ON GROUPS WHICH ARE THE PRODUCT OF ABELIAN SUBGROUPS

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If the group $G = AB$ is the product of two abelian subgroups A and B, then G is metabelian by a well-known result of Itô [**8**], so that the commutator subgroup G' of G is abelian. In the following we are concerned with the following condition:

There exists a normal subgroup $N \neq 1$ *of* $G = AB \neq 1$ *which is contained in* A *or* B. (*)

Recently, Holt and Howlett in [**7**] have given an example of a countably infinite p-group $G = AB$, which is the product of two elementary abelian subgroups A and B with $\mathrm{Core}(A) = \mathrm{Core}(B) = 1$, so that in this group (*) does not hold. Also, Sysak in [**13**] gives an example of a product $G = AB$ of two free abelian subgroups A and B with $\mathrm{Core}(A) = \mathrm{Core}(B) = 1$.

On the positive side, Itô has already shown in [**8**] that (*) holds for every finite group G. Cohn has proved in [**5**] that (*) holds if A and B are infinite cyclic. If A or B is artinian, the validity of (*) was shown by Sesekin in [**11**]. That (*) also is valid if A or B is noetherian was proved in [**1**] and [**12**]. The strongest positive result was obtained by Zaicev in [**15**], who showed that (*) holds if A or B has finite sectional rank.

In this note some further sufficient conditions for (*) are added. For instance, (*) holds if A or B is a torsion group with at least one nontrivial artinian p-component for some prime $p \in \pi R$, where R is the Hirsch–Plotkin radical of G. Further, it is sufficient for (*) that G/G' has finite sectional rank or G' is a torsion group with artinian primary components.

Note that, even for a finite p-group $G = AB$, condition (*) becomes false in general, for there exist finite p-groups $G = AB$ with $\mathrm{Core}(A) = \mathrm{Core}(B) = 1$: see [**1**], [**4**] or [**6**].

NOTATION.

G' = commutator subgroup of the group G
$Z(G)$ = center of G
πG = set of all primes p for which there is an element of order p in G
$C(X)$ = centralizer of the subset X in G
$N(X)$ = normalizer of the subgroup X in G

A group is called *artinian* (*noetherian*) if its subgroups satisfy the minimum (maximum) condition. An abelian group G has *finite sectional rank* if it has finite torsionfree rank and each primary component of G has finite rank. A soluble group has *finite sectional rank* if all its abelian factors (sections) have finite sectional rank. If N is a normal subgroup of the

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factorized group $G=AB$, the *factorizer* of N in $G=AB$ is the subgroup $X(N)=AN\cap BN$; it is easy to see that $X(N)$ has the 'triple factorization'

$$X(N)=N(A\cap BN)=N(B\cap AN)=(A\cap BN)(B\cap AN).$$

If the group $G=AB$ is the product of two abelian subgroups A and B with finite torsionfree ranks, then by Zaicev [**15**] G is a metabelian group with finite torsionfree rank. The following lemma gives a condition for the Hirsch-Plotkin radical of such a group to be 'factorized' as a product of a subgroup of A and a subgroup of B.

LEMMA. *Let the group $G=AB$ be the product of two abelian subgroups A and B with finite torsionfree ranks. If the Hirsch–Plotkin radical $R=R(G)$ is nilpotent, then R is factorized.*

Proof. By [**15**, Theorem 3.2], G is a metabelian group with finite torsionfree rank. By hypothesis $R=R(G)$ is nilpotent. It may be assumed that $R\neq 1$ and hence $R'\subset R$. The factorizer $X(R)$ has the triple factorization

$$X=X(R)=R(A\cap BR)=R(B\cap AR)=(A\cap BR)(B\cap AR).$$

By Zaicev [**16**, Theorem 2], X/R' is locally nilpotent. Since R is nilpotent, application of Robinson [**9**] yields that X is locally nilpotent. Since A and B are abelian, X is normal in G. Hence $X=R$ is factorized.

