(\gamma)\mathbf{s}(u\gamma^{-1}))\, du \quad \text{by Proposition 3.5} \\
&= \int_{U(F)\backslash U(\mathbb{A})} \varphi(\mathbf{s}(\gamma); \mathbf{s}(\gamma)\mathbf{s}(\gamma^{-1}u))\, du \quad \text{by change of variables } \gamma u\gamma^{-1} \mapsto u \\
&= \int_{U(F)\backslash U(\mathbb{A})} \varphi(\mathbf{s}(\gamma); \mathbf{s}(\gamma)\mathbf{s}(\gamma^{-1})\mathbf{s}(u))\, du \quad \text{by Proposition 3.5} \\
&= \int_{U(F)\backslash U(\mathbb{A})} \varphi(\mathbf{s}(\gamma); \mathbf{s}(u))\, du \quad \text{by Proposition 3.5.}
\end{aligned}
$$

We would like to show this is equal to zero. For this purpose, recall that for each γ, $\varphi(\mathbf{s}(\gamma))$ is in the space $V_{\pi_\omega^{(n)}}$ and hence is (a finite sum of functions) of the form $f_1\widetilde{\otimes}\cdots\widetilde{\otimes} f_k$ with $f_i \in V_{\pi_i}$ and each f_i is a cusp form. We may assume $\varphi(\mathbf{s}(\gamma))$ is a simple tensor $f_1\widetilde{\otimes}\cdots\widetilde{\otimes} f_k$. Now we can write $U = U_1 \times \cdots \times U_k$, where each U_i is a unipotent subgroup of GL_{r_i} with at least one of U_i non-trivial, and accordingly we denote each element $u \in U$ by $u = \operatorname{diag}(u_1, \ldots, u_k)$. Then by definition of $\mathbf{s}$, we have

$$
\mathbf{s}(u) = \Big(u, \prod_i s_{r_i}(u_i)^{-1}\Big),
$$

and

$$
\begin{aligned}
\varphi\big(\mathbf{s}(\gamma); \mathbf{s}(u)\big) &= (f_1\widetilde{\otimes}\cdots\widetilde{\otimes} f_k)\big(\mathbf{s}(u)\big) \\
&= (f_1\widetilde{\otimes}\cdots\widetilde{\otimes} f_k)\Big(u, \prod_i s_{r_i}(u_i)^{-1}\Big) \\
&= \Big(\prod_i s_{r_i}(u_i)^{-1}\Big) f_1(u_1, 1)\cdots f_k(u_k, 1) \quad \text{by definition of } f_1\widetilde{\otimes}\cdots\widetilde{\otimes} f_k \\
&= f_1\big(u_1, s_{r_i}(u_i)^{-1}\big) \cdots f_k\big(u_k, s_{r_k}(u_k)^{-1}\big) \quad \text{because each } f_i \text{ is genuine} \\
&= f_1\big(\mathbf{s}_{r_1}(u_1)\big) \cdots f_k\big(\mathbf{s}_{r_k}(u_k)\big) \quad \text{by definition of } \mathbf{s}_{r_i}.
\end{aligned}
$$

Hence,

$$
\int_{U(F)\backslash U(\mathbb{A})} \varphi\big(\mathbf{s}(\gamma); \mathbf{s}(u)\big)\, du = \prod_{i=1}^{k} \int_{U_i(F)\backslash U_i(\mathbb{A})} f_i\big(\mathbf{s}_{r_i}(u_i)\big)\, du_i.
$$

This is equal to zero, because each f_i is cuspidal and at least one of U_i is non-trivial. ■

Next let us take care of the square-integrability.

Theorem 5.13 *Assume that $\pi_1, \ldots, \pi_k$ are all square-integrable modulo center. Then the metaplectic tensor product $\pi_\omega = (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$ is square-integrable modulo center.*

We need a few lemmas for the proof of this theorem.

Lemma 5.14 *Let S be a finite set of places including all the infinite places and let $\mathcal{O}_S = \prod_{v\notin S} \mathcal{O}_{F_v}$. Then the group $F^\times \mathbb{A}^{\times n} \mathcal{O}_S^\times \backslash \mathbb{A}^\times$ is finite.*

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Proof Let $F_S := \prod_{v\in S} F_v$. It suffices to show that the subgroup $F^\times F_S^{\times n}\mathcal{O}_S^\times \subseteq F^\times \mathbb{A}^{\times n}\mathcal{O}_S^\times$ has a finite index in $\mathbb{A}^\times$. But it is well known that the group $F^\times F_S^\times \mathcal{O}_S^\times$ has a finite index in $\mathbb{A}^\times$. (Indeed, if $S = S_\infty$, the quotient $F^\times F_S^\times \mathcal{O}_S^\times\backslash\mathbb{A}^\times$ is isomorphic to the class group of F, and hence for general S, the group $F^\times F_S^\times \mathcal{O}_S^\times\backslash\mathbb{A}^\times$ is a quotient of the class group.) Also, $F^\times F_S^{\times n}\mathcal{O}_S^\times$ has a finite index in $F^\times F_S^\times \mathcal{O}_S^\times$, because $F_S^{\times n}$ has a finite index in $F_S^\times$. Hence, $F^\times F_S^{\times n}\mathcal{O}_S^\times$ has a finite index in $\mathbb{A}^\times$. ■

Remark 5.15 One can show that the group $F^\times\mathbb{A}^{\times n}\mathcal{O}_S^\times\backslash\mathbb{A}^\times$ surjects onto $Cl(F)/Cl(F)^n$, where $Cl(F)$ is the class group of F. (See [K1, Proposition 1, Appendix].) Hence this quotient group is not trivial in general. Occasionally, however, it can be shown to be the trivial group depending on F and n. This is the case for example if $n = 2$ and $F = \mathbb{Q}$. An interested reader might want to look at [K1, Appendix].

Lemma 5.16 *Let G be a locally compact group and let $H, N \subset G$ be closed subgroups such that NH is a closed subgroup. Further assume that the quotient measures for $N\backslash G, H\backslash NH$ and $NH\backslash G$ all exist. (Recall that in general the quotient measure for $N\backslash G$ exists if the modular characters of G and N agree on N.) Then*

$$\int_{N\backslash G} f(g)\,dg = \int_{NH\backslash G}\int_{N\backslash NH} f(hg)\,dh\,dg$$
$$= \int_{NH\backslash G}\int_{N\cap H\backslash H} f(hg)\,dh\,dg$$

for all $f \in L^1(N\backslash G)$.

Proof The first equality is [Bo, Cor. 1 VII 47], and the second equality follows from the natural identification $N\backslash NH \cong N\cap H\backslash H$. ■

Now let $f\colon \widetilde{M}(\mathbb{A}) \to \mathbb{C}$ be any function. Then the absolute value $|f|$ is non-genuine in the sense that it factors through $M(\mathbb{A})$. Also, we let

$$Z_M^{(n)}(\mathbb{A}) := \left\{ \begin{pmatrix} a_1^n I_{r_1} & & \\ & \ddots & \\ & & a_k^n I_{r_k} \end{pmatrix} : a_i \in \mathbb{A}^\times \right\}.$$

This is a closed subgroup by Lemma 2.4 and 2.10. Note the inclusions

$$Z_M^{(n)}(\mathbb{A}) \subseteq p\big(Z_{\widetilde{M}}(\mathbb{A})\big) \subseteq Z_M(\mathbb{A}),$$

where all the groups are closed subgroups of $M(\mathbb{A})$. Then we have the following lemma.

Lemma 5.17 *Let $f\colon M(F)\backslash\widetilde{M}(\mathbb{A}) \to \mathbb{C}$ be an automorphic form with a unitary central character. Then f is square-integrable modulo the center $Z_{\widetilde{M}}(\mathbb{A})$ if and only if $|f| \in L^2(Z_M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A}))$, where $|f|$ is viewed as a function on $M(\mathbb{A})$ as noted above.*

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Proof Let f be an automorphic form on $\widetilde{M}(\mathbb{A})$ with a unitary central character. Since $|f|$ is non-genuine, we have

$$\int_{Z_{\widetilde{M}}(\mathbb{A})M(F)\backslash\widetilde{M}(\mathbb{A})} |f(\widetilde{m})|^2\, d\widetilde{m} = \int_{p(Z_{\widetilde{M}}(\mathbb{A}))M(F)\backslash M(\mathbb{A})} |f(\kappa(m))|^2\, dm,$$

where recall that $p\colon \widetilde{M}(\mathbb{A}) \to M(\mathbb{A})$ is the canonical projection. Note that the quotient measure on the right-hand side exists because the group $p(Z_{\widetilde{M}}(\mathbb{A}))M(F)$ is closed by [MW, Lemma I.1.5, p.8] and is unimodular, because $p(Z_{\widetilde{M}}(\mathbb{A}))$ is unimodular and $M(F)$ is discrete and countable. By Lemma 5.16, we have

$$\int_{Z_M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A})} |f(\kappa(m))|^2\, dm = \int_{p(Z_{\widetilde{M}}(\mathbb{A}))M(F)\backslash M(\mathbb{A})} \int_{Z_M^{(n)}(\mathbb{A})p(Z_{\widetilde{M}}(F))\backslash p(Z_{\widetilde{M}}(\mathbb{A}))} |f(\kappa(zm))|^2\, dz\, dm.$$

