# GENERALIZATIONS OF MONTEL-LINDELÖF'S THEOREM ON ASYMPTOTIC VALUES

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## Introduction

Montel [10] proved in 1912 the following theorem: Let $w = f(z)$ be an analytic function in the horizontal strip $B: 0 < x < +\infty,\ 0 < y < 1\ (z = x + iy)$ which is continuous on $0 < x < +\infty,\ 0 \leqq y < 1$ and omits at least two values. If $f(x)$ converges to a value $w_0$ as $x \to +\infty$, then $f(z)$ converges to $w_0$ as $z$ tends to $\infty$ in $0 < x < +\infty,\ 0 \leqq y < 1 - \varepsilon$ for any $\varepsilon$ such that $0 < \varepsilon < 1$.

Next the following fact was proved by Lindelöf [9] in 1915: If a function $f(z)$, bounded and analytic in $B$, converges to a value $w_0$ as $z$ tends to $\infty$ along a curve $L$ in $B$, then $f(z)$ converges to $w_0$ as $z$ tends to $\infty$ in any strictly narrower substrip[1]: $0 < x < +\infty,\ \varepsilon < y < 1 - \varepsilon\ (0 < \varepsilon < 1/2)$.

In 1918, Gross [6] generalized this theorem. He called in [5] a meromorphic function $w = f(z)$ defined in $B$ *exceptionally ramified* (ausnahmsverzweigt) if there exist a finite number of points $\{w_k\}$ in the extended $w$-plane and integers $\mu_k \geqq 2$ with $\sum(1 - 1/\mu_k) > 2$, such that, with at most a finite number of exceptions, the roots of the equations $f(z) = w_k$ have multiplicities divisible by $\mu_k$. If the equation $f(z) = w_k$ has only a finite number of roots we may set $\mu_k = \infty$. Thus, for instance, if $f(z)$ excludes at least three values, it is exceptionally ramified. The result obtained by him is as follows: If $f(z)$ is meromorphic and exceptionally ramified in $B$ and converges to $w_0$ as $z$ tends to $\infty$ along a curve $L$ in $B$, then $f(z)$ converges to $w_0$ as $z$ tends to $\infty$ in any strictly narrower substrip.

He did not explicitly include the case in which $L$ coincides with the positive $x$-axis, but it is easily seen that the same conclusion as in the above mentioned Montel's theorem can be obtained. Conversely, once the theorem is established both in the case that $L$ is identical with the positive $x$-axis and in the case that

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1) We say that a substrip is strictly narrower than $B$ if both its sides are in $B$.

$L$ coincides with the upper side of $B$, it can be proved for any $L$ inside $B$. To see this, we suppose that $L$ is a curve in $B$ which starts from a point $iy$ $(0 < y < 1)$ and extends to $z = \infty$, and denote by $D_1$ and $D_2$ the domains between $L$ and the upper and lower sides respectively. We map $D_1$ conformally onto a strip $B_\zeta : 0 < \xi < +\infty,\ 0 < \eta < 1$ so that $L$ and the upper side of $B$ are transformed into the positive $\xi$-axis and the upper side of $B_\zeta$ respectively. Then the inverse image of the line $0 < \xi < +\infty,\ \eta = 1 - \varepsilon$ for any $\varepsilon$ such that $0 < \varepsilon < 1/2$ is included in the domain between the line $y = 1 - \varepsilon$ and the line $y = 1$. This follows from the fact that the bounded harmonic function in $D_1$ equal to 0 on $L$ and to 1 on the upper side and with vanishing normal derivative on the rest of the boundary, is smaller than the harmonic function $y$. Therefore, if the theorem is true in $B_\zeta$, the convergence is concluded in the part of $B$ between $L$ and the line $y = 1 - \varepsilon$. The same reasoning applies to $D_2$ and we see that the convergence holds for $\varepsilon < y < 1 - \varepsilon$.

The theorem is, however, no longer always true for the class of ordinary meromorphic functions of bounded type as an example shows (see [8], p. 44). We might then raise the question as to whether the finiteness of the area of the Riemann surface of an inverse function is enough for the conclusion. This is answered affirmatively if we observe that, in this case, at least three values are taken at most finite times so that Gross's result can be applied.[2)]

In our paper, we shall refine the Montel-Lindelöf's theorem from other several general points of view. In particular, we shall stress the question as to what size of a set on the real axis is needed in order to conclude, from the convergence of a function along the set, the convergence of the function as the variable tends to $\infty$ in any strictly narrower substrip. We shall not treat the problem when a set along which a function tends to a limit is given inside $B$. It seems rather difficult to give it a decisive answer.

Chapter I will be devoted to $(\mathfrak{L})$ parabolic transformations. An $(\mathfrak{L})$ parabolic transformation is such as the number of sheets of the covering surface associated with the inverse transformation is under a certain restriction above neighborhoods of an element $\mathfrak{L}$, which is in a sense small. Such transformations were defined and used in n° 6 of [15]. After mentioning some notions defined in [15] and defining $(\mathfrak{L})$ parabolic transformations of schlicht type, a generalization of

2) I owe this remark to Professor Noshiro and Mr. Oikawa (Tokyo University).

the Montel-Lindelöf's theorem, will be stated in n° 1 for these transformations. Four lemmas on extremal length and one more lemma will be given in n$^{os}$ 2-3. We shall prove Theorem 1 in n° 4 and remark that the result can be applied to the problem of conformal mappings. In n° 5, we shall consider transformations which are not necessarily of schlicht type, and define $(\mathfrak{L})$ parabolic transformations. Theorem 2 gives an example of $(\mathfrak{L})$ parabolic transformation. The condition in it is the same as imposed in n° 7 of [15] to obtain an extension of a theorem of Beurling. We shall prove in n° 6 a generalization of the Montel-Lindelöf's theorem for $(\mathfrak{L})$ parabolic transformations. Throughout this chapter, the condition that a function tends to a limit along the real axis will be relaxed to the condition that the function tends to a limit *along a part of the axis*, whose size is characterized in terms of *logarithmic capacity*. The convergence will be concluded *even on the line* $y=1$ outside a small set.

In Chapter II, we shall deal with the analytic functions taking values on Riemann surfaces whose universal covering surfaces are of hyperbolic type, or, more generally, the analytic functions which are exceptionally ramified in a generalized sense. The precise definition of such functions will be given at the beginning of n° 1. We shall introduce a new element to a Riemann surface with the aid of a superharmonic function on it, and sets on the $x$-axis having positive average linear measure near $x=+\infty$. With these notions, a theorem of the Montel-Lindelöf type will be stated for a Riemann surface with positive boundary, generalizing a special case proved by Kuramochi [7] and by the author [14]. Here, the limit will be the element just introduced, and the set on the real axis, which ensures the convergence in any strictly narrower substrip, will have positive average linear measure near $x=+\infty$. The condition on the linear measure of the set is less restrictive than the condition in Chapter I on the logarithmic capacity of the set. One may compare this situation with the refinements of Fatou's theorem and Riesz's theorem by Beurling under additional conditions. In n° 2, some properties and examples of the above defined element will be given, and, in n° 3, the theorem stated in n° 1 will be proved. Theorem 5 will show that the condition that we obtain in the theorem is in a sense the best possible. In n$^{os}$ 4-5, Riemann surfaces with null boundary will be the object of discussion. In this case, in order to conclude the convergence in any strictly narrower substrip, we must take *the whole axis* as the set along which the function tends to a limit. It will be proved in Theorem 7.

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To extend our results to pseudo-analytic functions we need some properties of quasi-conformal mappings of one strip onto another. A similar discussion was carried out by Yûjôbô [24; 25] for his class of quasi-conformal mappings, but we are interested in the class of functions defined by Pfluger [16] and Ahlfors [1], which is more general than Yûjôbô's. We shall mention this question at the end of our paper; his result follows from ours.

## Chapter I. ($\mathfrak{Q}$) Parabolic Transformations

**1.** First we shall define the *extremal length* of a family of systems of curves and the *dilatation* of a transformation.

Let $\mathfrak{F}^*$ be a connected topological space, and $\mathfrak{F}$ a subset of $\mathfrak{F}^*$, composed of a countable number of Riemann surfaces. A set of at most countably many curves on $\mathfrak{F}^*$ will be called a *system of curves*. We shall say that a system of curves *separates* two given mutually disjoint sets on $\mathfrak{F}^*$ with respect to an open subset of $\mathfrak{F}^*$ if it intersects all curves, if any, which connect the two sets in the open set.[3] A covariant quantity $\rho$, $0 \leqq \rho \leqq +\infty$, defined on $\mathfrak{F}$ will be called *admissible* for a family of systems $\{c\}$ in $\mathfrak{F}$ if $\underline{\int_c} \rho ds \geqq 1$[4] for all $c \in \{c\}$. Given a real-valued function $\pi(P)$, $0 \leqq \pi(P) \leqq +\infty$, on $\mathfrak{F}$, we set $M_\pi\{c\} = \inf_\rho \overline{\iint_{\mathfrak{F}}} \pi\rho^2 d\tau$ for admissible $\rho$, where $d\tau$ is the area of surface element, and set $\lambda_\pi\{c\} = 1/M_\pi\{c\}$; the latter quantity will be called the *extremal length of* $\{c\}$ *with weight* $\pi$. In case $\pi(P) \equiv 1$, $M\{c\}$ and $\lambda\{c\}$ will represent $M_1\{c\}$ and $\lambda_1\{c\}$ respectively and $\lambda\{c\}$ will be called simply *extremal length*. The *extremal distance* of two sets $X_1$ and $X_2$ on a Riemann surface with respect to an open set $G$ is defined by the extremal length of the family of all curves which connect points of $X_1$ with points of $X_2$ in $G$, and we shall denote it by $\mu_G(X_1, X_2)$; if there is no such curve we set the extremal distance equal to $\infty$.

Let $f(P)$ be a homeomorphism of $\mathfrak{F}$ onto another countable set $\mathfrak{F}_1$ of Riemann surfaces, and $P \in \mathfrak{F}$ a point at which $u(f(P(x, y)))$ and $v(f(P(x, y)))$ are totally differentiable, where $z = x + iy$ is a local parameter at $P$ and $w = u + iv$ is a local parameter at $f(P)$; we shall say simply that $f(P)$ is *totally differentiable* at $P$. We set, at this point, $D_\theta w = D_\theta w(z) = \lim_{r \to 0} (w(z + re^{i\theta}) - w(z))/re^{i\theta}$

[3] Curves may terminate at boundary points of the open set. We shall omit this remark hereafter.

[4] For the precise definition of this lower integral, see n$^{os}$ 1-2 of [15].

$(0 \leqq \theta < 2\pi)$ and $J(z) = u_x v_y - u_y v_x$; the *dilatation* $q(P)$ is defined as $\max\limits_{0 \leqq \theta < 2\pi} |D_\theta w|^2/|J(z)|$ at points $P$ where $f(P)$ is totally differentiable and $J(z) \neq 0$. It is set equal to $+\infty$ at other points of $\mathfrak{F}$.[5] The value of the dilatation does not depend on the choice of local parameters.

We shall call $f(P)$ *absolutely continuous on a system of curves*[6] if some arc of the system is not rectifiable, or, otherwise, if, for a local parameter $z$ at any point $P$ on the system and for a local parameter $w$ at $f(P)$, $w(f(P(z)))$ is absolutely continuous on the $z$-image of the system so far as the function is well-defined. The inequality: $\lambda\{c_1\} \leqq \lambda_{1/q}\{c\}$( e) in n° 2 of [15]) is valid for a family of systems $\{c\}$ of curves in $\mathfrak{F}$, on each of which $f(P)$ is absolutely continuous and totally differentiable almost everywhere (a.e.),[7] and for the family of their images $\{c_1\}$ in $\mathfrak{F}_1$. This inequality will be used later.

Now let $\mathfrak{B}$ be a filter with a countable base $\{D_n\}$, composed of open sets in $\mathfrak{F}^*$.[8] We let $\mathfrak{B}$ *define a new element* $\mathfrak{L}$ and introduce a topology into $\mathfrak{F}^* + \{\mathfrak{L}\}$

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5) We can show by an elementary calculation that the dilatation of $f(P)$ and that of its inverse function at corresponding points are equal. We remark also that we can define dilatation similarly even if $f(P)$ is not schlicht.

6) We did not define this notion clearly in [15] but there too the notion should be understood in this sense.

7) We mean by this that, for any arc of $c$ which corresponds to a rectifiable arc $c_z$ in a parameter circle $|z| < 1$, $f(P)$ is totally differentiable at the point $P(z(s))$ for almost every value of $s$, where $z(s)$ is the representation of $c_z$ in terms of the arc-length (see [18], p. 258). This property does not depend on the choice of local parameter.

We shall show that, in case $c$ is a simple rectifiable curve in a plane, this is equivalent to saying that the exceptional set on $c$ has vanishing outer length in the sense of Carathéodory. If we use the representation of $c$ in terms of the arc-length and if we define an interval function and then an outer measure $s^*(A)$ for sets $A$ in $(0, s_0)$ as in n° 1 of [15], where $s_0$ denotes the total length of $c$, then $s^*(A)$ is equal to the outer Lebesgue linear measure. Since it is shown (see p. 155 of [20]) that the $s^*(A)$-value of any set $A$ in $(0, s_0)$ is equal to the outer length of the corresponding set in the plane in the sense of Carathéodory, our assertion follows.