The following proposition gives some information about groups $G=AB$ that do not satisfy $(*)$: see [**1**, Lemma 4.1].

PROPOSITION. *If the group $G=AB\neq 1$ is the product of two abelian subgroups A and B and if 1 is the only normal subgroup of G which is contained in A or B, then the following hold.*

(1) $A\cap B=Z(G)=1$.

(2) $A\cap C(G')=B\cap C(G')=1$; *in particular* $A\cap G'=B\cap G'=1$.

(3) $A=N(A)$ *and* $B=N(B)$.

(4) *No non-identity element of A is conjugate to an element of B.*

(5) *If $X(N)=(A\cap BN)(B\cap AN)$ is the factorizer of the normal subgroup N of G contained in G', then $A\cap BN$ and $B\cap AN$ are isomorphic.*

(6) *If the normal subgroup N of G contains G', then 1 is the only normal subgroup of $X(N)$ which is contained in A or B.*

(7) *G' is not a minimal normal subgroup of G.*

(8) *G' is not contained in the FC-center of G.*

(9) *G' is not a torsion group with artinian primary components.*

(10) *If $(A\cap BG')\simeq(B\cap AG')$ is a π-group for some set of primes π, then $X(G')$ is a locally finite-nilpotent π-group and no nontrivial primary component of $(A\cap BG')\simeq(B\cap AG')$ is artinian.*

(11) *$(A\cap BG')\simeq(B\cap AG')$ does not have finite sectional rank.*

(12) *If $(A\cap BG')\simeq(B\cap AG')$ has finite torsionfree rank, then G' is not torsionfree.*

(13) *If G/G' is a π-group for some set of primes π, then G is a π-group.*

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(14) *G/G' does not have finite sectional rank.*
(15) *If G/G' has finite torsionfree rank, then G' is not torsionfree.*
(16) *G' is not noetherian.*

Proof. (1) Assume that $Z(G)\neq 1$. Let $z=ab\neq 1$, with $a\in A$ and $b\in B$, be an element of $Z(G)$. Without loss of generality $a\neq 1$. By [**4**, Lemma 2.1], $Z(G)$ is factorized, so that $a\in Z(G)$. Hence $\langle a\rangle$ is a nontrivial normal subgroup of G which is contained in A. This contradiction shows $Z(G)=1$. Since A and B are abelian, $A\cap B$ is contained in $Z(G)$, so that also $A\cap B=1$.

(2) Assume that $A\cap C(G')\neq 1$. The subgroup $S=C(A\cap C(G'))$ of G contains G', so that S is normal in G. Since A is abelian, A is contained in S. Since $A\cap C(G')$ is contained in the center of S, it follows that $Z(G)\neq 1$. As a characteristic subgroup of the normal subgroup S of G the center $Z(S)$ is a nontrivial normal subgroup of G. Hence $Z(S)$ is not contained in A. By the modular law $AZ(S)=AZ(S)\cap AB=A(AZ(S)\cap B)$, and $AZ(S)\cap B\subseteq Z(G)=1$. It follows that $AZ(S)=A$ and $Z(S)\subseteq A$. This contradiction shows that $A\cap C(G')=1$. Similarly $B\cap C(G')=1$. Then also $A\cap G'=B\cap G'=1$. Thus (2) holds. (See Sesekin [**10**].)

(3) First it is shown that $Z(G)=1$ implies $A=C(A)$ and $B=C(B)$. Assume that $A\subset C(A)$. By the modular law $C(A)=C(A)\cap AB=A(C(A)\cap B)$, where $C(A)\cap B\neq 1$. Now $C(B\cap C(A))=G$, so that $Z(G)\neq 1$. This contradiction shows that $A=C(A)$. Similarly $B=C(B)$.