Since for each fixed $m \in M(\mathbb{A})$, the function $z \mapsto f(\kappa(zm))$ is a smooth function on $p(Z_{\widetilde{M}}(\mathbb{A}))$, there exists a finite set S of places such that for all $z' \in p(Z_{\widetilde{M}}(\mathcal{O}_S)) = Z_M(\mathcal{O}_S) \cap p(Z_{\widetilde{M}}(\mathbb{A}))$, we have $f(\kappa(z'zm)) = f(\kappa(zm))$. Hence the inner integral of the above integral is written as

$$\int_{Z_M^{(n)}(\mathbb{A})p(Z_{\widetilde{M}}(\mathcal{O}_S))p(Z_{\widetilde{M}}(F))\backslash p(Z_{\widetilde{M}}(\mathbb{A}))} |f(\kappa(zm))|^2\, dz. \tag{5.5}$$

Note that we have the inclusion

$$Z_M^{(n)}(\mathbb{A})p\big(Z_{\widetilde{M}}(\mathcal{O}_S)\big)\, p(Z_{\widetilde{M}}(F))\backslash p\big(Z_{\widetilde{M}}(\mathbb{A})\big) \subseteq Z_M^{(n)}(\mathbb{A})Z_M(\mathcal{O}_S)Z_M(F)\backslash Z_M(\mathbb{A}),$$

because

$$p\big(Z_{\widetilde{M}}(\mathcal{O}_S)\big) \cap p\big(Z_{\widetilde{M}}(F)\big) = Z_M(\mathcal{O}_S) \cap Z_M(F) = 1,$$

and note that $Z_M^{(n)}(\mathbb{A})Z_M(\mathcal{O}_S)Z_M(F)\backslash Z_M(\mathbb{A})$ can be identified with the product of k copies of $F^\times\mathbb{A}^{\times n}\ \mathcal{O}_S^\times\backslash\mathbb{A}^\times$. By Lemma 5.14, we know that this is a finite group, and hence the integral in (5.5) is just a finite sum. Thus, for some finite $z_1, \dots, z_N \in p(Z_{\widetilde{M}}(\mathbb{A}))$, we have

$$\begin{aligned}
\int_{Z_M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A})} \big| f\big(\kappa(m)\big)\big|^2\, dm &= \int_{p(Z_{\widetilde{M}}(\mathbb{A}))M(F)\backslash M(\mathbb{A})} \sum_{i=1}^N \big| f\big(\kappa(z_i m)\big)\big|^2\, dm\\
&= \sum_{i=1}^N \int_{p(Z_{\widetilde{M}}(\mathbb{A}))M(F)\backslash M(\mathbb{A})} \big| f\big(\kappa(m)\big)\big|^2\, dm\\
&= N \int_{p(Z_{\widetilde{M}}(\mathbb{A}))M(F)\backslash M(\mathbb{A})} \big| f\big(\kappa(m)\big)\big|^2\, dm,
\end{aligned}$$

where for the second equality we used

$$\big| f\big(\kappa(z_i m)\big)\big| = \big| f\big(\big(\kappa(z_i)\kappa(m)\big)\big)\big| = \big|\omega\big(\kappa(z_1)\big)\big|\big| f\big(\kappa(m)\big)\big| = \big| f\big(\kappa(m)\big)\big|,$$

where ω is the central character of f which is assumed to be unitary. The lemma follows from this. ∎

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Lemma 5.18 *Assume that $\pi_1, \ldots, \pi_k$ are as in Theorem 5.13. Let $\varphi_i \in \pi_i^{(n)}$ for $i = 1, \ldots, k$ and $\varphi = \varphi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \varphi_k \in \pi^{(n)}$, which is a function on $\widetilde{M}^{(n)}(\mathbb{A})$. Then*

$$\int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(F)\backslash M^{(n)}(\mathbb{A})} \left| \varphi\big(\kappa(m)\big) \right|^2 dm < \infty.$$

Proof Write each element $m \in M(\mathbb{A})$ as $m = \operatorname{diag}(g_1, \ldots, g_k)$, where $g_i \in \mathrm{GL}_{r_i}(\mathbb{A})$. Then $\operatorname{diag}(g_1, \ldots, g_k) \in M^{(n)}(\mathbb{A})$ if and only if $g_i \in \mathrm{GL}_{r_i}^{(n)}(\mathbb{A})$ for all i. Hence the integral in the lemma is the product of integrals

$$\int_{Z_{\mathrm{GL}_{r_i}}^{(n)}(\mathbb{A})\,\mathrm{GL}_{r_i}^{(n)}(F)\backslash\,\mathrm{GL}_{r_i}^{(n)}(\mathbb{A})} \left| \varphi_i\big(\kappa(g_i)\big) \right|^2 dg_i,$$

where $Z_{\mathrm{GL}_{r_i}}^{(n)}(\mathbb{A})$ consists of the elements of the form $a_i I_{r_i}$ with $a_i \in \mathbb{A}^{\times n}$. So we have to show that this integral converges. But with Lemma 5.17 applied to $M = \mathrm{GL}_{r_i}$, we know that

$$\int_{Z_{\mathrm{GL}_{r_i}}^{(n)}(\mathbb{A})\,\mathrm{GL}_{r_i}(F)\backslash\,\mathrm{GL}_{r_i}(\mathbb{A})} \left| \varphi_i\big(\kappa(g_i)\big) \right|^2 dg_i < \infty,$$

because each φ_i is square-integrable modulo center. By Lemmas 5.16 and A.5, this is written as

$$\int_{Z_{\mathrm{GL}_{r_i}}^{(n)}(\mathbb{A})\,\mathrm{GL}_{r_i}^{(n)}(\mathbb{A})\,\mathrm{GL}_{r_i}(F)\backslash\,\mathrm{GL}_{r_i}(\mathbb{A})} \int_{Z_{\mathrm{GL}_{r_i}}^{(n)}(\mathbb{A})\,\mathrm{GL}_{r_i}^{(n)}(F)\backslash\,\mathrm{GL}_{r_i}^{(n)}(\mathbb{A})} \left| \varphi_i\big(\kappa(m_i' m_i)\big) \right|^2 dm_i'\, dm_i < \infty.$$

In particular, the inner integral converges, which proves the lemma. ∎

Now we are ready to prove Theorem 5.13.

Proof of Theorem 5.13 By Lemma 5.17, we have only to show

$$\int_{Z_M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A})} \left| \widetilde{\varphi}\big(\kappa(m)\big) \right|^2 dm < \infty.$$

By Lemma 5.16, we have

(5.6)

$$\begin{aligned}
&\int_{Z_M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A})} \left| \widetilde{\varphi}\big(\kappa(m)\big) \right|^2 dm \\
&= \int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A})} \int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(F)\backslash M^{(n)}(\mathbb{A})} \left| \widetilde{\varphi}\big(\kappa(m'm)\big) \right|^2 dm'\,dm \\
&= \int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A})} \int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(F)\backslash M^{(n)}(\mathbb{A})} \left| \sum_{\gamma} \varphi\big(\kappa(\gamma m'm); 1\big) \right|^2 dm'\,dm \\
&= \int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A})} \int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(F)\backslash M^{(n)}(\mathbb{A})} \left| \sum_{\gamma} \varphi\big(\kappa(\gamma m); \kappa(\gamma m'\gamma^{-1})\big) \right|^2 dm'\,dm.
\end{aligned}$$

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Let us show that the inner integral converges. Note that

$$\int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(F)\backslash M^{(n)}(\mathbb{A})} \Big| \sum_{\gamma} \varphi\big(\kappa(\gamma m); \kappa(\gamma m'\gamma^{-1})\big) \Big|^2 \, dm' \leq \int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(F)\backslash M^{(n)}(\mathbb{A})} \sum_{\gamma} \big| \varphi\big(\kappa(\gamma m); \kappa(\gamma m'\gamma^{-1})\big) \big|^2 \, dm',$$

and the map $m' \mapsto \big|\varphi(\kappa(\gamma m); \kappa(\gamma m'\gamma^{-1}))\big|^2$ is invariant under $Z_M^{(n)}(\mathbb{A})M^{(n)}(F)$ on the left. Hence to show the inner integral converges, it suffices to show the integral

$$\int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(F)\backslash M^{(n)}(\mathbb{A})} \big| \varphi\big(\kappa(\gamma m); \kappa(\gamma m'\gamma^{-1})\big) \big|^2 \, dm'$$

converges. But this follows from Lemma 5.18.