The author made a misstatement at lines 20-24, p. 203 in [15]; it follows from property i) only that $g^{-1}(P)$ is totally differentiable a.e. on $c(u_n)$ for almost every value $u_n$. In order to correct this error, we require the function to be totally differentiable a.e. on a system of curves instead of requiring it to be totally differentiable everywhere on the system, at line 4, p. 193; line 22, p. 194; line 18, p. 196 in [15]. We notice that Lemma 1 and e) in n° 2 still hold and that the subsequent statements remain valid. At line 20, p. 203, we define $C_n'$ to be the subfamily of $C_n$ such that, on each element $c(u_n) \in C_n'$, the restriction of $g^{-1}(P)$ to $\mathfrak{F}$ is totally differentiable a.e. and absolutely continuous. Then, from properties i) and iii), it follows that $c(u_n) \in C_n'$ for almost every $u_n$ such that $0 < u_n < 1$.

8) The intersection $\bigcap\limits_n D_n$ needs not be empty. Therefore, it can happen, for instance, that $\mathfrak{B}$ is the filter of the neighborhoods of an inner point of $\mathfrak{F}^*$.

by taking $\{D_n+\{\mathfrak{Q}\}\}$ as a base of the neighborhoods of $\mathfrak{Q}$ and preserving the original bases of the neighborhoods of the points of $\mathfrak{F}^*$. A continuous transformation $f(\tilde{P})$ of a space $\tilde{\mathfrak{F}}$, composed of a countable number of Riemann surfaces, into $\mathfrak{F}^*$ will be called an $(\mathfrak{Q})$ *parabolic transformation of schlicht type*, if the restriction of $f(\tilde{P})$ to $\{\tilde{P}\in\tilde{\mathfrak{F}}\ ;f(\tilde{P})\in\mathfrak{F}\}$ is schlicht and if we can find for a base of $\mathfrak{V}$ a decreasing sequence $\{D_n\}$ of open sets, with mutually disjoint relative boundaries $\{c_n\}$, such that, for every pair $n$ and $m$ $(n<m)$, there exists a family of systems $\{c^{n,m}\}$ of curves in $\mathfrak{F}$, which separate $c_n$ and $c_m$ with respect to $\mathfrak{F}^*$, with the property that, on every $c^{n,m}$, $f^{-1}(P)$ is totally differentiable a.e. and absolutely continuous and that $\lambda_{1/q}\{c^{n,m}\}\to 0$ as $m\to\infty$ while $n$ is kept fixed, where $q=q(P)$ represents the dilatation of $f^{-1}(P)$.

To formulate the first theorem, we introduce one more notion. A closed set $F$ on the positive $x$-axis is said to *have positive average logarithmic capacity near* $x=+\infty$ if there exist $x_0>0$ and $a>0$ such that the logarithmic capacity of the part $F_a(x)$ of $F$ in the interval $(x-a,\ x+a)$ is greater than a finite constant $d>0$ for all $x>x_0$.

Now we state the following extension of the Montel-Lindelöf's theorem for $(\mathfrak{Q})$ parabolic transformations of schlicht type.

THEOREM 1. *Let $\mathfrak{F}^*$ be a connected topological space, $\mathfrak{F}$ a subset of $\mathfrak{F}^*$, composed of a countable number of Riemann surfaces, and $\mathfrak{Q}$ a new element defined by means of a filter $\mathfrak{V}$ on $\mathfrak{F}^*$. Suppose that there exists a base $\{D'_n\}$ of $\mathfrak{V}$, composed of open sets, such that every relative boundary $c'_n$ of $D'_n$ is nonempty and consists of a countable number of mutually disjoint Jordan closed curves or open arcs in $\mathfrak{F}$[9] of which at most a finite number are compact in $\mathfrak{F}$, and introduce a topology into $\mathfrak{F}^*+\{\mathfrak{Q}\}$ in the customary way. Let $B$ be the strip $0<x<+\infty$, $0<y<1$ in the $z$-plane, and $F$ a closed set on the positive $x$-axis having positive average logarithmic capacity near $x=+\infty$. Let $f(z)$ be an $(\mathfrak{Q})$ parabolic transformation of schlicht type of $B$ into $\mathfrak{F}^*$, which is a continuous mapping of $B+F$ into $\mathfrak{F}^*+\{\mathfrak{Q}\}$. If $f(x)\to\mathfrak{Q}$ as $F\ni x\to+\infty$, then we can find a set $\Omega$, relatively closed in $B$ and approaching the boundary of $B$ as $z\to\infty$, such that $f(z)\to\mathfrak{Q}$ as $z\to\infty$ outside of $\Omega$, with the property that the extremal distance of $F$ and $\Omega$ with respect to any open set $G\subset B$ tends to $+\infty$ as $G$ as a whole recedes to the point at infinity.*

[9] Each end of every open arc terminates at a point in $\mathfrak{F}$ or tends to the boundary of $\mathfrak{F}^*$.

*If, in addition,* $f(z)$ *is continuous on the line* $y=1$, *then* $f(x+i)$ *converges to* $\mathfrak{L}$ *as* $x \to +\infty$ *outside of a closed set whose part in an interval of definite length, whatever this length may be, has a logarithmic capacity tending to zero as the interval recedes to* $\infty$.

**2.** In this section we shall give four lemmas concerning extremal length.

First we mention the following result by Brelot and Choquet ([3], p. 243) which will be used in the proofs of lemmas:

*Let* $\mathfrak{F}$ *be a Riemann surface with positive boundary,* $K$ *a compact set in* $\mathfrak{F}$ *bounded by a finite number of closed analytic curves,* $u(P)$ *the harmonic measure of the boundary of* $\mathfrak{F}$ *with respect to* $\mathfrak{F}-K$, *and* $v(P)$ *its locally defined conjugate. Then almost every* $v$*-level curve starts from a point of* $K$ *and tends to the boundary of* $\mathfrak{F}$, *and* $u(P)$ *increases monotonously from* 0 *to* 1 *on it.*

We shall prove

LEMMA 1. *Let* $R$ *be a rectangle* $0<x<a$, $0<y<\pi$ *in the* $z$*-plane,* $z=x+iy$, *and* $F$ *a closed set on the right side of* $R$ *with positive logarithmic capacity. Let* $u(z)$ *be the bounded harmonic function in* $R$, *equal to* 0 *on the left side* $I$ *and to* 1 *on* $F$ *except for a set of logarithmic capacity zero and with vanishing normal derivative on the rest of the boundary.*[10] *Then for the extremal distance* $\mu_R(I, F)$ *between* $I$ *and* $F$, *there holds*

$$\mu_R(I, F) = \frac{1}{D[u]} = \frac{1}{\int_I dv},^{11)}$$

*where* $D[u]$ *is the Dirichlet integral of* $u(z)$ *in* $R$ *and* $v(z)$ *is the conjugate of* $u(z)$.

*Proof.*[12] We surround $F$ by a finite number of closed analytic curves and denote by $c_1$ their parts inside $R$. We surround $F$ again by a finite number of closed analytic curves which lie inside the curves taken the first time, and denote by $c_2$ their parts inside $R$. In this way we obtain an approximation $\{R_n\}$ of $R$

10) On $R$, $u(z)$ equals the harmonic measure of $F$ with respect to the double of $R$ minus $I$.

11) The variation $\int dv$ is taken in the positive sense always in this paper.

12) We may apply Theorem 2 of Strebel [21] to obtain this relation.

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such that $c_n$ is the part of the boundary of $R_n$ in $R$ and converges to $F$ as $n \to \infty$. Let $u_n(P)$ be the bounded harmonic function in $R_n$, equal to 0 on $I$ and to 1 on $c_n$ and having vanishing normal derivative on the rest of the boundary, and let $v_n(P)$ be its conjugate. We proved in Theorem 4 in [15] that $\mu_R(I, F) = \lim_{n\to\infty} 1/D[u_n]$. Since $D[u_n] \to D[u]$ and $D[u_n] = \int_I dv_n \to \int_I dv$ as $n \to \infty$, our lemma is obtained.

Secondly we give

LEMMA 2. *Let $\mathfrak{F}$ be a Riemann surface, $c_1$ and $c_2$ two nonempty disjoint closed sets which consist of countably many mutually disjoint Jordan closed curves or open arcs in $\mathfrak{F}$, and $\varDelta$ an open set with relative boundary $c_1 + c_2$. We take an exhaustion $\{B_n\}$ of $\mathfrak{F}$ such that, for every $n$, the boundary $\Gamma_n$ of $B_n$ consists of a finite number of closed analytic curves, let $u_n(P)$ be the bounded harmonic function in $\varDelta \cap B_n$, equal to 0 on $c_1 \cap (B_n + \Gamma_n)$ and to 1 on $c_2 \cap (B_n + \Gamma_n)$ except for sets of logarithmic capacity zero and having vanishing normal derivative on $\varDelta \cap \Gamma_n$, and let $v_n(P)$ be its conjugate. Then we have*

$$\mu_{\Delta \cap B_n}(c_1, c_2) = \frac{1}{D_{\Delta \cap B_n}[u_n]} = \frac{1}{\int dv_n},$$

*and this common value tends to $\mu_\Delta(c_1, c_2)$ as $n \to \infty$, where $\int dv_n$ is taken along the $u_n$-level curve for an arbitrary value $u_n$ such that $0 < u_n < 1$.*

*Proof.* The fact seems simple but the proof will be tediously long on account of the general character of $c_1$ and $c_2$.

We form the double $\hat{B}_n$ of $B_n$ along $\Gamma_n$, and denote by $\hat{c}_n$ and $\hat{c}'_n$ the respective doubles of $c_1 \cap (B + \Gamma_n)$ and $c_2 \cap (B_n + \Gamma_n)$ and by $\hat{\varDelta}_n \subset \hat{B}_n$ the double of $\varDelta \cap B_n$ along $\varDelta \cap \Gamma_n$. Then $\hat{\varDelta}_n$ is bounded by $\hat{c}_n + \hat{c}'_n$. We may suppose that $\hat{c}_n \neq \phi$ and $\hat{c}'_n \neq \phi$ for all $n \geqq 1$. The harmonic measure of $\hat{c}'_n$ with respect to $\hat{\varDelta}_n$ is equal to $u_n(P)$ in $\varDelta \cap B_n$ and will be denoted by the same notation $u_n(P)$.

For an arbitrarily fixed value $u_0$, $0 < u_0 < 1$, we take a regular piece of the $u_n$-level curve $c_0 : u_n(P) = u_0$, start from the points of this piece and trace $v_n$-level curves in both directions until we meet multiple points or points on $\hat{c}_n$ or $\hat{c}'_n$. Since $u_n(P)$ varies monotonously on our route, it is not a closed curve. If there is a route which terminates at a point of $\hat{c}_n + \hat{c}'_n$ and along which $u_n(P)$ tends to a positive value less than 1, then this point is an irregular boundary point of the open set $\hat{\Delta}_n$ in Dirichlet problem. It is well known that such irregular points form an $F_\sigma$-set of logarithmic capacity zero in $B_n$, and hence applying Theorem 2 of [14] we see that, on almost all routes, $u_n(P)$ increases monotonously from 0 to 1. Since this is true for any regular piece of the $u_n$-level curve $c_0$, it follows that, on almost all $v_n$-level curves passing $c_0$, $u_n(P)$ increases monotonously from 0 to 1.

Let now $P_0$ be any point of $\hat{\Delta}_n$ around which $u_n(P)$ is not constant. In a similar way we see that the set of $v_n$-level curves on which $u_n(P)$ increases monotonously from 0 to 1 covers a neighborhood of $P_0$ except for a set of $(u_n, v_n)$-measure zero. But each one of such $v_n$-level curves cuts the level curve $c_0$. Thus the set $E_n$ of all points, lying in $\hat{\Delta}_n$ on the $v_n$-level curves along which $u_n(P)$ varies from 0 to 1, covers the part of $\hat{\Delta}_n$ in which $u_n(P)$ is not constant, except for a set of $(u_n, v_n)$-measure zero. Therefore, by Fubini's theorem, we obtain

$$D_{\Delta\cap B_n}[u_n] = \iint_{\Delta\cap E_n} du_n\, dv_n = \int dv_n,$$

the integral $\int dv_n$ being taken along any $u_n$-level curve $c^{(n)}$ : $u_n(P) = \text{const.}\ u_n$, $0 < u_n < 1$ in $\Delta \cap B_n$.

For any admissible $\rho$ for the family of all curves, which connect $c_1$ and $c_2$ in $\Delta \cap B_n$, with respect to $u_n + iv_n$, we have by Schwarz's inequality that $1 \leqq \overline{\int}\rho^2 du_n \cdot \int_0^1 du_n = \overline{\int}\rho^2 du_n$, where the integrals are taken along a $v_n$-level curve in $\Delta \cap B_n$ on which $u_n(P)$ increases from 0 to 1. Since this relation is true for almost all $v_n$-level curves, it follows that $\int_{c^{(n)}} dv_n \leqq \iint \rho^2 du_n\, dv_n$ and hence $\mu_{\Delta\cap B_n}(c_1, c_2) \leqq D_{\Delta\cap B_n}[u_n]^{-1}$. To obtain the inverse inequality, we take an exhaustion $\{\hat{B}_n^p\}$ of $\hat{\Delta}_n$ such that $\hat{B}_n^p$ is bounded by a finite number of closed analytic curves. For large $p$, these curves are separated into two disjoint families and these families approach $\hat{c}_n$ and $\hat{c}_n'$ respectively as $p \to \infty$. Let us define the harmonic measure $u^{(p)}(P)$ of the latter family of curves with respect to $\hat{B}_n^p$. It is immediate to see that $\mu_{\Delta\cap B_n}(c_1, c_2) \geqq D_{\Delta\cap \hat{B}_n^p}[u^{(p)}]^{-1}$. Since $D_{\Delta\cap \hat{B}_n^p}[u^{(p)}] \to D_{\Delta\cap B_n}[u_n]$ as $p \to \infty$, the inequality $\mu_{\Delta\cap B_n}(c_1, c_2) \geqq D_{\Delta\cap B_n}[u_n]^{-1}$ follows. Thus we have the required equality.[12)]

We shall prove that $1/D_{\Delta\cap B_n}[u_0] = \mu_{\Delta\cap B_n}(c_1, c_2) \to \mu_\Delta(c_1, c_2)$ as $n \to \infty$.