Assume that $A\subset N(A)$. Let $E=N(A)\cap G'$. If $E=1$, then $N(A)\simeq N(A)G'/G'$ is abelian. If $E\neq 1$, then $[A,E]\subseteq(A\cap G')=1$ by (2), so that AE is abelian. In both cases $A\subset C(A)$, a contradiction. Hence $A=N(A)$. Similarly $B=N(B)$. This proves (3).

(4) Assume that for some a in A there is a conjugate $a^g=b$ which is in B. Let $g=a^*b^*$, where $a^*\in A$ and $b^*\in B$. Then it follows that $a=b\in A\cap B=1$. Therefore (4) holds.

(5) This follows from (2) and [**4**, Lemma 1.2].

(6) If $X=X(N)=AN\cap BN=1$, then $G'\subseteq N\subseteq X=1$, so that G is abelian, a contradiction. Hence $X\neq 1$. Assume there exists a normal subgroup $M\neq 1$ of X, which is contained in A or B. As a subgroup of N and X, the group G' normalizes M. Without loss of generality let M be contained in A. Then by (2)

$$[M,G']\subseteq(G'\cap M)\subseteq(G'\cap A)=1.$$

Hence G' is centralized by M, so that by (2)

$$M\subseteq(A\cap C(G'))=1.$$

It follows that $M=1$, a contradiction. This proves (6). (See [**1**] and also [**15**].)

Thus the factorizer $X=X(G')$ has a triple factorization with the following properties.

$X=G'A^*=G'B^*=A^*B^*$ where $A^*=A\cap BG'$ and $B^*=B\cap AG'$ and by (5) $A^*\simeq B^*$. By (6) 1 is the only normal subgroup of X which is contained in A^* or B^*. By (2) $A^*\cap C(X')=B^*\cap C(X')=1$. By (1) $Z(X)=1$. (†)

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The examples of Holt and Howlett and Sysak show that in this situation we need additional conditions to obtain a contradiction.

(7) If G' is a minimal normal subgroup of G, then $X = X(G')$ is abelian by [**1**]: see [**4**, Remark 3.3(b)]. This contradicts (†), so that (7) is proved.

(8) Assume that G' is contained in the FC-center F of G. By [**4**, Lemma 2.1], F is factorized, so that $F = AF \cap BF = (A \cap BF)(B \cap AF)$. By (6) 1 is the only normal subgroup of F which is contained in A or B. By (3) $A \cap BF = N_F(A \cap BF)$ and $B \cap AF = N_F(B \cap AF)$, so that $A \cap BF$ and $B \cap AF$ are Carter subgroups of F. Since F is an FC-group, the Carter subgroups of F are locally conjugate by [**14**, p. 159]. This contradicts (4). Hence G' is not contained in F. This proves (8).

(9) If G' is a torsion group with artinian primary components it is covered by finite normal subgroups of G. Hence G' is contained in the FC-center of G. This contradicts (8). Thus G' is not a torsion group with artinian primary components. This proves (9).

(10) If $(A \cap BG') \simeq (B \cap AG')$ is a π-group, by [**2**, p. 118, Theorem 5.4], $X = X(G') = A^*B^*$, with $A^* = A \cap BG'$ and $B^* = A \cap AG'$, is also a π-group. By [**4**, Corollary 2.6], the Hirsch–Plotkin radical $R = R(X)$ is factorized. Since $G' \subseteq R \subseteq X$ and since X is the smallest factorized subgroup of G containing G' it follows that $R = X$.

By [**3**, p. 234, Hilfssatz 3.4], for every prime p the p-component X_p of the locally nilpotent group X has the factorization $X_p = (A^* \cap X_p)(B^* \cap X_p)$. Every normal subgroup of X_p is also a normal subgroup of X. Thus, by (†) 1 is the only normal subgroup of X_p contained in A or B. In particular $Z(X_p) = 1$. If $A^* \cap X_p$ and $B^* \cap X_p$ are artinian, by [**1**] or [**2**, p. 112, Corollary 3.3], the normal subgroups of $X_p = (A^* \cap X_p)(B^* \cap X_p)$ satisfy the minimum condition. This implies that X_p is a hypercentral Černikov group. Assuming that $X_p \neq 1$, this implies that $Z(X_p) \neq 1$. This contradiction shows that no nontrivial primary component of $(A \cap BG') \simeq (B \cap AG')$ is artinian. This proves (10).