To show the outer integral converges, note that the map $m \mapsto |\widetilde{\varphi}(\kappa(m'm))|^2$ is smooth, and hence there exists a finite set of places S so that $\widetilde{\varphi}(\kappa(m'mk)) = \widetilde{\varphi}(\kappa(m'mk))$ for all $k \in M(\mathcal{O}_S)$. Thus, the integral in (5.6) is (a scalar multiple of)

$$\int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A})/M(\mathcal{O}_S)} \int_{Z_M^{(n)}(\mathbb{A})M^{(n)}(F)\backslash M^{(n)}(\mathbb{A})} \big| \widetilde{\varphi}\big(\kappa(m'm)\big) \big|^2 \, dm' \, dm.$$

Now the set theoretic map

$$F^{\times}\mathbb{A}^{\times n}\underbrace{\mathcal{O}_S^{\times}\backslash\mathbb{A}^{\times} \times \cdots \times F^{\times}}_{k \text{ copies}}\mathbb{A}^{\times n}\mathcal{O}_S^{\times}\backslash\mathbb{A}^{\times} \to Z_M^{(n)}(\mathbb{A})M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A})/M(\mathcal{O}_S)$$

given by

$$(a_1, \ldots, a_k) \longmapsto \begin{pmatrix} \iota_1(a_1) & & \\ & \ddots & \\ & & \iota_k(a_k) \end{pmatrix},$$

where ι_i is as in (2.9) is a well-defined surjection. Hence Lemma 5.14 implies that the set

$$Z_M^{(n)}(\mathbb{A})M^{(n)}(\mathbb{A})M(F)\backslash M(\mathbb{A})/M(\mathcal{O}_S)$$

is a finite set. Therefore, the outer integral of the above integral is a finite sum and hence converges. This completes the proof. ■

5.4 Twists by Weyl Group Elements

Just as we saw in Section 5.4 for the local case, the global metaplectic tensor product behaves in the expected way under the action of the Weyl group elements in W_M.

Theorem 5.19 *Let $w \in W_M$ be such that*

$${}^{w}(\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}) = \mathrm{GL}_{r_{\sigma(1)}} \times \cdots \times \mathrm{GL}_{r_{\sigma(k)}}.$$

Then we have

$${}^{w}(\pi_1 \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_k)_{\omega} \cong (\pi_{\sigma(1)} \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_{\sigma(k)})_{\omega},$$

where w is viewed as an element in $\mathrm{GL}_r(F)$.

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Proof Note that each $\mathbf{s}(w) \in \widetilde{\mathrm{GL}}_r(\mathbb{A})$ can be written as $\prod_v (w, s_{r,v}(w))$, where we view $(w, s_{r,v}(w)) \in \widetilde{\mathrm{GL}}_r(F_v)$ as an element of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ via the natural embedding $\widetilde{\mathrm{GL}}_r(F_v) \hookrightarrow \widetilde{\mathrm{GL}}_r(\mathbb{A})$, and the product $\prod_v$ is literally the product inside $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. Then one can see that

$$^w(\pi_1 \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_k)_\omega = \widetilde{\otimes}'_v \, {}^w(\pi_{1,v} \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_{k,v})_{\omega_v}.$$

Hence the theorem follows from the local counterpart (Theorem 4.8). ∎

The following proposition is immediate.

Proposition 5.20 *Let $\pi_\omega = (\pi_1 \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_k)_\omega$. For w as in the theorem and each automorphic form $\widetilde{\varphi} \in \pi_\omega$, define ${}^w\widetilde{\varphi}\colon {}^w\widetilde{M}(A) \to \mathbb{C}$ by*

$$^w\widetilde{\varphi}(m) = \widetilde{\varphi}\big(\mathbf{s}(w)^{-1} m \mathbf{s}(w)\big)$$

for $m \in {}^w\widetilde{M}(\mathbb{A})$. Then the representation ${}^w\pi_\omega$ is realized in the space

$$\{{}^w\widetilde{\varphi} : \widetilde{\varphi} \in V_{\pi_\omega}\}.$$

Let us mention the following subtle point. Here we have (at least) two different realizations of ${}^w\pi_\omega$ in a space of automorphic forms on ${}^w\widetilde{M}(\mathbb{A})$; the one is in the space $\{{}^w\widetilde{\varphi} : \widetilde{\varphi} \in V_{\pi_\omega}\}$ as in the proposition, and the other as in the definition of the metaplectic tensor product $(\pi_{\sigma(1)} \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_{\sigma(k)})_\omega$ by choosing an appropriate $A_{{}^w\widetilde{M}}$ that satisfies Hypothesis $(*)$ with respect to the Levi subgroup ${}^w\widetilde{M}$ (if possible at all). Without the multiplicity one property for the group ${}^w\widetilde{M}$, we do not know if they coincide. But one can see that if $A_{\widetilde{M}}$ satisfies Hypothesis $(*)$ with respect to $\widetilde{M}$, then the group ${}^wA_{\widetilde{M}} := wA_{\widetilde{M}}w^{-1}$ satisfies Hypothesis $(*)$ with respect to ${}^w\widetilde{M}$. Then if we define $(\pi_{\sigma(1)} \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_{\sigma(k)})_\omega$ by choosing $A_{{}^w\widetilde{M}} = {}^wA_{\widetilde{M}}$, one can see from the construction of our metaplectic tensor product that the space of $(\pi_{\sigma(1)} \,\widetilde{\otimes} \cdots \widetilde{\otimes}\, \pi_{\sigma(k)})_\omega$ is indeed a space of automorphic forms of the form ${}^w\widetilde{\varphi}$ for $\widetilde{\varphi} \in V_{\pi_\omega}$.

5.5 Compatibility with Parabolic Induction

Just as in the local case, we have the compatibility with parabolic inductions. But before stating the theorem, let us mention the following lemma.

Lemma 5.21 *Let $P = MN$ be the standard parabolic subgroup of GL_r. Then $\widetilde{M}(\mathbb{A})$ normalizes $N(\mathbb{A})^*$, where $N(\mathbb{A})^*$ is the image of $N(\mathbb{A})$ under the partial section $\mathbf{s}\colon \mathrm{GL}_r(\mathbb{A}) \to \widetilde{\mathrm{GL}}_r(\mathbb{A})$.*

Proof One can prove this by using the local analogue (Lemma 4.10). Namely, let $\widetilde{m} = (m, 1) \in \widetilde{M}(\mathbb{A})$, so $\widetilde{m}^{-1} = (m^{-1}, \tau_r(m, m^{-1})^{-1})$. Also let $n^* = (n, s_r(n)^{-1}) \in N(\mathbb{A})^*$. Then

$$\begin{aligned}\widetilde{m} n^* \widetilde{m}^{-1} &= (m,1)\big(n, s_r(n)^{-1}\big)\big(m^{-1}, \tau_r(m, m^{-1})^{-1}\big)\\ &= \big(mnm^{-1}, s_r(n)^{-1}\tau_r(m,n)\tau(m,m^{-1})^{-1}\tau_r(mn, m^{-1})\big).\end{aligned}$$

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Then one needs to show

$$s_r(n)^{-1}\tau_r(m,n)\tau_r(m,m^{-1})^{-1}\tau_r(mn,m^{-1}) = s(mnm^{-1})^{-1},$$

so that $\widetilde{m}n^*\widetilde{m}^{-1} = (mnm^{-1})^* \in N(\mathbb{A})^*$. This can be done by arguing "semi-locally". Namely, for a sufficiently large finite set S of places, we have

$$\begin{aligned}
&s_r(n)^{-1}\tau_r(m,n)\tau_r(m,m^{-1})^{-1}\tau_r(mn,m^{-1})\\
&\quad= \prod_{v\in S} s_r(n_v)^{-1}\tau_r(m_v,n_v)\tau_r(m_v,m_v^{-1})^{-1}\tau_r(m_vn_v,m_v^{-1})\\
&\quad= \prod_{v\in S} s_r(n_v)^{-1}\sigma_r(m_v,n_v)\frac{s_r(m_v)s_r(n_v)}{s_r(m_vn_v)}\\
&\qquad\cdot \sigma_r(m_v,m_v^{-1})^{-1}\frac{s_r(m_vm_v^{-1})}{s_r(m_v)s_r(m_v^{-1})}\sigma_r(m_vn_v,m_v^{-1})\frac{s_r(m_vn_v)s_r(m_v^{-1})}{s_r(m_vn_vm_v^{-1})}\\
&\quad= \prod_{v\in S} s_r(m_vn_vm_v^{-1})^{-1} = s_r(mnm^{-1})^{-1},
\end{aligned}$$

where for the second equality we used (2.3), for the third equality we used the same cocycle computation as in the proof of Lemma 4.10, and finally for the last equality we used $s_r(m_vn_vm_v^{-1}) = 1$ for all $v \notin S$. ∎

Let us mention that for the case at hand one can prove Lemma 5.5 as we did here. However, this lemma holds not just for our $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ but for covering groups in general (see [MW, I.1.3(4), p. 4]).