The proof will be somewhat similar to that of Theorem 3 of [15].[13] Since $\{u_n(P)\}$ are uniformly bounded we can choose a subsequence $\{u_{n_j}(P)\}$ which is uniformly convergent locally in $\Delta$. We shall show that the limiting function $u_0(P)$ is continuous on $\Delta+c_1+c_2$ and equal to 0 on $c_1$ and to 1 on $c_2$. Let $P_1$ be any point of $c_1 \cap B_n$, and $\alpha$ a Jordan arc of $c_1$ containing $P_1$. We draw a small Jordan domain around $P_1$, disjoint from $c_2$ in $B_n$, such that it is divided by $\alpha$ into two parts and at least one of them contains points of $\Delta$; if $P_1$ is an end point of a component of $c_1$, a domain slit along an arc of $\alpha$ is obtained. We add to $\Delta$ such one part or a domain slit along an arc of $\alpha$ and denote the enlarged open set thus obtained by $\Delta'$. The function $u'_n(P)$, similar to $u_n(P)$ and defined in $\Delta' \cap B_n$, is not less than $u_n(P)$. On account of the reflexion principle we can choose a subsequence of $\{u'_{n_j}(P)\}$ which converges uniformly in a neighborhood of $P_1$. Hence the limiting function, which is not less than $u_0(P)$, vanishes continuously at $P_1$. Thus $u_0(P)$ vanishes continuously at $P_1$. This is true for any point of $c_1$, and hence $u_0(P)$ is continuous and vanishes on $c_1$. In the same way, we can prove that $u_0(P)$ takes the value 1 continuously on $c_2$.

In view of an elementary property of extremal length, we see that $\mu_{\Delta\cap B_n}(c_1, c_2)$ is decreasing as $n\to\infty$ and $\lim_{n\to\infty} 1/D_{\Delta\cap B_n}[u_n] = \lim_{n\to\infty}\mu_{\Delta\cap B_n}(c_1, c_2) \geqq \mu_\Delta(c_1, c_2)$. Next we shall prove that $\lim_{n\to\infty} D_{\Delta\cap B_n}[u_n] \geqq D_\Delta[u_0]$ and that $\mu_\Delta(c_1, c_2) \geqq 1/D_\Delta[u_0]$. In fact, since $u_{n_j}(P)\to u_0(P)$ uniformly locally in $\Delta$, it is obvious that $D_\Delta[u_0] \leqq \varliminf_{j\to\infty} D_{\Delta\cap F_{n_j}}[u_{n_j}] = \lim_{n\to\infty} D_{\Delta\cap F_n}[u_n]$. On the other hand, $\rho\equiv 1$ is admissible for curves which connect $c_1$ and $c_2$ in $\Delta$ with respect to $u_0+iv_0$, and hence there holds $\mu_\Delta(c_1, c_2) \geqq 1/D_\Delta[u_0]$. These three relations together yield $\lim_{n\to\infty}\mu_{\Delta\cap B_n}(c_1, c_2) = \mu_\Delta(c_1, c_2)$, which is the required relation.

Thirdly we shall prove

LEMMA 3. *Let $R$ be a rectangle $0<x<a<a_1$, $0<y<\pi$ in the $z$-plane, $I$ its left side, and $F$ a closed set on the right side with logarithmic capacity greater than $k>0$. Then $\mu_R(I, F)$ has a finite majorant depending only on $a_1$ and $k$.*

*Proof.* We map $R$ by $w=e^z$ onto the upper half of an annulus $A: 1<|w|$

[13] We give here a correction of [15]: The assumption that both $c_1$ and $c_2$ are closed sets and every point of $c_1$ and $c_2$ has a neighborhood such that the part of $c_1$ and $c_2$ in it is a crossing arc of the neighborhood was left out by mistake at line 18, p. 197 and in the statement of Theorem 3 of [15].

$< e^a$. It is easy to see that the image $F_w$ of $F$ on the outer circle has a logarithmic capacity greater than a positive constant depending only on $k$, if we take into consideration the fact that the logarithmic capacity of any closed set is equal to the corresponding transfinite diameter. The same is obviously true of the union $\hat{F}_w$ of $F_w$ and its reflexion on the lower semicircle. It is immediately seen that $\mu_A(|w|=1, \hat{F}_w)$ is equal to the half of $\mu_R(I, F)$. The following proof of the fact that $\mu_A(|w|=1, \hat{F}_w)$ is dominated by a constant depending only on $a_1$ and $k$ is analogous to that of Theorem 1 of [17].

Let $U(w) = \int \log 1/|w-\omega|\, d\mu(\omega)$ be the equilibrium potential of $\hat{F}_w$ in the $w$-plane with equilibrium constant $\kappa$, and take the sum $\hat{U}(w) = U(w) + U(e^{2a}/\bar{w})$. Then on $|w|=1$, $\hat{U}(w) \geqq \log 1/\{e^a(1+e^a)^2\}$. Let us consider the set $\mathfrak{S}$ of all level curves of the conjugate $\hat{V}(w)$ of $\hat{U}(w)$ which are simple curves starting from the origin and on which $\hat{U}(w)$ increases monotonously from $-\infty$ to $2\pi$. Since $\hat{U}(w)$ is symmetric with respect to $|w| = e^a$, these level curves are located inside this circle. The set of the $\hat{V}$-values such that the corresponding $\hat{V}$-level curves belong to $\mathfrak{S}$ has linear measure $2\pi$ on account of the above mentioned result by Brelot-Choquet. For, if we take a value $u_0$ sufficiently large, the level curve $\hat{U}(w) = -u_0$ consists of two simple closed curves $\gamma$ and $\gamma'$ around $w=0$ and $w=\infty$ respectively, and the parts of the $\hat{V}$-level curves in the domain bounded by $\hat{F}_w + \gamma + \gamma'$ are identical with the orthogonal trajectories of the level curves of the harmonic measure of $\hat{F}_w$ with respect to the domain. We shall denote by $\{\gamma_{\hat{\Gamma}}\}$ the parts of the curves of $\mathfrak{S}$ between $\hat{F}_w$ and the $\hat{U}$-level curve: $\hat{U}(w) = \log 1/\{e^a(1+e^a)^2\}$. It is easy to see that $\lambda\{\gamma_{\hat{\Gamma}}\} = [2\kappa + \log\{e^a(1+e^a)^2\}]/2\pi$. This value is smaller than a certain constant $c(a_1, k)$ which depends only on $a_1$ and $k$. On the other hand, we have $\mu_A(|w|=1, \hat{F}_w) \leqq \lambda\{\gamma_{\hat{\Gamma}}\}$ in view of elementary properties of extremal length. Thus $\mu_A(|w|=1, \hat{F}_w) < c(a_1, k)$ and the lemma is proved.

The last lemma in this section is

LEMMA 4. *Let $G$ be a rectangle $-a_0 < x < a_0$, $0 < y < 1$ in the $z$-plane, $F_0$ a closed set in $-a_0/2 < x < a_0/2$ on the lower side with logarithmic capacity $k_0 > 0$, and suppose that, given a positive $\varepsilon < 1$, $\Omega$ is a continuum which contains at least one point of the interval $\varepsilon < y < 1-\varepsilon$ on the imaginary axis and one point on the boundary of $G$. Then the extremal length of any family of systems $\{c\}$ of curves in $G$ separating $F_0$ from $\Omega$ with respect to $G$ is greater* *than a positive finite number which depends only on* $a_0$, $k_0$ *and* $\varepsilon$.

*Proof.* We may assume that $\varepsilon < a_0/2$. Let $S$ be the domain between the sides of two rectangles $R_1$ : $-a_0/2 < x < a_0/2$, $\varepsilon < y < 1-\varepsilon$ and $R_2$ : $-a_0/2 - \varepsilon/2 < x < a_0/2 + \varepsilon/2$, $\varepsilon/2 < y < 1-\varepsilon/2$. Let $u(z)$ be the function bounded and harmonic in the rectangle $R_3$ : $-a_0/2 < x < a_0/2$, $0 < y < \varepsilon$, equal to 1 on $F_0$ except for a set of logarithmic capacity zero and to 0 on the upper side and with vanishing normal derivative on the rest of the boundary. The total variation of its conjugate is equal to the Dirichlet integral $D[u]$ and greater than a positive number $d_0$ depending only on $a_0$ and $k_0$, according to Lemmas 1 and 3. Almost every orthogonal trajectory of the $u$-level curve connects a point of $F_0$ with a point of the upper side of $R_3$ and $u(z)$ decreases montonously from 1 to 0 on it, on account of the above mentioned result by Brelot-Choquet. We set $\rho_1 = (u_x^2 + v_y^2)^{1/2}/D[u]$ in $R_3$ and $=0$ in $G - R_3$ with respect to $z = x + iy$. If a system $c$ of our family $\{c\}$ cuts all these trajectories, then the integral $\int_c \rho_1 ds \geqq 1$. Suppose that there exists a trajectory $\sigma$ which does not meet this system $c$. Let $R(y)$ be the side of the rectangle in $S$, passing the point $(0, y)$, where $\varepsilon/2 < y < \varepsilon$, and keeping the same distance $y - \varepsilon/2$ from the side of $R_2$. We start from the lower end point of $\sigma$, go along $\sigma$, turn to the left at the point intersecting $R(y)$ and proceed along $R(y)$ until we meet $\Omega$. By the hypothesis, $R(y)$ necessarily meets $\Omega$. Since $c$ separates $F_0$ from $\Omega$, it cuts our route. This is true for all $y$ in $(\varepsilon/2, \varepsilon)$, that is, $c$ intersects all $R(y)$, $\varepsilon/2 < y < \varepsilon$. Therefore, the ordinary length of $c$ is $\geqq \varepsilon/2$. So if we set $\rho_2 = 2/\varepsilon$ in $G$ with respect to $z = x + iy$ and set $\rho = \rho_1 + \rho_2$, then $\int_c \rho ds \geqq 1$ for all $c \in \{c\}$. Thus $1/\lambda\{c\} \leqq \iint_G \rho^2 dxdy \leqq 2\iint_G (\rho_1^2 + \rho_2^2)\, dxdy = 2(1/D[u] + 8a_0/\varepsilon^2) \leqq 2(1/d_0 + 8a_0/\varepsilon^2)$ and the lemma is proved.

3. We shall give one more lemma before we prove the theorem.

LEMMA 5. *Under the same conditions as in Theorem* 1, *for every* $n$, $\Omega_n = \{z \in B\ ;\ f(z) \in \mathfrak{F}^* - D_n'\}$ *approaches the boundary of* $B$ *as* $z \to \infty$ *and the extremal distance of* $F$ *and* $\Omega_n$ *with respect to any open set* $G \subset B$ *tends to infinity as* $G$ *as a whole recedes to the point at infinity.*

*Proof.* Let $\{D_n\}$ be a base which satisfies the conditions required in the definition of $(\mathfrak{Q})$ parabolic transformation of schlicht type. For every $n$, there exists $\nu(n)$ such that $D_{\nu(n)} \subset D'_n$. By assumption, there is a family of systems $\{c^{\nu(n), m}\}$ of curves in $\mathfrak{F}$, which separate $c_{\nu(n)}$ and $c_m$ $(m > \nu(n))$, with respect to $\mathfrak{F}^*$, such that $\lambda_{1/q}\{c^{\nu(n), m}\} \to 0$ as $m \to \infty$. For every $m > \nu(n)$ we can find $\pi(m)$ which satisfies $D'_{\pi(m)} \subset D_m$. Each system $c^{\nu(n), m}$ separates $c'_n$ and every $c'_k$ with $k \geqq \pi(m)$, with respect to $\mathfrak{F}^*$. Therefore, for every $m > n$, there eixsts a family of systems of curves in $\mathfrak{F}$ which separate $c'_n$ and $c'_m$ with respect to $\mathfrak{F}^*$, on each of which $f^{-1}(P)$ is totally differentiable a.e. and absolutely continuous, and whose extremal length with weight $1/q$ tends to zero as $m \to \infty$. We may now assume, without loss of generality, that $\{c_n\}$ and $\{c'_n\}$ are identical.