(11) If $(A \cap BG') \simeq (B \cap AG')$ has finite sectional rank, by [**15**, Theorem 3.5] $X = X(G') = (A \cap BG')(B \cap AG')$ also has finite sectional rank. Since X is also the factorizer of its Fitting subgroup, X is locally nilpotent by [**4**, Theorem 2.4]. As a locally nilpotent group with finite sectional rank, X is hypercentral. Assuming that $X \neq 1$ this implies $Z(X) \neq 1$. This contradicts (†) and proves (11).

(12) If $(A \cap BG') \simeq (B \cap AG')$ has finite torsionfree rank and if G' is torsionfree, then by Robinson [**10**, Theorem 4], $X = G'(A \cap BG') = G'(B \cap AG') = (A \cap BG')(B \cap AG')$ is nilpotent (with finite torsionfree rank). Assuming that $X \neq 1$ this implies $Z(X) \neq 1$. This contradicts (†) and proves (12).

(13) By (2) $A \simeq AG'/G'$ and $B \simeq BG'/G'$. Hence, if G/G' is a π-group, then A and B are π-groups. By [**2**, p. 118, Theorem 5.4], $G = AB$ also is a π-group.

(14) If G/G' has finite sectional rank, it follows as in the proof of (13) that A and B have finite sectional rank. This contradicts (11).

(15) If G/G' has finite torsionfree rank, it follows as in the proof of (13) that A and B have finite torsionfree rank. Hence G' cannot be torsionfree by (12).

(16) By a well-known theorem of Mal'cev every abelian group of automorphisms of a noetherian abelian group is noetherian. Hence, if G' is noetherian, $G/C(G')$ also is noetherian. By (2) $A \simeq AC(G')/C(G')$ and $B \simeq BC(G')/C(G')$. Hence also A and B are noetherian. This contradicts (11). The proposition is proved.

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The preceding proposition gives a number of sufficient conditions for the validity of (*). The most important ones are contained in the following theorem.

THEOREM. *If the group $G = AB \neq 1$ is the product of two abelian subgroups A and B, then there exists a nontrivial normal subgroup N of G which is contained in A or B if at least one of the following conditions holds.*

(a) *If A^* is a subgroup of A and B^* is a subgroup of B such that $A^* \simeq B^*$, then A^* and B^* have finite sectional rank.*

(b) *A or B is a torsion group with at least one nontrivial artinian p-component for some prime $p \in \pi R$, where R is the Hirsch–Plotkin radical of G.*

(c) *A or B has finite torsionfree rank and G' is torsionfree.*

(d) *G/G' has finite sectional rank.*

(e) *G/G' has finite torsionfree rank and G' is torsionfree.*

(f) *G' is noetherian or a torsion group with artinian primary components.*

REMARKS. (a) Since every normal subgroup of $G = AB$ which is contained in A or B is also contained in the Hirsch–Plotkin radical of G, the condition $p \in \pi R$ in (b) of the theorem is also necessary. Using the construction of Holt and Howlett [**7**] it is also possible to construct groups $G = AB$ which are the product of an elementary abelian p-subgroup A and a subgroup B which is the direct product of an elementary abelian p-subgroup and a finite q-subgroup, such that G does not satisfy (*) (here $p \neq q \notin \pi R$).

(b) Examples of Sysak in [**13**] show that (f) cannot be strengthened to G' having finite rank. Is it sufficient in (f) that G' is a minimax group or has finite sectional rank?

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