At any rate, Lemma 5.5 allows one to form the global induced representation

$$\mathrm{Ind}_{\widetilde{M}(\mathbb{A})N(\mathbb{A})^*}^{\widetilde{\mathrm{GL}}_r(\mathbb{A})}\pi$$

for an automorphic representation π of $\widetilde{M}(\mathbb{A})$, and hence one can form the Eisenstein series on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ as in the non-metaplectic case.

With this said, we have the following theorem.

Theorem 5.22 *Let $P = MN \subseteq \mathrm{GL}_r$ be the standard parabolic subgroup whose Levi part is $M = \mathrm{GL}_{r_1}\times\cdots\times\mathrm{GL}_{r_k}$. Further, for each $i = 1,\dots,k$, let $P_i = M_iN_i \subseteq \mathrm{GL}_{r_i}$ be the standard parabolic of GL_{r_i} whose Levi part is $M_i = \mathrm{GL}_{r_{i,1}}\times\cdots\times\mathrm{GL}_{r_{i,l_i}}$. For each i, assume we can find $A_{\widetilde{M}_i}$ that satisfies Hypothesis $(*)$ with respect to M_i (which is always the case if $n=2$), and we are given an automorphic representation*

$$\sigma_i := (\tau_{i,1}\widetilde{\otimes}\cdots\widetilde{\otimes}\tau_{i,l_i})_{\omega_i}$$

of $\widetilde{M}_i(\mathbb{A})$, which is given as the metaplectic tensor product of the unitary automorphic subrepresentations $\tau_{i,1},\dots,\tau_{i,l_i}$ of $\widetilde{\mathrm{GL}}_{r_{i,1}}(\mathbb{A}),\dots,\widetilde{\mathrm{GL}}_{r_{i,l_i}}(\mathbb{A})$, respectively. Assume that π_i is an irreducible constituent of the induced representation $\mathrm{Ind}_{\widetilde{P}_i(\mathbb{A})}^{\widetilde{\mathrm{GL}}_{r_i}(\mathbb{A})}\sigma_i$ and is realized as an automorphic subrepresentation. Then the metaplectic tensor product

$$\pi_\omega := (\pi_1\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\pi_k)_\omega$$

is an irreducible constituent of the induced representation

$$\mathrm{Ind}_{\widetilde{Q}(\mathbb{A})}^{\widetilde{M}(\mathbb{A})}(\tau_{1,1}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\tau_{1,l_1}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\tau_{k,1}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\tau_{k,l_k})_\omega,$$

where Q is the standard parabolic of M whose Levi part is $M_1 \times \cdots \times M_k$, where $M_i \subseteq \mathrm{GL}_{r_i}$ for each i.

Proof This follows from its local analogue (Theorem 4.11) and the local-global compatibility of the metaplectic tensor product $\pi_\omega \cong \widetilde{\bigotimes}' \pi_{\omega_v}$. ■

Remark 5.23 Just as we mentioned in Remark 4.13 for the local case, in the above theorem one can replace "constituent" by "irreducible subrepresentation" or "irreducible quotient", and the analogous statement still holds.

5.6 Restriction to a Smaller Levi subgroup

Finally, let us mention an important property of the metaplectic tensor product that one needs to compute constant terms of metaplectic Eisenstein series (see [T2]).

Both locally and globally, let

$$M_2 = \mathrm{GL}_{r_2} \times \cdots \times \mathrm{GL}_{r_k} = \left\{ \begin{pmatrix} I_{r_1} & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_k \end{pmatrix} \in M : g_i \in \mathrm{GL}_{r_i} \right\}$$

be viewed as a subgroup of M in the obvious way. We view GL_{r-r_1} as a subgroup of GL_r embedded in the right lower corner, and so M_2 can be also viewed as a Levi subgroup of GL_{r-r_1} embedded in this way.

Both locally and globally, we let

$$\tau_{M_2} \colon M_2 \times M_2 \to \mu_n$$

be the block-compatible 2-cocycle on M_2 defined analogously to τ_M. One can see that the block-compatibility of τ_M and τ_{M_2} implies

$$\tau_{M_2} = \tau_M|_{M_2 \times M_2},$$

which gives the embeddings

$$\widetilde{M}_2 \subseteq \widetilde{M} \hookrightarrow \widetilde{\mathrm{GL}}_r.$$

(Note that the last map is not the natural inclusion because here $\widetilde{M}$ is actually ${}^c\widetilde{M}$, and that is why we use $\hookrightarrow$ instead of $\subseteq$.)

For each automorphic form $\widetilde{\varphi} \in V_{\pi_\omega}$ in the space of the metaplectic tensor product, one would like to know which space the restriction $\widetilde{\varphi}|_{\widetilde{M}_2(\mathbb{A})}$ belongs to. Just as the non-metaplectic case, it would be nice if this restriction were simply in the space of the metaplectic tensor product of $\pi_2, \ldots, \pi_k$ with respect to the character ω restricted to, say, $A_{\widetilde{M}} \cap \widetilde{M}_2$. But as we will see, this is not necessarily the case. The metaplectic tensor product is more subtle.

Let us first introduce the subgroup $A_{\widetilde{M}_2}$ of $\widetilde{M}_2$ which plays the role analogous to that of $A_{\widetilde{M}}$:

$$A_{\widetilde{M}_2}(R) := \left\{ \left(\begin{pmatrix} I_{r_1} & \\ & A_2 \end{pmatrix}, \xi \right) : \left(\begin{pmatrix} a_1 I_{r_1} & \\ & A_2 \end{pmatrix}, \xi \right) \in A_{\widetilde{M}}(R) \text{ for some } a_1 \in R^{\times n} \right\}.$$

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Note that $A_{\widetilde{M}}(R) \cap \widetilde{M}_2(R) \subseteq A_{\widetilde{M}_2}(R)$, but the equality might not hold in general. Also note $A_{\widetilde{M}_2}(R) \subseteq A_{\widetilde{M}}(R)$. The following lemma implies that $A_{\widetilde{M}_2}$ is abelian.

Lemma 5.24 *Let* $\left(\begin{pmatrix} I_{r_1} & \\ & A_2 \end{pmatrix}, \xi\right), \left(\begin{pmatrix} I_{r_1} & \\ & A_2' \end{pmatrix}, \xi'\right) \in A_{\widetilde{M}_2}(R)$. *Then*

$$\tau_{M_2}(A_2, A_2') = \tau_{M_2}(A_2', A_2).$$

Proof This follows by the block-compatibility of τ_M and the fact that $A_{\widetilde{M}}(R)$ is abelian. ■

Also, one can see that the image of $A_{\widetilde{M}_2}(R)$ under the canonical projection is closed, and hence $A_{\widetilde{M}_2}(R)$ is closed.

Lemma 5.25 *For $R = \mathbb{A}$ or F_v, we have*

$$A_{\widetilde{M}_2}(R)\widetilde{M}_2^{(n)}(R) = A_{\widetilde{M}}(R)\widetilde{M}^{(n)}(R) \cap \widetilde{M}_2(R).$$

Also for global F we have

$$A_{\widetilde{M}_2}\widetilde{M}_2^{(n)}(F) = A_{\widetilde{M}}\widetilde{M}^{(n)}(F) \cap \mathbf{s}\big(M_2(F)\big),$$

where by definition

$$A_{\widetilde{M}_2}\widetilde{M}_2^{(n)}(F) := A_{\widetilde{M}_2}(\mathbb{A})\widetilde{M}_2^{(n)}(\mathbb{A}) \cap \mathbf{s}\big(M(F)\big),$$

which is not necessarily the same as $A_{\widetilde{M}_2}(F)\widetilde{M}_2^{(n)}(F)$.

Proof This can be verified by direct computation. Note that for both cases, the inclusion $\subseteq$ is immediate. For the reverse inclusion, we need to show that if $a \in A_{\widetilde{M}}(R)$ and $m \in \widetilde{M}^{(n)}(R)$ are such that $am \in A_{\widetilde{M}}(R)\widetilde{M}^{(n)}(R) \cap \widetilde{M}_2(R)$, one can always write $a = a_2 a_1$ with $a_2 \in A_{\widetilde{M}_2}(R)$ such that $a_1 m \in \widetilde{M}_2^{(n)}(R)$, and hence

$$am = a_2(a_1 m) \in A_{\widetilde{M}_2}(R)\widetilde{M}_2^{(n)}(R) \subseteq A_{\widetilde{M}_2}\widetilde{M}_2^{(n)}(F). \qquad ■$$

Now assume that our group $A_{\widetilde{M}}$ satisfies the following hypothesis.

Hypothesis $(**)$ (i) $A_{\widetilde{M}}$ satisfies Hypothesis $(*)$
(ii) $A_{\widetilde{M}_2}$ as defined above contains the center $Z_{\widetilde{\mathrm{GL}}_{r-r_1}}$.
(iii) $A_{\widetilde{M}_2}$ satisfies Hypothesis $(*)$ with respect to $\widetilde{M}_2$.