Suppose that there exists a sequence of points $\{z_p\}$, $z_p = x_p + iy_p$, in $\Omega_n$ such that $\varepsilon < y_p < 1 - \varepsilon$ for a certain $\varepsilon > 0$ and $x_p \to +\infty$ as $p \to \infty$. Let $G_p$ be the rectangle $x_p - 2a < x < x_p + 2a$, $0 < y < 1$, where $a > 0$ is a number for which the logarithmic capacity of $F_a(x)$ is greater than $d > 0$ for all $x > x_0 > 0$. Given $m > n$, if we take $p$ sufficiently large, then $x_p > x_0$ and the image of $F_{2a}(x_p)$ lies in $D_m + \{\mathfrak{Q}\}$. We may suppose that the boundary of $\Omega_n$ has no compact component in $G_p$, because $c_n$ has at most a finite number of compact components and $f(z)$ is schlicht on $\{z \in B\ ;\ f(z) \in \mathfrak{F}\}$. Denote by $c_p^{n, m}$ the inverse image in $G_p$ of $c^{n, m}$. Each $c_p^{n, m}$ intersects all curves in $G_p$ which connect $F$ and the component of $\Omega_n$ that contains $z_p$. Hence, by Lemma 4, $\lambda\{c_p^{n, m}\} \geqq \lambda_0 > 0$ where $\lambda_0$ is a constant depending only upon $a$, $d$ and $\varepsilon$. On the other hand, if $c_B^{n, m}$ denotes the inverse image of $c^{n, m}$ in $B$, there holds $\lambda\{c_B^{n, m}\} \leqq \lambda_{1/q}\{c^{n, m}\}$ as pointed out in n° 1 ( $e$) of [15]). These two inequlities are, however, not compatible, because $\lambda\{c_B^{n, m}\} \geqq \lambda\{c_p^{n, m}\}$ and $\lambda_{1/q}\{c^{n, m}\} \to 0$ as $m \to \infty$. Thus the first part is proved.

Next let $G \subset B$ be an open set which is not disjoint from $\Omega_n$. Let $B_1$ be the smallest strip, containing $G$, of the form $x_1 < x < +\infty$, $0 < y < 1$. Since $\mu_{B_1}(F, \Omega_n) \leqq \mu_G(F, \Omega_n)$, it is enough to show that $\mu_{B_1}(F, \Omega_n) \to +\infty$ as $x_1 \to +\infty$. We take $x_1$ so large that the part of $F$ on the boundary of $B_1$ is mapped into $D_m + \{\mathfrak{Q}\}$ $(m > n)$. We denote by $\delta_m$ the inverse image in $B_1$ of $D_m + c_m$. It is obvious that $\mu_{B_1}(F, \Omega_n) \geqq \mu_{B_1}(\delta_m, \Omega_n)$. We approximate $B_1$ by an increasing sequence of rectangles $\{R_p\}$ with boundaries $\{\Gamma_p\}$ such that the closure of $R_p$ is included in $R_{p+1}$. Let $u_p(z)$ be the harmonic function in $R_p - \Omega_n - \delta_m$, equal to the harmonic measure of the double of $\delta_m \cap (R_p + \Gamma_p)$ with respect to the double of $R_p - \Omega_n - \delta_m$, which is a part of the double of $R_p$ formed along $\Gamma_p$. If $v_p(z)$ denotes the conjugate of $u_p(z)$, then almost all $v_p$-level curves connect

$\Omega_n$ to $\delta_m$, as we have seen in the proof of Lemma 2. Therefore, the inverse image $c_{B_1}^{n,m}$ in $B_1$ of every $c^{n,m}$ intersects all these $v_p$-level curves. Consequently, if we set $\rho = 1/D[u_p] = 1/\int_{u_p=\text{const.}} dv_p$ with respect to $u_p + iv_p$ in $R_p - \Omega_n - \delta_m$ and to 0 in $B_1 - R_p$, then $\rho$ is admissible for $\{c_{B_1}^{n,m}\}$, and hence $\lambda\{c_{B_1}^{n,m}\} \geqq D[u_p]$. Then we apply Lemma 2 and obtain $\lim 1/D[u_p] = \mu_{B_1 - \Omega_n - \delta_m}(\delta_m, \Omega_n) = \mu_{B_1}(\delta_m, \Omega_n)$. If we combine these relations with the already obtained inequality $\mu_{B_1}(\delta_m, \Omega_n) \leqq \mu_{B_1}(F, \Omega_n)$, then it follows that $1/\lambda\{c_{B_1}^{n,m}\} \leqq \mu_{B_1}(F, \Omega_n)$. Since $\lambda\{c_{B_1}^{n,m}\} \leqq \lambda_{1/q}\{c^{n,m}\}$ and $\lambda_{1/q}\{c^{n,m}\} \to 0$ as $m \to \infty$, we see that $\mu_{B_1}(F, \Omega_n) \to +\infty$ as $x_1 \to +\infty$.

4. *Proof of Theorem* 1. We take $0 < x_1 < x_2 < \ldots$ so that, $B_n$ being the strip $x_n < x < +\infty$, $0 < y < 1$, the $y$-coordinate of the points of $B_n \cap \Omega_n$ satisfies $0 < y < 1/n$ or $1 - 1/n < y < 1$ and so that $\mu_{B_n}(F, \Omega_n) > 2^n$, and we determine $0 < x_1' < x_2' < \ldots$ so that $x_n' - x_n > 2^n$. We set $\Omega_n \cap \{(x, y)\ ;\ x_n' \leqq x \leqq x_{n+1}',\ 0 < y < 1\} = \Omega_n'$ and $\bigcup_{n=1}^{\infty} \Omega_n' = \Omega$. Then $\Omega$ approaches the boundary of $B$ as $z \to \infty$ and $f(z) \to \mathfrak{Q}$ as $\Omega \ni z \to \infty$.

In order to prove that $\mu_G(F, \Omega) \to +\infty$ as $G \to \infty$, it is sufficient to show that $\mu_{B_n'}(F, \Omega)$ tends to $+\infty$ as $n \to \infty$, where $B_n'$ is the strip: $x_n' < x < +\infty$, $0 < y < 1$. For each $k \geqq n$, we divide the family $\{\gamma_k\}$ of all curves in $B_n'$, which connect points of $\Omega_k'$ and points of $F \cap [x_n', +\infty)$, into two subfamilies: one part $\{\gamma_k'\}$ consists of the curves situated entirely in $B_k$ except for their end points and the other part $\{\gamma_k''\}$ consists of the rest of the curves. Then $\lambda\{\gamma_k'\} \geqq \mu_{B_k}(F, \Omega_k) > 2^k$ and $\lambda\{\gamma_k''\} \geqq x_k' - x_k > 2^k$. We shall use the following general property of extremal length: $M(\bigcup_{n=1}^{\infty}\{c_n\}) \leqq \sum_{n=1}^{\infty} M\{c_n\}$; this follows from the relation $\overline{\iint} \rho^2 d\tau \leqq \sum_{n=1}^{\infty} \overline{\iint} \rho_n^2 d\tau$, where $\rho_n$ is admissible for $\{c_n\}$ and $\rho = \sup \rho_n$ at every point, because then $\rho$ is admissible for $\bigcup_{n=1}^{\infty}\{c_n\}$. Thus we have $1/\mu_{B_n'}(F, \Omega) \leqq \sum_{k=n}^{\infty} 1/\lambda\{\gamma_k\} \leqq \sum_{k=n}^{\infty} 1/\lambda\{\gamma_k'\} + \sum_{k=n}^{\infty} 1/\lambda\{\gamma_k''\} \leqq 2\sum_{k=n}^{\infty} 1/2^k = 1/2^{n-2}$. This relation shows that $\mu_{B_n'}(F, \Omega) \to +\infty$ as $n \to \infty$.

To prove the last relation under the assumption that $f(z)$ is continuous in $0 < x < +\infty$, $0 < y \leqq 1$, first we shall show that the extremal distance of the closed set $\delta_n = \{x + i\ ;\ f(x+i) \in \mathfrak{F}^* - D_n\}$ and the line $y = 1/2$ with respect to any rectangle $R$, which has two sides on the lines $y = 1/2$ and $y = 1$, tends to $+\infty$ as $R \to \infty$. Given $m > n$, we take $R$ sufficiently near to $z = \infty$ that the

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image of the lower side of $R$ is contained in $D_m$. Let $u(z)$ be the bounded harmonic function in $R$ which is equal to 0 on the lower side and to 1 on $\delta_n$ except for a set of logarithmic capacity zero and whose normal derivative vanishes on the rest of the boundary. As we have shown several times, the extremal length of the inverse images $\{c_R^{n,m}\}$ in $R$ of $\{c^{n,m}\}$ is not less than $D_R[u]$ which is equal to $1/\mu_R(\delta_n, y=1/2)$ by Lemma 1. Since $\lambda\{c_R^{n,m}\}$ $\leqq \lambda_{1/q}\{c^{n,m}\} \to 0$ as $m \to \infty$, it is concluded that $\mu_R(\delta_n, y=1/2)$ tends to $+\infty$ as $R \to \infty$. Then the same reasoning as above shows that we can find a closed set $\delta$ on the line $y=1$ outside of which $f(z) \to \mathfrak{L}$ and which has $\mu_R(\delta, y=1/2)$ tending to $+\infty$ as $G \to \infty$. Hence by Lemma 3 the logarithmic capacity of the part of $\delta$ in $[x, x+a]$ tends to 0 as $x \to +\infty$ for any $a>0$. Thus the proof is completed.

*Remark* 1. If $f(z)$ is continuous on the $x$-axis, it is concluded that $f(x) \to \mathfrak{L}$ as $x \to +\infty$ outside of a set whose part in $[x, x+a]$ has a logarithmic capacity tending to 0 as $x \to +\infty$ for any finite $a>0$, just for the same reason as on the line $y=1$. Thus in this case, the convergence of $f(x)$ to $\mathfrak{L}$ along comparatively small set, which may be of linear measure zero, ensures the convergence of $f(x)$ to $\mathfrak{L}$ as $x \to +\infty$ along a fairly large set.

*Remark* 2. Let $D$ be the unit square $0<\xi<1$, $0<\eta<1$ $(\zeta=\xi+i\eta)$, slit along $s_n : \xi=1/n$, $0<\eta<1-1/n$ $(n=2, 3, \ldots)$. We map $D$ conformally in a one-to-one manner onto $B$ such that $z=\infty$ corresponds to the point $\zeta=i$ and that the upper side of $D$ is transformed to the positive real axis. We may take the $\zeta$-plane for $\mathfrak{F}^*=\mathfrak{F}$, concentric circular domains converging to $\zeta=i$ for $\{D_n\}$ and the whole positive $x$-axis for $F$. Then we can apply Theorem 1 and see that the function $\zeta=f(z)$ mapping $B$ onto $D$ tends to the value $\zeta=i$ as $z\to\infty$ on the line $y=1$ outside a certain small set. This shows that the image of the parts of the slits $\{s_n\}$ outside any neighborhood of $\zeta=i$ is quite small near $z=\infty$ on the line $y=1$.

Let us consider another example. Let $D^*$ be the unit square $0<\xi<1$, $0<\eta<1$, slit along $s_n : \xi=1/n$, $0<\eta<1-1/n$ and $s_n' : \xi=\{1/n+1/(n+1)\}/2$, $1/n<\eta<1$ $(n=2, 3, \ldots)$. This is a simply-connected domain. The left side is a boundary element in the sense of Carathéodory and no point on it is accessible. We map $D^*$ conformally in a one-to-one manner onto $B$ such that $z=\infty$ corresponds to the left side. Applying Theorem 1, we see that the image of the upper side of $D^*$ has not positive average logarithmic capacity near $x = +\infty$. This shows that the image of the slits $\{s'_n\}$ is not so small near $x = +\infty$ on the real axis.

*Remark* 3. Even if the domain has a more complicated form than a strip, or even if the set $F$ on the $x$-axis along which $f(x)$ does not satisfy the condition required to its size in the theorem, the reasoning in the above proof allows, in some cases, to conclude the convergence of $f(z)$ to $\mathfrak{L}$ as $z \to \infty$ along a certain part of the domain near $F$.

*Remark* 4. The reasoning may be utilized also in the case that a set along which $f(z)$ tends to a limit lies inside $B$. For instance, let $F$ be a closed set on the line $y = 1/2$ which has positive average logarithmic capacity near $z = \infty$. Under the same condition as in the theorem, if $f(z) \to \mathfrak{L}$ along $F$ then $f(z) \to \mathfrak{L}$ as $z \to \infty$ in any strictly narrower substrip of $B$ and hence along the line $y = 1/2$ with no exception.

**5.** We shall consider, in the rest of this chapter, continuous transformations which have not necessarily schlicht character. An $(\mathfrak{L})$ parabolic transformation of a space $\tilde{\mathfrak{F}}$, composed of a countable number of Riemann surfaces, into a Riemann surface $\underline{\mathfrak{R}}$ is defined as follows, as in n° 6 of [15]:

Let $f(\tilde{P})$ be a continuous transformation of $\tilde{\mathfrak{F}}$ into $\underline{\mathfrak{R}}$ which is locally pseudo-analytic in the sense of Pfluger-Ahlfors outside a closed set $\tilde{E} \subset \tilde{\mathfrak{F}}$ with image $\underline{E}$ in $\underline{\mathfrak{R}}$ of linear measure zero. Let $\underline{\mathfrak{L}}$ be an element which is defined by means of a filter with a countable base which consists of open sets in $\underline{\mathfrak{R}}$. We suppose that we can find a decreasing sequence $\{\underline{D}_n\}$ of open sets, which form a base of the filter, in such a manner that each relative boundary $\underline{c}_n$ is composed of a countable number of mutually disjoint Jordan closed curves or open arcs,[9] that $\{\underline{c}_n\}$ are disjoint from each other and from $\underline{E}$ and that, for every pair $n$ and $m$ $(n < m)$ there exists a harmonic function $\underline{u}_{n,m}(\underline{P})$ in $D_n - D_m - \underline{c}_m$, with $\overline{\lim\limits_{\underline{c}_n}}\, \underline{u}_{n,m}(\underline{P}) \leqq 0$ and $\underline{\lim\limits_{\underline{c}_m}}\, \underline{u}_{n,m}(\underline{P}) \geqq 1$ for which

$$\int_0^1 \frac{d\underline{u}_{n,m}}{\int d\underline{v}_{n,m}} \to \infty \tag{1}$$

as $m \to \infty$ while $n$ is kept fixed, where $q(\tilde{P})$ denotes the dilatation of $f(\tilde{P})$,[14]

14) See footnote 6).