As an example of $A_{\widetilde{M}}$ satisfying the above hypothesis, we have the following lemma.

Lemma 5.26 *If $n = 2$, the choice of $A_{\widetilde{M}}$ as in Proposition A.6 satisfies Hypothesis $(**)$. Moreover, one has $A_{\widetilde{M}_2} = A_{\widetilde{M}} \cap \widetilde{M}_2$ both locally and globally.*

Proof This can be checked case-by-case. ■

Next, for each $\delta \in \mathrm{GL}_{r_1}(F)$, define $\omega_\delta \colon A_{\widetilde{M}_2}(F)\backslash A_{\widetilde{M}_2}(\mathbb{A}) \to \mathbb{C}^1$ by

$$\omega_\delta(a) = \omega\big(\mathbf{s}(\delta)a\mathbf{s}(\delta^{-1})\big).$$

Since $\mathbf{s}(\delta)A_{\widetilde{M}_2}(\mathbb{A})\mathbf{s}(\delta^{-1}) = A_{\widetilde{M}_2}(\mathbb{A})$ and $A_{\widetilde{M}_2}(\mathbb{A}) \subseteq A_{\widetilde{M}}(\mathbb{A})$, this is well defined, and since $\mathbf{s}$ is a homomorphism on $M(F)$, ω_δ is a character. Indeed, one can compute

$$\omega_\delta(a) = (\det\delta, \det a)^{1+2c}\omega(a),$$

because one can see $\mathbf{s}(\delta)a\mathbf{s}(\delta^{-1}) = (1, (\det\delta, \det a)^{1+2c})a$ and ω is genuine. Hence for each $a \in A_{\widetilde{M}_2}(\mathbb{A}) \cap A_{\widetilde{M}_2}\widetilde{M}_2^{(n)}(F)\widetilde{M}_2^{(n)}(\mathbb{A})$, because $(\det\delta, \det a) = 1$, we have $\omega_\delta(a) = \omega(a)$, namely

$$\omega_\delta|_{A_{\widetilde{M}_2}(\mathbb{A})\cap A_{\widetilde{M}_2}\widetilde{M}_2^{(n)}(F)\widetilde{M}_2^{(n)}(\mathbb{A})} = \omega|_{A_{\widetilde{M}_2}(\mathbb{A})\cap A_{\widetilde{M}_2}\widetilde{M}_2^{(n)}(F)\widetilde{M}_2^{(n)}(\mathbb{A})}.$$

Therefore, using $\pi_2, \dots, \pi_k$ and ω_δ, one can construct the metaplectic tensor product representation of $\widetilde{M}_2(\mathbb{A})$ with respect to $A_{\widetilde{M}_2}$, namely,

$$\pi_{\omega_\delta} := (\pi_2\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_{\omega_\delta}. \tag{5.7}$$

Then we have the following proposition.

Proposition 5.27 *Assume that $A_{\widetilde{M}}$ satisfies Hypothesis (∗∗). For each $\widetilde{\varphi} \in \pi_\omega = (\pi_1\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_\omega$,*

$$\widetilde{\varphi}|_{\widetilde{M}_2(\mathbb{A})} \in \bigoplus_\delta m_\delta \pi_{\omega_\delta},$$

where $\pi_{\omega_\delta} = (\pi_2\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_{\omega_\delta}$ as in (5.7) and δ runs through a finite subset of $\mathrm{GL}_{r_1}(F)$, and $m_\delta \in \mathbb{Z}^{>0}$ is a multiplicity. (Note that which δ appears in the sum could depend on φ.)

Proof Recall that

$$\widetilde{\varphi}(m) = \sum_{\gamma\in A_M M^{(n)}(F)\backslash M(F)} \varphi(\mathbf{s}(\gamma)m; 1),$$

where the sum is finite, but by Lemma 5.8 we know that which γ contributes to the sum depends only on the class in $\widetilde{M}(\mathbb{A})/\widetilde{M}^{(n)}(\mathbb{A})\kappa(M(\mathcal{O}_S))$ for some finite set S of places. Note that $A_M M^{(n)}(F)$ is a normal subgroup of $M(F)$, and hence $A_M M^{(n)}(F)\backslash M(F)$ is a group. (This is actually an abelian group because it is a subgroup of the abelian group $A_M(\mathbb{A})M^{(n)}(\mathbb{A})\backslash M(\mathbb{A})$.) By Lemma 5.25 we have the inclusion

$$A_{M_2}M_2^{(n)}(F)\backslash M_2(F) \hookrightarrow A_M M^{(n)}(F)\backslash M(F).$$

Hence we have

$$\begin{aligned}\widetilde{\varphi}(m) &= \sum_{\gamma\in A_M M^{(n)}(F)\backslash M(F)} \varphi(\mathbf{s}(\gamma)m; 1)\\ &= \sum_{\delta\in M_2(F)A_M M^{(n)}(F)\backslash M(F)} \ \sum_{\mu\in A_{M_2}M_2^{(n)}(F)\backslash M_2(F)} \varphi(\mathbf{s}(\mu)\mathbf{s}(\delta)m; 1).\end{aligned}$$

By using Lemma 5.25, one can see that the map on $\widetilde{M}_2(\mathbb{A})$ defined by $m_2 \mapsto \varphi(m_2\mathbf{s}(\delta)m)$ is in the induced space $\text{c-Ind}_{A_{\widetilde{M}_2}(\mathbb{A})\widetilde{M}_2^{(n)}(\mathbb{A})}^{\widetilde{M}_2(\mathbb{A})} \pi_{\omega,2}^{(n)}$, where

$$\pi_{\omega,2}^{(n)} := \omega\big(\pi_2^{(n)}\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\pi_k^{(n)}\big)$$

and ω is actually the restriction of ω to $A_{\widetilde{M}_2}(\mathbb{A})$. Now since we are assuming that $A_{\widetilde{M}}$ satisfies Hypothesis $(**)$, the inner sum is finite. Since the sum over $\gamma \in A_M M^{(n)}(F)\backslash M(F)$ is finite, the outer sum is also finite.

Since $\delta \in M_2(F)A_M M^{(n)}(F)\backslash M(F)$ can be chosen to be in $\mathrm{GL}_{r_1}(F)$, we have $\mathbf{s}(\mu)\mathbf{s}(\delta) = \mathbf{s}(\delta)\mathbf{s}(\mu)$. So we can write

$$\widetilde{\varphi}(m) = \sum_{\delta \in M_2(F)A_M M^{(n)}(F)\backslash M(F)} \ \sum_{\mu \in A_{M_2} M_2^{(n)}(F)\backslash M_2(F)} \varphi\big(\mathbf{s}(\delta)\mathbf{s}(\mu)m; 1\big).$$

One can see by using Lemma 5.25 that for each δ the map on $\widetilde{M}_2(\mathbb{A})$ defined by

$$m_2 \mapsto \varphi\big(\mathbf{s}(\delta)m_2; 1\big)$$

is in the induced space $\text{c-Ind}_{A_{\widetilde{M}_2}(\mathbb{A})\widetilde{M}_2^{(n)}(\mathbb{A})}^{\widetilde{M}_2(\mathbb{A})} \pi_{\omega_\delta}^{(n)}$, where

$$\pi_{\omega_\delta}^{(n)} = \omega_\delta(\pi_2^{(n)} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k^{(n)}).$$

Hence the function on $\widetilde{M}_2(\mathbb{A})$ defined by

$$\widetilde{\varphi}_\delta \colon m_2 \mapsto \sum_{\mu \in A_{M_2} M_2^{(n)}(F)\backslash M_2(F)} \varphi\big(\mathbf{s}(\delta)\mathbf{s}(\mu)m_2; 1\big)$$

belongs to a space of π_{ω_δ}. Hence we can write

$$\widetilde{\varphi}(m_2) = \sum_{\delta \in M_2(F)A_M M^{(n)}(F)\backslash M(F)} \widetilde{\varphi}_\delta(m_2). \tag{5.8}$$

for all $m_2 \in \widetilde{M}_2(\mathbb{A})$.

Now we will show that this sum can be written as a finite sum independent of m_2. First, as we noted above, the δ's that contribute to the sum depend only on the classes in $\widetilde{M}(\mathbb{A})/\widetilde{M}^{(n)}(\mathbb{A})\kappa(M(\mathcal{O}_S))$. Hence, for each coset in $\widetilde{M}_2(\mathbb{A})/\widetilde{M}_2^{(n)}(\mathbb{A})\kappa(M_2(\mathcal{O}_S))$ the δ's that contribute to the sum are all equal. Also, since $\widetilde{\varphi}_\delta$ is left invariant on $\mathbf{s}(M_2(F))$, the δ's that contribute to the sum in (5.8) depend only on the double cosets in

$$\mathbf{s}\big(M_2(F)\big)\backslash \widetilde{M}_2(\mathbb{A})/\widetilde{M}_2^{(n)}(\mathbb{A})\kappa\big(M_2(\mathcal{O}_S)\big).$$

But one can see that this double coset space can be identified with the product of $k-1$ copies of

$$F^\times\backslash\mathbb{A}^\times/\mathbb{A}^{\times n}\,\mathcal{O}_S^\times = F^\times\mathbb{A}^{\times n}\,\mathcal{O}_S^\times\backslash\mathbb{A}^\times,$$

which is finite by Lemma 5.14. Hence, there are only finitely many δ's such that $\widetilde{\varphi}_\delta(m_2) \neq 0$ for some m_2.

Hence there exists finitely many $\delta_1, \dots, \delta_N \in M_2(F)A_M M^{(n)}(F)\backslash M(F)$ such that

$$\widetilde{\varphi}|_{\widetilde{M}_2(\mathbb{A})} = \sum_{i=1}^N \widetilde{\varphi}_{\delta_i}.$$

Since we do not know the multiplicity one property for the group $\widetilde{M}_2$, we might have a possible multiplicity m_δ. This completes the proof. ∎

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Theorem 5.28 *Assume that the metaplectic tensor product $(\pi_1\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_\omega$ is realized with the group $A_{\widetilde{M}}$ that satisfies Hypothesis $(**)$. Then we have*

$$(\pi_1\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_\omega\|_{\widetilde{M}_2(\mathbb{A})}\subseteq\bigoplus_{\delta\in\mathrm{GL}_{r_1}(F)}m_\delta(\pi_2\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_{\omega_\delta},$$

where $m_\delta\in\mathbb{Z}^{\geq 0}$

Proof This is immediate from the above proposition. ∎

Now we can restrict the metaplectic tensor product "from the bottom" and get the same result. Let

$$M_{k-1}=\mathrm{GL}_{r_1}\times\mathrm{GL}_{r_{k-1}}=\left\{\begin{pmatrix}g_1&&&\\&\ddots&&\\&&g_{k-1}&\\&&&I_{r_k}\end{pmatrix}\in M:g_i\in\mathrm{GL}_{r_i}\right\},$$

and embed M_{k-1} in GL_r in the upper left corner. Then define $A_{\widetilde{M}_{k-1}}$ and the character ω_δ analogously. Also consider the analogue of Hypothesis $(**)$.

Hypothesis $(***)$

(i) $A_{\widetilde{M}}$ satisfies Hypothesis $(*)$

(ii) $A_{\widetilde{M}_{k-1}}$ as defined above contains the center $Z_{\widetilde{\mathrm{GL}}_{r-r_k}}$.

(iii) $A_{\widetilde{M}_{k-1}}$ satisfies Hypothesis $(*)$ with respect to $\widetilde{M}_{k-1}$.

Then we have the following theorem.

Theorem 5.29 *Assume that the metaplectic tensor product $(\pi_1\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_\omega$ is realized with the group $A_{\widetilde{M}}$ that satisfies Hypothesis $(***)$. Then we have*

$$(\pi_1\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_\omega\|_{\widetilde{M}_{k-1}(\mathbb{A})}\subseteq\bigoplus_{\delta\in\mathrm{GL}_{r_k}(F)}m_\delta(\pi_1\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_{k-1})_{\omega_\delta},$$

where $m_\delta\in\mathbb{Z}^{>0}$

Proof The proof is essentially the same as the case for the restriction to $\widetilde{M}_2$. We will leave the verification to the reader. ∎

Also, for the case $n=2$, we can do even better.

Theorem 5.30 *Assume that $n=2$.*

(i) *Choose $A_{\widetilde{M}}$ to be as in Proposition A.6. For $j=2,\dots,k$, let*

$$M_j=\mathrm{GL}_{r_j}\times\cdots\times\mathrm{GL}_{r_k}\subseteq M,$$

embedded into the right lower corner. Then

$$(\pi_1\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_\omega\|_{\widetilde{M}_j(\mathbb{A})}\subseteq\bigoplus_{\omega'}m_{\omega'}(\pi_j\widetilde{\otimes}\cdots\widetilde{\otimes}\pi_k)_{\omega'},$$

where ω' runs through a countable number of characters on $A_{\widetilde{M}_j}=A_{\widetilde{M}}\cap\widetilde{M}_j$.

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(ii) *Choose* $A_{\widetilde{M}}$ *to be as in Proposition A.7. For* $j = 1, \ldots, k-1$, *let* $M_{k-j} = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_{k-j}} \subseteq M$, *embedded into the left upper corner. Then*

$$(\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega \|_{\widetilde{M}_{k-j}(\mathbb{A})} \subseteq \bigoplus_{\omega'} m_{\omega'} (\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{k-j})_{\omega'},$$

where ω' *runs through a countable number of characters on* $A_{\widetilde{M}_{k-j}} = A_{\widetilde{M}} \cap \widetilde{M}_{k-j}$.

Proof For (i), one can inductively show that $A_{\widetilde{M}_j} = A_{\widetilde{M}_{j-1}} \cap \widetilde{M}_{j-1}$ satisfies Hypotheses $(*)$ and $(**)$ for the Levi subgroup M_j. Thus one can successively apply the above theorem for $j = 2, \ldots, k$, which proves the theorem. Case (ii) can be treated similarly. ■

Remark 5.31 In the above theorem, we choose different $A_{\widetilde{M}}$ for the two cases to define $(\pi_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_k)_\omega$. They are, however, equivalent, because, though ω is a character on $A_{\widetilde{M}}$, the metaplectic tensor product is dependent only on the restriction $\omega|_{Z_{\widetilde{\mathrm{GL}}_r}}$ to the center.

Appendix A On the Discreteness of the Group $A_M M^{(n)}(F) \backslash M(F)$

In this appendix, we will discuss the issue of when $A_{\widetilde{M}}$ can be chosen so that the group $A_M M^{(n)}(F) \backslash M(F)$ is a discrete subgroup of $A_{\widetilde{M}}(\mathbb{A}) \widetilde{M}^{(n)}(\mathbb{A}) \backslash \widetilde{M}(\mathbb{A})$, and hence the metaplectic tensor product can be defined. In particular, we will show that if $n = 2$, one can always choose such $A_{\widetilde{M}}$, and hence all the global results hold without any condition. If $n > 2$, the author does not know if it is always possible to choose such nice $A_{\widetilde{M}}$, though he suspects that this is always the case.

Throughout this appendix the field F is a number field. Also, for topological groups $H \subseteq G$, we always assume $H \backslash G$ is equipped with the quotient topology.

The crucial fact is the following proposition.

Proposition A.1 *For any positive integer* m, *the image of* $F^\times$ *in* $\mathbb{A}^{\times m} \backslash \mathbb{A}^\times$ *is discrete in the quotient topology.*

Proof Let $K = \prod_v K_v \subseteq \mathbb{A}^\times$ be the open neighborhood of the identity defined by $K_v = \mathcal{O}_{F_v}^\times$ for all finite v and $K_v = F_v^\times$ for all infinite v. To prove the discreteness of the image of $F^\times$, it suffices to show that the set $\mathbb{A}^{\times m} K \cap \mathbb{A}^{\times m} F^\times$ has only finitely many points modulo $A^{\times m}$. This is because the image of $F^\times$ in $\mathbb{A}^{\times m} \backslash \mathbb{A}^\times$ will then have an open neighborhood of the identity in the subspace topology for $\mathbb{A}^{\times m} \backslash \mathbb{A}^{\times m} F^\times$ containing finitely many points, and the quotient $\mathbb{A}^{\times m} \backslash \mathbb{A}^\times$ is Hausdorf, since $\mathbb{A}^{\times m}$ is closed.

Now, let $a^m \in \mathbb{A}^{\times m}$ and $u \in F^\times$ be such that $a^m u \in \mathbb{A}^{\times m} K \cap \mathbb{A}^{\times m} F^\times$. Then $u \in \mathbb{A}^{\times m} K$, and so for each finite v, we have $u_v \in F_v^{\times m} K_v$, which implies the fractional ideal (u) generated by u is m-th power in the group I_F of fractional ideals of F. Namely $(u) \in P_F \cap I_F^m$, where P_F is the group of principal fractional ideals. On the other hand for any $(u) \in P_F \cap I_F^m$, one can see that $u \in \mathbb{A}^{\times m} K$.