$\underline{v}_{n,m}(\underline{P})$ is the conjugate of $\underline{u}_{n,m}(\underline{P})$ and $\int q d\underline{v}_{n,m}$ means the integral $\int q(\tilde{P}) d\underline{v}_{n,m}(f(\tilde{P}))$ taken along the inverse image of the level curve $\underline{u}_{n,m}(\underline{P}) = \text{const.}\ \underline{u}_{n,m},\ 0 < \underline{u}_{n,m} < 1$ which has no point in common with $\underline{E}$. Then $f(\tilde{P})$ will be called an $(\mathfrak{L})$ *parabolic transformation* of $\tilde{\mathfrak{F}}$ into $\mathfrak{R}$.

To give an example of such transformation, we consider the special case where $\mathfrak{L}$ may be identified with an inner point $\underline{P}$ of $\mathfrak{R}$. Let $\omega = te^{i\varphi}$ be a local parameter such that $\omega = 0$ corresponds to $\underline{P}$. We set $\underline{D}_n$ equal to the image on $\mathfrak{R}$ of $|\omega| < 1/n$ and $\underline{c}_n$ to that of $|\omega| = 1/n$. The following fact was proved in n° 7 of [15].

THEOREM 2. *Let $f(\tilde{P})$ be a continuous transformation of a space $\tilde{\mathfrak{F}}$, composed of a countable set of Riemann surfaces, into a Riemann surface $\mathfrak{R}$, which is locally pseudo-analytic in the sense of Pfluger-Ahlfors outside a closed set $\tilde{E}$ with image in $\mathfrak{R}$ of linear measure zero. Let the filter of the neighborhoods of an inner point $\underline{P}$ of $\mathfrak{R}$ define an element $\mathfrak{L}$. Let $\omega = te^{i\varphi}$ be a local parameter such that $\omega = 0$ corresponds to $\underline{P}$, denote by $\mathfrak{F}_\rho$ the part, lying over $|\omega| < \rho$, of the covering Riemann surface which is homeomorphic to $B - E$, and denote by $S(\rho)$ the area of $\mathfrak{F}_\rho$ : $\int_0^\rho \int q t d\varphi dt$, measured with density equal to the dilatation $q$. If we can find*

$$1 > \rho_1 > \rho_1' \geqq \rho_2 > \rho_2' \geqq \ldots \to 0$$

*such that*

$$\sum_{\nu=\mu}^{\infty} \frac{(\rho_\nu - \rho_\nu')^2}{S(\rho_\nu) - S(\rho_\nu')} = +\infty$$

*for every integer $\mu > 0$,*[15] *then $f(P)$ is an $(\mathfrak{L})$ parabolic transformation.*

**6.** We shall establish a theorem of the Montel-Lindelöf type for $(\mathfrak{L})$ parabolic transformations.

THEOREM 3. *Let $\mathfrak{R}$ be a Riemann surface, and $\mathfrak{L}$ an element defined by means of a filter $\mathfrak{V}$ on $\mathfrak{R}$. Let $B$ be the strip $0 < x < \infty$, $0 < y < 1$, and $F$ a closed set on the positive x-axis which has positive average logarithmic capacity near $x = +\infty$. Let $f(z)$ be an $(\mathfrak{L})$ parabolic transformation of $B$ into $\mathfrak{R}$, which is a continuous transformation of $B + F$ into $\mathfrak{R} + \{\mathfrak{L}\}$, and suppose that we can*

[15] The corresponding statement in Theorem 6 of [15] should be corrected in this way.

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*find a base* $\{\underline{D}'_n\}$ *of* $\mathfrak{B}$, *composed of open sets, with relative boundaries* $\{\underline{c}'_n\}$ *which are disjoint from each other and from* $\underline{E}$ *and each of which is composed of a countable number of mutually disjoint Jordan closed curves or open arcs*[9] *and has only a finite number of compact components. If* $f(x) \to \mathfrak{Q}$ *as* $F \ni x \to +\infty$, *then we have the same conclusions as in Theorem* 1.

*Proof.* We form a kind of covering surface $\mathfrak{F}^*$ over $\mathfrak{R}$, as in n° 6 of [15], in such a way that a subspace $\mathfrak{F}$ of $\mathfrak{F}^*$, composed of a countable number of Riemann surfaces, on one hand corresponds to $B-E$ in a one-to-one manner and, on the other hand, is ordinary Riemann covering surfaces of $\mathfrak{R}-\underline{E}$. Let $\{\underline{D}_n\}$ be the open sets taken in the definition of ($\mathfrak{Q}$) parabolic transformation. Let $D_n$ be the part of $\mathfrak{F}^*$ which is projected into $\underline{D}_n$, $c_n$ be the relative boundary of $\underline{D}_n$, and the filter having $\{D_n\}$ as its base define an element $\mathfrak{Q}$. As is shown in [15], condition (1) implies that, for each $n$, $\lambda_{1/q}\{c^{n,m}\} \to 0$ as $m \to \infty$ while $n$ is kept fixed, where $c^{n,m}$ is a system of curves in $\mathfrak{F}^*$ projected into a level curve $\underline{u}_{n,m}(\underline{P}) = \underline{u}_{n,m}$, $0 < \underline{u}_{n,m} < 1$, disjoint from $\underline{E}$.

Contrary to the conclusion, we assume that there exists $n_0$ such that the inverse image of $\mathfrak{R}-\underline{D}'_{n_0}$ contains a sequence of points $\{z_p\}$ tending to $\infty$ in a strictly narrower substrip of $B$. We can find $\nu(n_0)$ such that $\underline{D}_{\nu(n_0)} \subset \underline{D}'_{n_0}$. If we take $m > \nu(n_0)$ sufficiently large, then the integral in (1) with $\nu(n_0)$ and $m$ is positive. Let $m_0$ be any number such that $\underline{D}'_{m_0} \subset \underline{D}_m$. If it is shown that there are at most a finite number of compact components of the inverse images of $\underline{c}'_{m_0}$, containing at least one of $\{z_p\}$ in each inside, then we can apply the reasoning in Lemma 5 to $\mathfrak{F}^*$ and $\mathfrak{Q}$ and a contradiction will be led. Other conclusions can be obtained in the same way as in Theorem 1.

We suppose that there exists a closed curve $c$ in $B$ whose image is contained in $\underline{c}'_{m_0}$ and which contains $z_p$ in its inside. We connect $z_p$ with a point of $c$ by a curve $l$ inside $c$, and consider its image $f(l)$ in $\mathfrak{R}$. There is a part $\underline{L}$ of $f(l)$ which lies in a component $\underline{D}'_0$ of $\underline{D}'_{n_0} - \underline{D}'_{m_0} - \underline{c}'_{m_0}$ and connects a point $\underline{P}'_{n_0}$ of $\underline{c}'_{n_0}$ with a point of $\underline{c}'_{m_0}$. Denote by $z'$ the inverse image of $\underline{P}'_{n_0}$ on $l$. The connected component, passing $z'$, of the inverse image of $\underline{c}'_{n_0}$ is a complete image and compact inside $c$, and hence the component of $\underline{c}'_{n_0}$ which contains $\underline{P}'_{n_0}$ is compact. Since we can connect any point of $\underline{D}'_0 - \underline{E}$ and a point of $\underline{L}$ by a curve in $\underline{D}'_0$ which does not meet $\underline{E}$, the part in $\underline{D}'_0$ of almost every level curve of $\underline{u}_{\nu(n_0),m}(\underline{P})$ has a complete inverse image inside $c$.

Suppose now that there are an infinite number of closed curves $\{c^i\}$ in $B$ each of which contains at least one of $\{z_p\}$ in its inside and whose images are contained in $\underline{c}'_{m_0}$. For each $c^i$ there exists at least one component like $\underline{D}'_0$ of $\underline{D}'_{n_0} - \underline{D}'_{m_0} - \underline{c}'_{m_0}$. Since $\underline{c}'_{n_0}$ has only a finite number of closed components, we can find a component $\underline{D}^*_0$ of $\underline{D}_{n_0} - \underline{D}_{m_0} - \underline{c}_{m_0}$ like $\underline{D}'_0$ and an infinite subsequence $\{c^{i_j}\}$ such that the part in $\underline{D}^*_0$ of almost every level curve of $\underline{u}_{\nu(n_0), m}(\underline{P})$ has a complete inverse image inside each $c^{i_j}$. Then the integral $\int q d\underline{v}_{\nu(n_0), m} \geqq \int d\underline{v}_{\nu(n_0), m}$ $= +\infty$ along almost every $\underline{u}_{\nu(n_0), m}$-level curve, and hence the integral in (1) is zero. This contradicts our assumption and the theorem is proved.

The same remarks as Remarks 1, 3 and 4 in n° 4 may be given to Theorem 3.

Let $w = f(z)$ be an ordinary meromorphic function in $B$ which is continuous at a closed set $F$ on the $x$-axis having positive average logarithmic capacity near $x = +\infty$ and which tends to a value along $F$. If the covering Riemann surface of the inverse function of $f(z)$ satisfies the condition on $S(\rho)$ required in Theorem 2, then the conclusions in Theorem 3 are valid for $f(z)$. However, it is an open question whether the finiteness of the Dirichlet integral of $f(z)$, instead of the condition on $S(\rho)$, is sufficient to have the same conclusions or not.

## Chapter II. Exceptionally Ramified Transformations

**1.** The condition for a transformation to be $(\mathfrak{L})$ parabolic has a character that restricts the number of sheets of the covering surface associated with the inverse transformation. In Chapter II, we shall deal with transformations with the property that the universal covering surfaces of their ranges of values are of hyperbolic type, or, more generally, with exceptionally ramified transformations; bounded analytic functions are examples. We shall be concerned only with analytic transformations in the sequel except at the end.

First we shall give the definition for analytic transformations to be *exceptionally ramified in the generalized sense.*[16] Let $f(z)$ be an analytic transformation of a plane domain into a Riemann surface $\underline{\mathfrak{R}}$. When $\underline{\mathfrak{R}}$ is planar, we may suppose that $f(z)$ is a meromorphic function assuming values in the extended $w$-plane. We shall then call $f(z)$ exceptionally ramified, with Gross [5], if $f(z)$ satisfies the condition stated in the introduction. When $\underline{\mathfrak{R}}$ is of genus

[16] It may be more adequate to define exceptionally ramified covering surfaces instead of defining exceptionally ramified functions. But here we follow the Gross's definition in [5].

1, we regard $f(z)$ as a transformation into a torus. If there exists at least one point $\underline{P}_0$ of the torus such that every point, situated above $\underline{P}_0$, of the Riemann surface of the inverse function of $f(z)$ is a branch point of multiplicity divisible by an integer $\mu_0 \geqq 2$ possibly with a finite number of exceptions, then we shall call $f(z)$ exceptionally ramified. If at most finitely many points cover $\underline{P}_0$, we set $\mu_0 = +\infty$. When the genus of $\underline{\mathfrak{R}}$ is greater than one, $f(z)$ will be called so unconditionally. It is to be remarked that, if $\underline{\mathfrak{R}}$ has a positive boundary, $f(z)$ is always exceptionally ramified.

We now map $B$ into a Riemann surface $\underline{\mathfrak{R}}$ by an exceptionally ramified analytic transformation $f(z)$. If $f(z)$ is continuous on $B^* : 0 < x < +\infty$, $0 \leqq y < 1$ and $f(x)$ tends to an inner point $\underline{P}$ of $\underline{\mathfrak{R}}$ as $x \to +\infty$, then it is easily seen that $f(z)$ tends to $\underline{P}$ as $z \to \infty$ in any narrower strip $0 < x < +\infty$, $0 \leqq y < 1 - \varepsilon$. The difficulty lies in the case that $f(x)$ tends to the boundary of $\underline{\mathfrak{R}}$ as $x \to +\infty$. We proved in Lemma 4 of [14] an extension of the Montel-Lindelöf's theorem in the case when $f(z)$ tends to a boundary component $\underline{P}_C$ of harmonic measure zero and when there exists a closed curve $\underline{\gamma}$ surrounding $\underline{P}_C$ such that the part of the boundary of $\underline{\mathfrak{R}}$ which is separated by $\underline{\gamma}$ from $\underline{P}_C$ is of positive harmonic measure.[17] A proof of a special case, which essentially covers the full case, was given already in [7], using the idea in pp. 65–66 of [12], and the proof was simpler than that of [14].

To extend these results, we shall introduce notions corresponding to an element $\mathfrak{L}$ (or $\underline{\mathfrak{L}}$) and a set of positive average logarithmic capacity near $x = +\infty$, which were frequently used in Chapter I.