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Accordingly, if we define

$$G := \{u \in F^{\times} : (u) \in P_F \cap I_F^m\},$$

we have the surjection

$$F^{\times m}\backslash G \longrightarrow \mathbb{A}^{\times m}\backslash(\mathbb{A}^{\times m}K \cap \mathbb{A}^{\times m}F^{\times}),$$

given by $u \mapsto \mathbb{A}^{\times m}u$. So we have only to show that the group $F^{\times m}\backslash G$ is finite. But note that the map $u \mapsto (u)$ gives rise to the short exact sequence

$$0 \longrightarrow U_F^m\backslash U_F \longrightarrow F^{\times m}\backslash G \longrightarrow P_F^m\backslash P_F \cap I_F^m \longrightarrow 0,$$

where U_K is the group of units for F. Now the group $U_F^m\backslash U_F$ is finite by Dirichlet's unit theorem. The group $P_F^m\backslash P_F \cap I_F^m$ is isomorphic to the group of m-torsions in the class group of F via the map

$$P_F^m\backslash P_F \cap I_F^m \longrightarrow P_F\backslash I_F, \quad \mathfrak{A}^m \longmapsto \mathfrak{A}$$

for each fractional ideal $\mathfrak{A}^m \in I_F^m$, and hence finite. Therefore, $F^{\times m}\backslash G$ is finite. ■

As a first consequence of this, we have the following proposition.

Proposition A.2 *The image of $M(F)$ in $M^{(n)}(\mathbb{A})\backslash M(\mathbb{A})$ is discrete.*

Proof Let

$$\mathrm{Det}_M \colon M(\mathbb{A}) \to \underbrace{\mathbb{A}^{\times n}\backslash\mathbb{A}^{\times} \times \cdots \times \mathbb{A}^{\times n}\backslash\mathbb{A}^{\times}}_{k-\text{times}}$$

be the map defined by

$$\mathrm{Det}_M\big(\mathrm{diag}(g_1, \ldots, g_k)\big) = \big(\det(g_1), \ldots, \det(g_k)\big).$$

Then $\ker(\mathrm{Det}_M) = M^{(n)}(\mathbb{A})$. Moreover, the map Det_M is continuous. Hence we have a continuous group isomorphism

$$M^{(n)}(\mathbb{A})\backslash M(\mathbb{A}) \to \mathbb{A}^{\times n}\backslash\mathbb{A}^{\times} \times \cdots \times \mathbb{A}^{\times n}\backslash\mathbb{A}^{\times}.$$

Moreover, one can construct the continuous inverse by sending each $a_i \in \mathbb{A}^{\times n}\backslash\mathbb{A}^{\times}$ to the first entry of the i-th block $\mathrm{GL}_{r_i}(\mathbb{A})$. But the image of $M(F)$ in $\mathbb{A}^{\times n}\backslash\mathbb{A}^{\times} \times \cdots \times \mathbb{A}^{\times n}\backslash\mathbb{A}^{\times}$ under Det_M is discrete by the above proposition. The proposition follows. ■

We then have the following corollary.

Corollary A.3 *If the center $Z_{\widetilde{\mathrm{GL}}_r}(\mathbb{A})$ is contained in $\widetilde{M}^{(n)}(\mathbb{A})$, which is the case if n divides nr_i/d for all $i = 1, \ldots, k$ where $d = \gcd(n, r-1+2cr)$, then Hypothesis $(*)$ is satisfied, and the metaplectic tensor product can be defined.*

Proof If the center is already in $\widetilde{M}^{(n)}(\mathbb{A})$, one can choose $A_{\widetilde{M}}(\mathbb{A}) = Z_{\widetilde{\mathrm{GL}}_r}(\mathbb{A})$ and then $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A}) = \widetilde{M}^{(n)}(\mathbb{A})$, and so $A_M M^{(n)}(F) = M^{(n)}(F)$. Then by the above proposition, $A_M M^{(n)}(F)\backslash M(F)$ is discrete in $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(n)}(\mathbb{A})\backslash\widetilde{M}(\mathbb{A})$. ■

Proposition A.2 also implies the following proposition.

Proposition A.4 *The group $M(F)M^{(n)}(\mathbb{A})$ (resp. $M(F)^*\widetilde{M}^{(n)}(\mathbb{A})$) is a closed subgroup of $M(\mathbb{A})$ (resp. $\widetilde{M}(\mathbb{A})$).*

Proof It suffices to show it for $M(F)M^{(n)}(\mathbb{A})$, because the canonical projection is continuous. But for this, one can apply the following lemma with $G = M(\mathbb{A}), Y = M^{(n)}(\mathbb{A})$, and $\Gamma = M(F)$, which will complete the proof. ∎

Lemma A.5 *Let G be a Hausdorf topological group. If $\Gamma \subset G$ is a discrete subgroup and $Y \subset G$ a closed normal subgroup such that the image of Γ in G/Y is discrete in the quotient topology, then the group ΓY is closed in G.*

Proof Let $p\colon G \to G/Y$ be the canonical projection. By our assumption, the image $p(\Gamma)$ of Γ is discrete in the quotient topology. Now since Y is closed, the quotient G/Y is a Hausdorf topological group. Hence, $p(\Gamma)$ is closed by Lemma 9.1.3(b) of [D-E]. To show that ΓY is closed, it suffices to show that every net $\{\gamma_i y_i\}_{i\in I}$ that converges in G, where $\gamma_i \in \Gamma$ and $y_i \in Y$, converges in ΓY. But since p is continuous, the net $\{p(\gamma_i y_i)\}$ converges in G/Y, but $p(\gamma_i y_i) = p(\gamma_i)$ and $p(\gamma_i) \in p(\Gamma)$. Since $p(\Gamma)$ is closed and discrete, in order for the net $\{p(\gamma_i)\}$ to converge, there exists $\gamma \in \Gamma$ such that $p(\gamma_i) = p(\gamma)$ for all sufficiently large $i \in I$; namely, the net $\{p(\gamma_i)\}$ is eventually constant. Hence for sufficiently large i, we have $\gamma_i y_i = \gamma y_i'$ for some $y_i' \in Y$. This means that the net $\{\gamma_i y_i\}$ is eventually in the set γY. But since Y is closed, so is γY, which implies that the net $\{\gamma_i y_i\}$ converges in $\gamma Y \subset \Gamma Y$. ∎

Finally in this appendix, we will show that if $n = 2$, one can always choose $A_{\widetilde{M}}$ so that the group $A_M M^{(n)}(F)\backslash M(F)$ is discrete, and hence the metaplectic tensor product is defined, and, moreover, the metaplectic tensor product can be realized in such a way that it behaves nicely with the restriction to the smaller rank groups.

First, let us note that for any r, the center $Z_{\widetilde{\mathrm{GL}}_r}(\mathbb{A})$ is given by

$$Z_{\widetilde{\mathrm{GL}}_r}(\mathbb{A}) = \{(aI_r, \xi) : a \in \mathbb{A}^{\times\varepsilon}\}, \quad \varepsilon = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 2 & \text{if } r \text{ is even.} \end{cases}$$

Accordingly, one can see

$$Z_{\widetilde{\mathrm{GL}}_r}(\mathbb{A})\widetilde{\mathrm{GL}}_r^{(2)}(\mathbb{A}) = \begin{cases} \widetilde{\mathrm{GL}}_r(\mathbb{A}) & \text{if } r \text{ is odd,} \\ \widetilde{\mathrm{GL}}_r^{(2)}(\mathbb{A}) & \text{if } r \text{ is even.} \end{cases}$$

Proposition A.6 *Assume $n = 2$. Let*

$$\widetilde{Z}_i(\mathbb{A}) = Z_{\widetilde{\mathrm{GL}}_{r_i+\cdots+r_k}}(\mathbb{A}) \subseteq \widetilde{\mathrm{GL}}_{r_i}(\mathbb{A}) \widetilde{\times} \cdots \widetilde{\times} \widetilde{\mathrm{GL}}_{r_k}(\mathbb{A}) \subseteq \widetilde{M}(\mathbb{A})$$

and

$$A_{\widetilde{M}}(\mathbb{A}) = \widetilde{Z}_1(\mathbb{A})\widetilde{Z}_2(\mathbb{A}) \cdots \widetilde{Z}_k(\mathbb{A}).$$

Then $A_{\widetilde{M}}(\mathbb{A})$ is a closed abelian subgroup of $\widetilde{Z_M}(\mathbb{A})$. Furthermore, the group $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(2)}(\mathbb{A})$ is closed and the image of $M(F)$ in $A_M(\mathbb{A})M^{(2)}(\mathbb{A})\backslash M(\mathbb{A})$ as well as in $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(2)}(\mathbb{A})\backslash\widetilde{M}(\mathbb{A})$ is discrete.

Proof It is clear that $A_{\widetilde{M}}(\mathbb{A})$ is abelian, since for each $i = 1, \ldots, k$, $\widetilde{Z}_i$ is the center of $\widetilde{\mathrm{GL}}_{r_i+\cdots+r_k}(\mathbb{A})$, and hence commutes pointwise with $\widetilde{Z}_j(\mathbb{A}) \subseteq \widetilde{\mathrm{GL}}_{r_i+\cdots+r_k}(\mathbb{A})$ for all $j \geq i$. To show that $A_{\widetilde{M}}(\mathbb{A})$ is closed, it suffices to show that $A_M(\mathbb{A}) := p(A_{\widetilde{M}}(\mathbb{A}))$ is closed. Now one can write $A_M(\mathbb{A}) = \prod'_v A_M(F_v)$, where $A_M(F_v)$ is defined analogously to the global case. Then one can see that $Z_M^{(2)}(F_v) \subseteq A_M(F_v) \subseteq Z_M(F_v)$, and since $Z_M^{(2)}(F_v)$ is closed and of finite index in $Z_M(F_v)$, so is $A_M(F_v)$. But $Z_M(F_v)$ is closed in $M(F_v)$ and so $A_M(F_v)$ is closed in $M(F_v)$. Then one can show that $A_M(\mathbb{A})$ is closed in $M(\mathbb{A})$ by Lemma 2.10.

Now one can show by induction on k that the group $A_M(\mathbb{A})M^{(2)}(\mathbb{A})$ is the kernel of the map

$$\mathrm{Det}_M \colon M(\mathbb{A}) \longrightarrow \mathbb{A}^{\times\varepsilon_1}\backslash\mathbb{A}^{\times} \times \cdots \times \mathbb{A}^{\times\varepsilon_k}\backslash\mathbb{A}^{\times},$$

where ε_i is either 1 or 2. Hence one has a continuous group isomorphism

$$A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(2)}(\mathbb{A})\backslash\widetilde{M}(\mathbb{A}) \longrightarrow \mathbb{A}^{\times\varepsilon_1}\backslash\mathbb{A}^{\times} \times \cdots \times \mathbb{A}^{\times\varepsilon_k}\backslash\mathbb{A}^{\times},$$

where the space on the right is Hausdorff. Hence the space on the left is Hausdorf as well, which shows that $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(2)}(\mathbb{A})$ is closed. One can also show that the image of $M(F)$ is discrete as we did for Proposition A.2. ∎

Proposition A.7 *Assume $n = 2$. Let*

$$\widetilde{Z}_j(\mathbb{A}) = Z_{\widetilde{\mathrm{GL}}_{r_1+\cdots+r_{k-j}}}(\mathbb{A}) \subseteq \widetilde{\mathrm{GL}}_{r_1}(\mathbb{A})\widetilde{\times}\cdots\widetilde{\times}\widetilde{\mathrm{GL}}_{r_{k-j}}(\mathbb{A}) \subseteq \widetilde{M}(\mathbb{A})$$

and

$$A_{\widetilde{M}}(\mathbb{A}) = \widetilde{Z}_1(\mathbb{A})\widetilde{Z}_2(\mathbb{A})\cdots\widetilde{Z}_k(\mathbb{A}).$$

Then $A_{\widetilde{M}}(\mathbb{A})$ is a closed abelian subgroup of $\widetilde{Z_M}(\mathbb{A})$. Furthermore, the group $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(2)}(\mathbb{A})$ is closed and the image of $M(F)$ in $A_M(\mathbb{A})M^{(2)}(\mathbb{A})\backslash M(\mathbb{A})$ as well as in $A_{\widetilde{M}}(\mathbb{A})\widetilde{M}^{(2)}(\mathbb{A})\backslash\widetilde{M}(\mathbb{A})$ is discrete.

Proof The proof is identical to that of the previous proposition. ∎

Remark A.8 The above proposition and Corollary A.3 imply Proposition 3.13. Also for $n > 2$, if n and $r = r_1 + \cdots + r_k$ are such that n divides nr_i/d for all $i = 1 \cdots k$, where $d = \gcd(n, r-1+2cr)$ and n divides nr_i/d_2 for all $i = 2 \cdots k$, where $d_2 = \gcd(n, r - r_1 - 1 + 2c(r - r_2))$, then $A_{\widetilde{M}} = Z_{\widetilde{\mathrm{GL}}_r}$ satisfies Hypothesis $(**)$, and hence one has the restriction property to the smaller rank group. Moreover, this is always the case, for example, if $\gcd(n, r-1+2cr) = \gcd(n, r-r_1-1+2c(r-r_1)) = 1$. Similarly, one can satisfy Hypothesis $(***)$ if n divides nr_i/d for all $i = 1 \cdots k$ and divides nr_i/d_{k-1} for all $i = 1 \cdots k-1$, where $d_{k-1} = \gcd(n, r-r_{k-1}-1+2c(r-r_{k-1}))$. Those conditions are indeed often satisfied especially when n is a prime.

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References

[AT] E. Artin and J. Tate, *Class field theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[B] W. D. Banks, *Twisted symmetric-square L-functions and the nonexistence of Siegel zeros on* $\mathrm{GL}(3)$. Duke Math. J. **87**(1997), no. 2, 343–353. http://dx.doi.org/10.1215/S0012-7094-97-08713-5

[BBL] W. Banks, D. Bump, and D. Lieman, *Whittaker-Fourier coefficients of metaplectic Eisenstein series*. Compositio Math. **135**(2003), no. 2, 153–178. http://dx.doi.org/10.1023/A:1021763918640

[BLS] W. D. Banks, J. Levy, and M. Sepanski, *Block-compatible metaplectic cocycles*. J. Reine Angew. Math. **507**(1999), 131–163. http://dx.doi.org/10.1515/crll.1999.507.131

[Bo] N. Bourbaki, *Integration. II.* Chapters 7–9, translated from the 1963 and 1969 French originals by Sterling K. Berberian, Elements of Mathematics, Springer-Verlag, Berlin, 2004.

[BFH] D. Bump, S. Friedberg, and J. Hoffstein, *p-adic Whittaker functions on the metaplectic group*. Duke Math. J. **63**(1991), no. 2, 379–397. http://dx.doi.org/10.1215/S0012-7094-91-06316-7

[BG] D. Bump and D. Ginzburg, *Symmetric square L-functions on* $\mathrm{GL}(r)$. Ann. of Math. **136**(1992), no. 1, 137–205. http://dx.doi.org/10.2307/2946548

[BH] D. Bump and J. Hoffstein, *On Shimura's correspondence*. Duke Math. J. **55**(1987), no. 3, 661–691. http://dx.doi.org/10.1215/S0012-7094-87-05533-5

[C] J. W. Cogdell, *Lectures on L-functions, converse theorems, and functoriality of* $\mathrm{GL}(n)$. In: Lectures on automorphic L-functions, Fields Institute Monographs, 20, American Mathematical Society, Providence, RI, 2004, pp. 1–96.

[D-E] A. Deitmar and S. Echterhoff, *Principles of harmonic analysis.* Universitext, Springer, New York, 2009.

[F] Y. Z. Flicker, *Automorphic forms on covering groups of* $\mathrm{GL}(2)$. Invent. Math. **57**(1980), no. 2, 119–182. http://dx.doi.org/10.1007/BF01390092

[FK] Y. Z. Flicker and D. A. Kazhdan, *Metaplectic correspondence*. Inst. Hautes Études Sci. Publ. Math. **64**(1986), 53–110.

[GO] G. Chinta and O. Offen, *A metaplectic Casselman-Shalika formula for* GL_r. Amer. J. Math. **135**(2013), no. 2, 403–441. http://dx.doi.org/10.1353/ajm.2013.0013

[K1] A. C. Kable, *Exceptional representations of the metaplectic double cover of the general linear group*. PH.D thesis, Oklahoma State University, 1997.

[K2] A. C. Kable, *The tensor product of exceptional representations on the general linear group*. Ann. Sci. École Norm. Sup. (4) **34**(2001), no. 5, 741–769. http://dx.doi.org/10.1016/S0012-9593(01)01075-8

[KP] D. A. Kazhdan and S. J. Patterson, *Metaplectic forms*. Inst. Hautes Études Sci. Publ. Math. **59**(1984), 35–142.

[Kub] T. Kubota, *On automorphic functions and the reciprocity law in a number field*. Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 2, Kinokuniya Book-Store Co., Ltd., Tokyo, 1969.

[Mat] H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*. Ann. Sci. É cole Norm. Sup. **2**(1969), 1–62.

[Me] P. Mezo, *Metaplectic tensor products for irreducible representations*. Pacific J. Math. **215**(2004), no. 1, 85–96. http://dx.doi.org/10.2140/pjm.2004.215.85

[MW] C. Moeglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*. Cambridge Tracts in Mathematics, 113, Cambridge University Press, Cambridge, 1995. http://dx.doi.org/10.1017/CBO9780511470905

[S] T. Suzuki, *Metaplectic Eisenstein series and the Bump-Hoffstein conjecture.* Duke Math. J. **90**(1997), no. 3, 577–630. http://dx.doi.org/10.1215/S0012-7094-97-09016-5

[T1] S. Takeda, *The twisted symmetric square L-function of* $\mathrm{GL}(r)$. Duke Math. J. **163**(2014), no. 1, 175–266. http://dx.doi.org/10.1215/00127094-2405497

[T2] S. Takeda, *On a certain metaplectic Eisenstein series and the twisted symmetric square L-function*. Math. Z. **281**(2015), no. 1–2, 103–157. http://dx.doi.org/10.1007/s00209-015-1476-x

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