Let $\underline{\mathfrak{B}}$ be a filter on $\underline{\mathfrak{R}}$ with a countable base which consists of open sets. We associate a new element $\underline{\mathfrak{L}}$ with $\underline{\mathfrak{B}}$, and introduce a topology into $\underline{\mathfrak{R}} + \{\underline{\mathfrak{L}}\}$ in the usual way (cf. n° 1 of Chapter I). The intersection of the sets of $\underline{\mathfrak{B}}$ will be called the *trace* of $\underline{\mathfrak{L}}$ on $\underline{\mathfrak{R}}$ and that of the closures, taken relatively to $\underline{\mathfrak{R}}$, of the sets of $\underline{\mathfrak{B}}$ the *closed trace* of $\underline{\mathfrak{L}}$ on $\underline{\mathfrak{R}}$. They will be denoted by $\tau(\underline{\mathfrak{L}})$ and $\bar{\tau}(\underline{\mathfrak{L}})$ respectively. These may be empty. Suppose that there exists a function

[17] In the statement of Lemma 4 of [14], it is required that there exists a set of positive logarithmic capacity which $f(z)$ does not assume near $z = \infty$ or that there exists a closed curve $\underline{\gamma}$ of the character just stated. But in the first case, we exclude from $\underline{\mathfrak{R}}$ a closed set of positive logarithmic capacity, not assumed by $f(z)$ near $z = \infty$, in a domain corresponding to a parameter circle, and thus the first case reduces to the second case, because the image of the circumference of the parameter circle may be considered to be a simple closed curve in $\underline{\mathfrak{R}}$ and taken for $\underline{\gamma}$ in the second case.

$v(\underline{P})$ on $\mathfrak{R}$ which satisfies:

i) $v(\underline{P})$ is superharmonic possibly except at a certain point $\underline{P}_0 \notin \bar{\tau}(\mathfrak{L})$,

ii) $v(\underline{P})$ is bounded from below everywhere or outside every neighborhood of $\underline{P}_0$ if this is exceptional,

iii) $v(\underline{P}) \to +\infty$ when and only when $\underline{P} \to \mathfrak{L}$.

Then we shall say that $\mathfrak{L}$ is *complete and of harmonic measure zero*, and call $v(\underline{P})$ a *function associated with* $\mathfrak{L}$. It is easy to see that $\tau(\mathfrak{L}) = \{\underline{P} \in \mathfrak{R}\ ;\ v(\underline{P}) = +\infty\}$. We shall give further properties and examples of such $\mathfrak{L}$ in the next section.

We shall say that a closed set $F$ on the positive $x$-axis *has positive average linear measure near* $x = +\infty$ if there exist finite numbers $x_0 > 0$ and $a > 0$ such that the part $F_a(x)$ of $F$ in the interval $(x-a, x+a)$ has linear measure greater than a certain positive number for all $x > x_0$.

We shall give theorems, distinguishing two cases; the case where Riemann surfaces have a positive boundary and the case where they have a null boundary. The reason why these two cases are distinguished will be explained by Theorem 7. In the first place, we shall be concerned with the first case.

THEOREM 4. *Let $\mathfrak{L}$ be a complete element of harmonic measure zero added to a Riemann surface $\mathfrak{R}$ with positive boundary, and $F$ a closed set having positive average linear measure near $x = +\infty$ on the $x$-axis. Let $f(z)$ be a continuous transformation of $B + F$ into $\mathfrak{R} + \{\mathfrak{L}\}$ which is analytic in $B$. If $f(x)$ tends to $\mathfrak{L}$ as $F \ni x \to +\infty$, then $f(z)$ tends to $\mathfrak{L}$ as $z \to \infty$ in any strictly narrower substrip.*

**2.** In this section we shall discuss on complete elements of harmonic measure zero in more details and give examples.

The following lemma will be used in the proof of Theorem 4.

LEMMA 6. *Let $\mathfrak{L}$ be an element which is complete and of harmonic measure zero. Then, for any point $\underline{P}_0 \notin \bar{\tau}(\mathfrak{L})$, we can find an associated function superharmonic outside $\underline{P}_0$. In case $\mathfrak{R}$ has a positive boundary, there exists a positive associated function superharmonic everywhere on $\mathfrak{R}$.*

*Proof.* First we consider the case in which $\mathfrak{R}$ has a null boundary. Let $v(\underline{P})$ be an associated function superharmonic outside a point $\underline{P}_0^* \notin \bar{\tau}(\mathfrak{L})$. We take a domain outside $\bar{\tau}(\mathfrak{L})$ corresponding to a parameter circle $|\omega| < 2$ such

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that $\omega=0$ corresponds to $\underline{P}_0^*$ and that $v(\underline{P})$ is positive outside the domain corresponding to $|\omega|<1$. We replace $v(\underline{P})$ by the solution of the Dirichlet problem in a ring domain $1<|\omega|<3/2$ for boundary value 0 on $|\omega|=1$ and $v(\underline{P}(\omega))$ on $|\omega|=3/2$, and add the function corresponding to $\alpha \log|\omega|$ in $|\omega|<1$, where $\alpha$ is a positive number. If $\alpha$ is sufficiently large, the resulting function $v^*(\underline{P})$ is superharmonic except at $\underline{P}_0^*$ and an associated function of $\underline{\mathfrak{L}}$. Now let $\underline{P}_0 \neq \underline{P}_0^*$ be any point not belonging to $\tau(\underline{\mathfrak{L}})$. There exists a function $h(\underline{P})$, which is bounded and harmonic outside neighborhoods of $\underline{P}_0^*$ and $\underline{P}_0$ and has positive and negative logarithmic singularities at $\underline{P}_0^*$ and $\underline{P}_0$ respectively. The sum $v^*(\underline{P})+\alpha h(\underline{P})$ is an associated function of $\underline{\mathfrak{L}}$, superharmonic outside $\underline{P}_0$.

If $\underline{\mathfrak{R}}$ has a positive boundary and if an associated function $v(\underline{P})$ is not superharmonic at $\underline{P}_0^*$, then we define $v^*(\underline{P})$ in the same way as above and add to $v^*(\underline{P})$ $\alpha$ times the Green's function with pole at $\underline{P}_0^*$ and also a certain large positive constant. Thus we have a positive associated function of $\underline{\mathfrak{L}}$, superharmonic everywhere on $\underline{\mathfrak{R}}$.

One way of obtaining $\underline{\mathfrak{L}}$ is as follows: Given a function $v(\underline{P})$, superharmonic on $\underline{\mathfrak{R}}$ possibly outside an isolated negative logarithmic singularity at a point $\underline{P}_0$ and bounded from below everywhere or outside a certain neighborhood of $\underline{P}_0$ if this is a singular point, we obtain an $\underline{\mathfrak{L}}$ which is complete and of harmonic measure zero if we define a base of a filter by $\{\underline{P}\,;\, v(\underline{P})>n\}$ $(n=1, 2, \ldots)$. We shall say that $v(\underline{P})$ determines $\underline{\mathfrak{L}}$.

We shall give more directly several examples of complete element of harmonic measure zero. Let $\underline{E}$ be a $G_\delta$-set of logarithmic capacity zero on $\underline{\mathfrak{R}}$. If $\underline{\mathfrak{R}}$ has a null boundary, we add the assumption that there is an outer point $\underline{P}_0$ of $\underline{E}$ on $\underline{\mathfrak{R}}$. In case $\underline{\mathfrak{R}}$ has a positive boundary, we take the Green's function of $\underline{\mathfrak{R}}$ as kernel of potential. In case $\underline{\mathfrak{R}}$ has a null boundary, we remove a small neighborhood of $\underline{P}_0$ and take the Green's function of the remaining surface as kernel. Then there exists a potential $U(\underline{P})$ such that $\underline{E}=\{\underline{P}\,;\, U(\underline{P})=+\infty\}$ as is remarked in [4]. In the case that $\underline{\mathfrak{R}}$ has a null boundary, we prolong $U(\underline{P})$ to a superharmonic function on $\underline{\mathfrak{R}}$ except at a negative logarithmic singularity located at $\underline{P}_0$. The potential $U(\underline{P})$ or this prolonged $U(\underline{P})$ determines a complete element of harmonic measure zero whose trace coincides with $\underline{E}$.

Next let $\underline{\mathfrak{R}}$ be a domain with positive boundary and relatively compact in a Riemann surface $\underline{\mathfrak{R}}_0$, and $\underline{F}$ a closed set on the boundary $\underline{\mathfrak{R}}^b$ of $\underline{\mathfrak{R}}$ with har-

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monic measure zero with respect to $\mathfrak{R}$. We shall show that there exists a finite-valued positive superharmonic function $v(P)$ in $\mathfrak{R}$ such that $v(P) \to +\infty$ when and only when $P \to F$.

We take a sequence $\{G_n\}$ of open sets decreasing to $F$ in $\mathfrak{R}_0$, whose boundaries $\{G_n^b\}$ are regular and disjoint from each other and pass no irregular boundary points of $\mathfrak{R}^b$. Since $F$ is of harmonic measure zero, there exists a positive superharmonic function $v_n(P) \leqq 1$ in $\mathfrak{R}$ tending to 1 as $P \to F$. We replace this function in $\mathfrak{R} - G_{n+1} - G_{n+1}^b$ by the solution of the Dirichlet problem for boundary value equal to $v_n(P)$ on $\mathfrak{R} \cap G_{n+1}^b$ and to 0 everywhere on $\mathfrak{R}^b - G_{n+1}$, and denote the resulting function defined in $\mathfrak{R}$ by $v_n'(P)$. Since $G_n^b$ has a positive distance from $G_{n+1}^b$ (with respect to a certain metric on $\mathfrak{R}_0$) and all points of $\mathfrak{R}^b \cap G_n^b$ are regular points, $v_n'(P)$ vanishes continuously at the points of $\mathfrak{R}^b \cap G_n^b$. So we take an open set $\Delta_n \supset \mathfrak{R}^b \cap G_n^b$ in $\mathfrak{R}_0$ bounded by a finite number of closed analytic curves such that $v_n'(P) < 1/n^2$ in $\Delta_n \cap \mathfrak{R}$. The difference $K_n = (\mathfrak{R} \cap G_n^b) - \Delta_n$ being compact in $\mathfrak{R}$, we can find a positive superharmonic function $v^*(P) \leqq 1$ in $\mathfrak{R}$ such that $\lim_{P \to F} v_n^*(P) = 1$ and $v_n^*(P) < 1/n^2$ on $K_n$. Preserving the boundary value, we harmonize $\inf(v_n'(P), v_n^*(P))$ in $\mathfrak{R} - G_n - G_n^b$ and denote the superharmonic function thus obtained in $\mathfrak{R}$ by $v_n(P)$. This function has the property that $0 < v_n(P) \leqq 1$ everywhere, $\lim_{P \to F} v_n(P) = 1$ and $v_n(P) < 1/n^2$ outside $G_n$. The sum $v(P) = \sum_{n=1}^{\infty} v_n(P)$ is again positive superharmonic and tends to $+\infty$ as $P \to F$, and $v(P) \leqq \sum_{n=1}^{N-1} v_n(P) + \sum_{n=N}^{\infty} 1/n^2 \leqq (N-1) + \sum_{n=1}^{N} 1/n^2 < +\infty$ in $\mathfrak{R} - G_N$. Thus $v(P) \to +\infty$ if and only if $P \to F$.

Another example is given when a filter defines a closed set $F_C$ of boundary components of harmonic measure zero of a Riemann surface $\mathfrak{R}$. Then there exists a base consisting of open sets $\{G_n\}$, having no point of accumulation in $\mathfrak{R}$ and bounded by closed analytic curves $\{G_n^b\}$ disjoint from each other, and the harmonic measure $\omega_n(P)$ of $G_n^b$ with respect to $G_1 - G_n - G_n^b$ tends to zero as $n \to \infty$.

We shall show the existence of a function associated with this $F_C$. First we set $n_1 = 1$ and shall define $\{n_k\}$ by induction. We choose $n_k$ such that $\omega_{n_k}(P) < 1/k^2$ on $G_1 - G_{n_{k-1}}$. We prolong $\omega_{n_k}(P)$ into $G_{n_k}$ by 1 so that it is superharmonic everywhere in $G_1$, and denote the function thus obtained again by $\omega_{n_k}(P)$. The sum $v_0(P) = \sum_{k=1}^{\infty} \omega_{n_k}(P)$ is positive superharmonic in $G_1$ and

$\lim v_0(\underline{P}) = +\infty$ if and only if $\underline{P}$ tends to $\underline{F}_C$. We draw a finite number of closed analytic curves $\underline{\mathfrak{c}}_1$ near $\underline{G}_1^b$ in $\underline{G}_1$ such that $\underline{\mathfrak{c}}_1$ and $\underline{G}_1^b$ enclose a finite number of annuli. We solve the Dirichlet problem in these annuli with boundary value equal to 0 on $\underline{G}_1^b$ and to $v_0(\underline{P})$ on $\underline{\mathfrak{c}}_1$. The function defined in $\underline{\mathfrak{R}} - \underline{G}_1 - \underline{G}_1^b$ by setting equal to a Green's function with pole at some point $\underline{P}_m$ in each component $\underline{D}_1^m$ will be denoted by $g(\underline{P})$. Then the function $v_1(\underline{P})$ defined by $-\alpha g(\underline{P})$ in $\underline{\mathfrak{R}} - \underline{G}_1$, and by $v_0(\underline{P})$ in $\underline{G}_1$ which is replaced by the above solution in the annuli is superharmonic on $\underline{\mathfrak{R}}$ except at $\underline{P}_m$, if $\alpha$ is taken sufficiently large. Let $h_m(\underline{P})$ $(m > 1)$ be harmonic on $\underline{\mathfrak{R}}$ outside a negative logarithmic singularity at $\underline{P}_1$ and a positive one at $\underline{P}_m$ of the form $\log 1/r$ and bounded outside neighborhoods of these points. The sum $V_1(\underline{P}) + \alpha \sum_{m>1} h_m(\underline{P})$ gives a required associated function.

3. To prove Theorem 4 we give one more lemma.

Lemma 7. *Let $v(z)$ be a positive superharmonic function, defined in $B$ and lower semicontinuous on $B+F$ (the value $+\infty$ is admitted), where $F$ is a closed set having positive average linear measure near $x = +\infty$ on the real axis. If $v(x) \to +\infty$ along $F$, then $v(z) \to +\infty$ as $z \to \infty$ in any strictly narrower substrip $0 < x < +\infty$, $0 < \varepsilon < y < 1-\varepsilon$. If, in particular, $F$ coincides with the positive real axis, then $v(z) \to +\infty$ as $z \to \infty$ in $0 < x < +\infty$, $0 \leqq y < 1-\varepsilon$.*

*Proof.* There exist $x_0 > 0$, $a > 0$ and $d > 0$ such that $m(F_a(x)) > d$ for all $x > x_0$. Let $R(x)$, $x > x_0$, be the rectangle with vertices $x-a$, $x+a$, $x+a+i$, $x-a+i$ and denote by $\omega_x(z)$ the harmonic measure of $F_a(x)$ with respect to $R(x)$. Since $m(F_a(x)) > d > 0$, there holds $\omega_x(x+iy) > \omega_0 > 0$ uniformly for $x > x_0$ and $y$, $\varepsilon < y < 1-\varepsilon$; this is seen by mapping $R(x)$ onto a disc. If $x$ is sufficiently large, then $v(x) > n$ on $F_a(x)$. Thus $v(x+iy) > n\omega_0$ for this $x$ and $y \in (\varepsilon, 1-\varepsilon)$. This shows that $v(z) \to +\infty$ as $z \to \infty$ in $\varepsilon < y < 1-\varepsilon$. The latter part of the lemma is obvious.

Now we give

*Proof of Theorem* 4. By Lemma 6 there exists a positive associated function $v(\underline{P})$ superharmonic on $\underline{\mathfrak{R}}$. The composed function $v(f(z))$ is positive superharmonic in $B$ and lower semicontinuous on $B+F$. Since $v(f(z)) \to +\infty$ along $F$, $v(f(z)) \to +\infty$ as $z \to \infty$ in any strictly narrower substrip by Lemma 7. Therefore, $f(z)$ tends to $\underline{\mathfrak{L}}$ as $z \to \infty$ in any strictly narrower substrip in view of property iii) of $v(P)$.

We shall show that the condition on $F$ in the theorem can not be replaced by any weaker condition.

THEOREM 5. *Let $F$ be any closed set on the positive x-axis which has not positive average linear measure near $x=+\infty$. Then there exists a nonconstant bounded analytic function $f(z)$ in $B$ which is continuous at $F$ such that $f(z)\to 0$ along $F$ but $|f(z)|\to \sup_B |f(z)|$ along a sequence of points tending to $\infty$ on the line $y=1/2$.*

*Proof.* First we determine some numbers with respect to $B'$ : $-\infty < x < +\infty$, $0<y<1$. Let $\omega_r$, $r>0$, be the maximum value, on the left half $x\leqq 0$ of $B'$, of the harmonic measure of the interval $[r, +\infty)$ on the real axis with respect to $B'$. As $r\to+\infty$, obviously $\omega_r\to 0$. We shall denote by $b_n$ the infimum of $r$ such that $\omega_r<1/2^n$ $(n=1, 2, \ldots)$.

By hypothesis, for any $x>0$, $a>0$ and $\varepsilon>0$, there exists a closed interval $I$ situated in $(x, +\infty)$ and of length $>a$ such that $m(F\cap I)<\varepsilon$. We take a closed interval $I_1$ of center $x_1>0$ and of length $2a_1$ such that $a_1>b_1$ and $m(F\cap I_1)<1/2$, and a closed interval $I_2$ of center $x_2$ and of length $2a_2$ such that $a_2>b_2$, $m(F\cap I_2)<1/2^2$ and $I_2$ lies in $(x_1+a_1+b_2, +\infty)$, and we continue this process. We cover each $F\cap I_n$ by an open set $G_n$ which consists of a finite number of intervals such that $G_n\cap G_{n+1}=\phi$ and $m(G_n)<1/2^{n-1}$.

We shall show that, given $x_0>0$ and $\delta>0$, there exists a number $d(x_0, \delta)>0$ such that if the linear measure of any set $A$ consisting of a finite number of open intervals in $(-x_0, x_0)$ is less than $d(x_0, \delta)$ then the harmonic measure of the set at $z=i/2$ with respect to $B'$ is less than $\delta$. We map $B'$ conformally onto the upper half plane by $\zeta=e^{\pi z}$. Then the linear measure $m(f(A))$ of the image of $A$ is given by $\pi\int_A e^{\pi x}dx$ and hence $\leqq \pi e^{\pi x_0}m(A)$. We know that, among sets of the same linear measure $m(f(A))$ each of which consists of a finite number of open intervals on the $\xi$-axis, the interval $(-m(f(A))/2, m(f(A))/2)$ has the largest harmonic measure at $\zeta=i$ with respect to the upper half plane. Since the harmonic measure at $\zeta=i$ of $f(A)$ is equal to the harmonic measure at $z=i/2$ of $A$, the latter becomes arbitrarily small if $m(A)$ is sufficiently small.

Now we determine $n_1<n_2<\ldots$ so that, for each $p$, $m(G_{n_p})>d(b_p, 1/2^p)$.

As we have just seen, the harmonic measure $h_p$ at $z_{n_p}=x_{n_p}+i/2$ of $G_{n_p}\cap(x_{n_p}-b_p, x_{n_p}+b_p)$ with respect to $B'$ is less than $1/2^p$. Let $J_p$ be the interval interposed between $I_{n_p}$ and $I_{n_{p+1}}$ and put $G=\bigcup_{p=1}^{\infty}(G_{n_p}\cap J_p)$. Obviously $F\subset G$. We define a nonnegative function $\varphi(x)$ on the $x$-axis to be equal to 0 outside $G$ and to a constant $c_j$ on each interval $K_j$ of $G$ such that $\varphi(x)\leqq p-1$ in $0<x<x_{n_p}$ and $\varphi(x)\to+\infty$ as $x\to+\infty$ along $G$. Let $u_j(z)$ be the harmonic measure of $K_j$ with respect to $B'$, and set $u(z)=\sum_{j=1}^{\infty}c_j u_j(z)$. We shall show that this is convergent. First notice that $a_{n_p}>b_{n_p}>b_p$, and that

$$\varphi(x)\leqq\sum_{p=1}^{\infty}p\{\chi(x\,;\,G_{n_p}\cap(x_{n_p}-b_p,\ x_{n_p}+b_p))$$
$$+\chi(x\,;\,[x_{n_p}+b_p,\ x_{n_{p+1}}-b_{p+1}])\},$$

where $\chi$ represents the characteristic function of sets. If we denote by $U_p(z)$ the harmonic measure of $[x_{n_p}+b_p,\ +\infty)$ with respect to $B'$, then there holds

$$u(z_{n_p})<pU_p(z_{n_p})+ph_p+(p+1)U_p(z_{n_p})+U_{p+1}(z_{n_p})+U_{p+2}(z_{n_p})+\ \ldots$$
$$<p/2p+p/2^p+(p+1)/2^p+1/2^{p+1}+1/2^{p+2}+\ \ldots\ =3p/2^p+2/2^p\to 0\ \text{ as }\ p\to\infty.$$

By a similar evaluation we see that the convergence of the series is uniform on any bounded set in $B'+G$. Therefore, $u(z)$ is harmonic in $B'$ and takes the value $c_j$ continuously at $K_j$.

We take any branch $v(z)$ of the conjugate of $u(z)$ and set $f(z)=e^{-u(z)-iv(z)}$. Then $|f(z)|\leqq 1$ and $|f(z_{n_p})|\to 1$ as $p\to\infty$. By the reflexion principle, $f(z)$ is continuous at $G$ and $|f(x)|=e^{-\varphi(x)}$. Since $F\subset G$, and $\varphi(x)\to+\infty$ as $x\to+\infty$, $f(z)$ is continuous at $F$ and $f(x)\to 0$ as $F\ni x\to+\infty$.

**4.** In the rest of the paper, we shall deal with the second case in which $\mathfrak{R}$ *has a null boundary.* First we state

THEOREM 6. *Let $\mathfrak{L}$ be a complete element of harmonic measure zero added to a Riemann surface $\mathfrak{R}$ with null boundary, and $f(z)$ a continuous transformation of $0<x<+\infty$, $0\leqq y<1$ into $\mathfrak{R}+\{\mathfrak{L}\}$ which is analytic and exceptionally ramified in B. If $f(x)$ tends to $\mathfrak{L}$ as $x\to+\infty$, then $f(z)$ tends to $\mathfrak{L}$ as $z\to\infty$ in any narrower substrip $0<x<+\infty$, $0\leqq y<1-\varepsilon$.*

In order to prove this theorem, we need two lemmas. In this section, we shall discuss the first lemma concerning the type problem. In the case that $\mathfrak{R}$ has genus $\geqq 2$, it is known that the Schottky covering surface of planar character of $\mathfrak{R}$ has a positive boundary, or is known something more (see [19], [22], [23]). It seems that the following lemma has not been proved in the case where $\mathfrak{R}$ is of infinite genus.

LEMMA 8. *Let $\mathfrak{R}$ be a Riemann surface of genus $\geqq 2$. If we draw two disjoint analytic loops $\underline{c}_1$ and $\underline{c}_2$ on $\mathfrak{R}$ which do not separate $\mathfrak{R}$, and if we join different replicas of $\mathfrak{R}$, having cuts at $\underline{c}_1$ and $\underline{c}_2$, along opposite shores of the replicas of $\underline{c}_1$ and $\underline{c}_2$ indefinitely, then the Riemann surface $\mathfrak{R}^{(\infty)}$ thus obtained has a positive bonndary.*[18]

*Proof.* Obviously we may suppose that $\mathfrak{R}$ has a null boundary. Let $c_0$ be a replica of $\underline{c}_1$. It separates $\mathfrak{R}^{(\infty)}$ into two parts. We shall show that the harmonic measure of $c_0$ with respect to any one $\mathfrak{R}_1^{(\infty)}$ of them is not a constant.

We denote by $\mathfrak{R}_0$ the replica of $\mathfrak{R}$ which has $c_0$ on its boundary and is contained in $\mathfrak{R}_1^{(\infty)}$, and by $c_{01}$, $c_{02}$ and $c_{03}$ the other shores of $\mathfrak{R}_0$ corresponding to $\underline{c}_1$ and $\underline{c}_2$. The three replicas adjoining to $\mathfrak{R}_0$ through $c_{01}$, $c_{02}$ and $c_{03}$ will be denoted by $\mathfrak{R}_{01}$, $\mathfrak{R}_{02}$ and $\mathfrak{R}_{03}$ respectively. We proceed in this manner and obtain the partial surface $\mathfrak{R}_1$. We define the harmonic measure $\omega_0(P)$ of $c_{01}+c_{02}+c_{03}$ with respect to $\mathfrak{R}_0$. Next on $\mathfrak{R}_{01}$, we define the harmonic measure $\omega_{01}(P)$ of the three shores $c_{011}$, $c_{012}$ and $c_{013}$ which are not identified with $c_{01}$, and define $\omega_{02}(P)$ and $\omega_{03}(P)$ similarly on $\mathfrak{R}_{02}$ and $\mathfrak{R}_{03}$ respectively. We continue this process.

Denote the conjugates of $\omega_0(P)$ and $\omega_{0j}(P)$ by $\bar{\omega}_0(P)$ and $\bar{\omega}_{0j}(P)$ respectively, and set $\int_{c_0} d\bar{\omega}_0 = b_0$, $\int_{c_{0j}} d\bar{\omega}_0 = b_{0j}$ and $\int_{c'_{0j}} d\bar{\omega}_{0j} = b'_{0j}$ $(j=1, 2, 3)$, where $c'_{0j}$ is the boundary of $\mathfrak{R}_{0j}$ identified with $c_{0j}$. We consider on $\mathfrak{R}_{0j}$ the level curves $\bar{\omega}_{0j}(P) = \text{const.}$, starting from $c'_{0j}$. They terminate at multiple points or tend to the ideal boundary of $\mathfrak{R}_{0j}$ or reach $c_{0jk}$ $(j, k=1, 2, 3)$. Take arbitrary points $\underline{P}_1 \in \underline{c}_1$ and $\underline{P}_2 \in \underline{c}_2$, denote the corresponding points on $c'_{0j}$ by $P'_{0j}$ and determine the branch of $\bar{\omega}_{0j}(P)$ so that $\bar{\omega}_{0j}(P'_{0j})=0$. By the function $\{b_{0j}/(b_0 b'_{0j})\}\{\omega_{0j}(P) + i\bar{\omega}_{0j}(P)\}$ the level curves are mapped onto a rectangle $R_{0j}: 0<\xi<b_{0j}/(b_0 b'_{0j})$, $0<\eta<b_{0j}/b_0$ with slits in the $\zeta=\xi+i\eta$-plane. These slits are parallel to the $\xi$-axis and have projection on the $\eta$-axis of linear measure zero on account of Theorem 2 of [14].

We determine a branch $\bar{\omega}_0^{(j)}(P)$ of $\bar{\omega}_0(P)$ so that it vanishes at the point

[18] We can prove, what is more, that there exists a nonconstant harmonic function with finite Dirichlet integral but no nonconstant analytic function with finite Dirichlet integral on $\mathfrak{R}^{(\infty)}$ if we follow the lines of the discussion at the end of n° 3 in [22].

$P_{0j}$ on $c_{0j}$ which corresponds to $\underline{P}_1$ or $\underline{P}_2$ and continue the function $b_0^{-1}\{\omega_0(P) - 1 + i\bar{\omega}_0^{(j)}(P)\}$ analytically along $c_{0j}$ in the direction such that the image of $c_{0j}$ coincides with the left side of $R_{0j}$. We choose $\alpha$, $0 < \alpha < 1$, sufficiently small that, on $\mathfrak{R}_0$ and $\mathfrak{R}_{0j}$ $(j = 1, 2, 3)$, the level curves $\omega_0(P) = \alpha$ and $\omega_{0j}(P) = \alpha$ enclose neat annuli together with $c_0$ and $c'_{0j}$ respectively. The point on $c'_{0j}$, corresponding to $P \in c_{0j}$ will be denoted by $P'$. We connect, by a straight line, every image $i\bar{\omega}_0^{(j)}(P)/b_0$ with $\{b_{0j}/(b_0 b'_{0j})\}\{\alpha + i\bar{\omega}_{0j}(P')\}$ in the partial rectangle $R_{0j}(\alpha) : 0 < \xi < b_{0j}/(b_0 b'_{0j})$, $0 < \eta < b_{0j}/b_0$. Thus $R_{0j}(\alpha)$ is transformed onto itself. If we leave the rest of $R_{0j}$ unchanged, then a continuous automorphism $T_{0j}$ of the slit $R_{0j}$ is obtained. It is continuously differentiable except on the segment $\omega_{0j} = \alpha b_{0j}/(b_0 b'_{0j})$ and has bounded dilatation.

Let us now map the first replica $\mathfrak{R}_0$ by the aid of $b_0^{-1}\{\omega_0(P) + i\bar{\omega}_0(P)\}$ onto a rectangle $R_0 : 0 < \xi < 1/b_0$, $0 < \eta < 1$ with slits parallel to the $\xi$-axis so that $c_0$ corresponds to the left side. This slit rectangle consists of the images of the regular $\bar{\omega}_0$-level curves on $\mathfrak{R}_0$, and the projection of the slits on the $\eta$-axis has linear measure zero on account of Theorem 2 of [14]. The right side contains the images of $\{c_{0j}\}$ $(j = 1, 2, 3)$ which are composed of a countable number of open intervals $\{I_\mu\}$ that have the total measure 1. We may now assume that the points $P_{0j}$, previously defined on $c_{0j}$, are the end points of some of $\{I_\mu\}$. We divide by the horizontal lines the $T_{0j}$-images of $R_{0j}$, into thin slit rectangles with the same widths whose left sides are congruent to $\{I_\mu\}$. We translate these rectangles and join them to $R_0$ by identifying the corresponding intervals. In such a manner, a continuous transformation of $\mathfrak{R}_0 + \mathfrak{R}_{01} + \mathfrak{R}_{02} + \mathfrak{R}_{03}$, slit along some curves, onto a collection of rectangles of width $1/b_0 + b_{01}/(b_0 b'_{01})$ or $1/b_0 + b_{02}/(b_0 b'_{02})$ or $1/b_0 + b_{03}/(b_0 b'_{03})$, with some slits parallel to the $\xi$-axis, is obtained. We continue this process and obtain a topological mapping $T(P)$ of $\mathfrak{R}_1^{(\infty)}$, slit along some curves, onto a domain $G$ of the form $0 < \xi < h(\eta) \leqq +\infty$, $0 < \eta < 1$, where, for almost all $\eta$, the image of $0 < \xi < h(\eta)$ tends to the boundary of $\mathfrak{R}_1^{(\infty)}$ while passing through $\mathfrak{R}_0$, one of $\mathfrak{R}_{0j}$, and so on. In addition, $T(P)$ is continuously differentiable except at a countable number of segments parallel to the $\eta$-axis, which correspond to the sides of the rectangles $R_0, R_{0j}, \ldots$ or to the level curves $\omega_0(P) = \alpha$, $\omega_{0j}(P) = \alpha, \ldots$, and the dilatation $q(P)$ of $T(P)$ at the points, where $T(P)$ is continuously differentiable, is uniformly bounded: $q(P) \leqq q_0 < +\infty$, because we have only 4 types of $\mathfrak{R}_0, \mathfrak{R}_{0j}, \ldots$.

We shall show that $h(\eta)$ is uniformly bounded. Because there are only 4 types of $\mathfrak{R}_0$, $\mathfrak{R}_{0j}$, . . . , the total variations of $\omega_0(P)$, $\omega_{0j}(P)$, . . . have at most 4 different values. We denote by $b$ the smallest number of them. Next the maximum of the 12 ratios $b_{0j}/b_0$ and $b_{0jk}/b'_{0j}$ $(j,\ k = 1, 2, 3)$ is denoted by $\beta$, where $b_{0jk} = \int_{c_{0jk}} d\bar{\omega}_{0j}$. Obviously $0 < \beta < 1$. The width of $R_0$ is $\leqq 1/b$ and the widths of $R_{0j}$ are $\leqq \beta/b$, the widths of the next ones are $\leqq \beta^2/b$, and so on. Therefore $h(\eta) \leqq b^{-1}(1 + \beta + \beta^2 + \ . \ . \ . ) = \{b(1-\beta)\}^{-1} < +\infty$.

Suppose that $\mathfrak{R}_1^{(\infty)}$ has ideal boundary of harmonic measure zero. We shall denote by $c^{(1)}$, $c^{(2)}$, . . . the new free edges appearing as we add new replicas to $\mathfrak{R}_0$, to $\mathfrak{R}_{01} + \mathfrak{R}_{02} + \mathfrak{R}_{03}$, and so on. The harmonic measure $\omega^{(n)}(P)$ $(n \geqq 1)$ of $c^{(n)}$ with respect to the domain between $c_0$ and $c^{(n)}$ tends to 0 as $n \to \infty$. The $T$-image in $G$ of the level curve of $\omega^{(n)}(P)$ intersects the segment $0 < \xi < h(\eta)$ for almost all $\eta$ such that $0 < \eta < 1$. Therefore, its length is $\geqq 1$. On using Schwarz's inequality, we have

$$1 \leqq \left(\int_{\omega^{(n)} = \text{const.}} |d\xi|\right)^2 \leqq \int \left|\frac{d\zeta}{d\bar{\omega}^{(n)}}\right|^2 \frac{d\bar{\omega}^{(n)}}{q} \int q d\bar{\omega}^{(n)}$$

and then

$$\int_0^1 \frac{d\omega^{(n)}}{\int_{\omega^{(n)} = \text{const.}} q d\bar{\omega}^{(n)}} \leqq \iint \left|\frac{d\zeta}{d\bar{\omega}^{(n)}}\right|^2 \frac{d\bar{\omega}^{(n)}}{q} d\omega^{(n)} \leqq \frac{1}{b(1-\beta)} < +\infty.$$

Since $q(P) \leqq q_0 < +\infty$, the left side $\geqq \left(q_0 \int_{c_0} d\bar{\omega}^{(n)}\right)^{-1}$. However, this tends to $+\infty$ as $n \to \infty$ and a contradiction arises.

To obtain a contradiction in another way, we may apply Theorem 5 of [15] after having known that $T(P)$ is everywhere quasi-conformal in the sense of Pfluger-Ahlfors in virtue of Theorem 2′ of [11].

*Remark.* Since any covering surface of a Riemann surface of hyperbolic type is of hyperbolic type, the Riemann surface obtained by joining indefinitely the replicas of $\mathfrak{R}$, which are cut along $p$ $(2 \leqq p \leqq +\infty)$ disjoint closed curves that do not separate $\mathfrak{R}$, has a positive boundary. The idea of the proof of our lemma will be used to discuss the type problem in general in another paper.

**5.** The second lemma for the proof of Theorem 6 is:

LEMMA 9. *Let $v(P)$ be a superharmonic function on a Riemann surface $\mathfrak{R}$. If there is a curve $l$, which may oscillate, such that $v(P) \to +\infty$ along $l$ and $l$ has at least one point $P_0$ of accumulation in $\mathfrak{R}$, then $l$ must terminate at $P_0$*

*and* $v(\underline{P}) = +\infty$.

*Proof*. We take a parameter circle $|\omega| < 1$ such that $\omega = 0$ corresponds to $\underline{P}_0$, and set $v(\underline{P}(\omega)) = V(\omega)$. If $V(0)$ were finite, there would exist a sequence of circles $|\omega| = \varepsilon_n$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and $V(\omega) \to V(0)$ along them (see [2]). This is impossible, because $V(\omega) \to +\infty$ along the image of $\underline{l}$ in $|\omega| < 1$ and $\underline{l}$ comes arbitrarily close to $\underline{P}_0$. Therefore $V(0) = +\infty$. If $\underline{l}$ oscillated, there would be a continuum in $|\omega| < 1$ at which $V(\omega) = +\infty$. This contradicts the fact that the set of points where a superharmonic function assumes $+\infty$ is of logarithmic capacity zero. Thus $\underline{l}$ terminates at $\underline{P}_0$ and $v(\underline{P}_0) = +\infty$.

Now we give

*Proof of Theorem* 6. Let $\mathfrak{R}$ be the covering surface of $\underline{\mathfrak{R}}$ which is defined by means of $f(z)$ and conformally equivalent to $B$. We shall say that a covering surface $\mathfrak{R}^*$ is inserted between $\mathfrak{R}$ and $\underline{\mathfrak{R}}$, if $\mathfrak{R}$ is a covering surface of $\mathfrak{R}^*$ and $\mathfrak{R}^*$ is that of $\underline{\mathfrak{R}}$. Suppose that a covering surface $\mathfrak{R}^*$ of positive boundary with a complete element $\mathfrak{L}^*$ of harmonic measure zero is inserted between $\mathfrak{R}$ and $\underline{\mathfrak{R}}$ such that the image of the positive $x$-axis converges to $\mathfrak{L}^*$ and that $\mathfrak{L}^*$ is projected into $\underline{\mathfrak{L}}$. By the last expression it is meant that every sequence of points on $\mathfrak{R}^*$ converging to $\mathfrak{L}^*$ is projected to a sequence on $\underline{\mathfrak{R}}$ converging to $\underline{\mathfrak{L}}$. Then by Theorem 4 we obtain the conclusion of Theorem 6.

We shall show that we can actually find such $\mathfrak{R}^*$ under the conditions of the theorem. First we consider the case that $\underline{\mathfrak{R}}$ is planar. We may suppose that $\underline{\mathfrak{R}}$ is a part of the extended $w$-plane, that $w = \infty$ is an inner point of $\underline{\mathfrak{R}}$ and that an associated function $v(w)$ of $\underline{\mathfrak{L}}$ has its negative logarithmic singularity at $w = \infty$. Since $\underline{\mathfrak{R}}$ is a domain outside a bounded closed set of logarithmic capacity zero and $v(w)$ is bounded from below near it, $v(w)$ can be extended so that it is superharmonic everywhere in the finite $w$-plane. By Lemma 9 it follows that $f(x)$ tends to a finite value $w_0$ as $x \to +\infty$ and that the convergence of a sequence of points of $\mathfrak{R}$ to $w_0$ implies the convergence to $\underline{\mathfrak{L}}$.

By our hypothesis, there exist points $\{w_k\}$ and associated integers $\{\mu_k\}$, $\mu_k \geqq 2$, such that $\sum_k (1 - 1/\mu_k) > 2$ and every point of $\mathfrak{R}$ situated above $w_k$ has multiplicity divisible by $\mu_k$ possibly with a finite number of exceptions, or, if $\mu_k = +\infty$, there exist at most a finite number of points of $\mathfrak{R}$ above $w_k$. If necessary, taking a substrip of the same height 1 of $B$, we may suppose that there is no exceptional point at all above any $w_k$. It is known that the regularly

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ramified simply-connected covering surface $\tilde{\mathfrak{R}}$ of the extended $w$-plane, which has branch points with multiplicity $\mu_k$ above $w_k$ (if $\mu_k = +\infty$, branch points are logarithmic), is of hyperbolic type. If $\underline{\mathfrak{R}}$ is not identical with the extended $w$-plane, then we take the part of $\tilde{\mathfrak{R}}$ which lies above $\underline{\mathfrak{R}}$. Then $\tilde{\mathfrak{R}}$ or this part may be regarded as a surface $\mathfrak{R}^*$ to be inserted between $\mathfrak{R}$ and $\underline{\mathfrak{R}}$. If $w_0$ does not coincide with $w_k$ for which $\mu_k = +\infty$, then the image of the positive $x$-axis terminates at an inner point $\tilde{P}_0$ of $\tilde{\mathfrak{R}}$. If we choose a sequence of domains $\{\tilde{D}_n\}$ around $\tilde{P}_0$ converging to it and take $\{\tilde{D}_n \cap \mathfrak{R}^*\}$ as a base of a filter to define an element $\mathfrak{L}^*$, then $\mathfrak{L}^*$ is a complete element of harmonic measure zero because the Green's function on $\tilde{\mathfrak{R}}$ with pole at $\tilde{P}_0$ is an associated function if it is considered on $\mathfrak{R}^*$. Thus all the requirements for $\mathfrak{R}^*$ are satisfied. If $w_0$ coincides with a $w_k$ for which $\mu_k = +\infty$, then the image of the positive $x$-axis terminates at a logarithmic branch point of $\tilde{\mathfrak{R}}$. In this case, we map $\tilde{\mathfrak{R}}$ conformally onto the upper half $\zeta$-plane ($\zeta = \xi + i\eta$) such that the logarithmic branch point corresponds to $\zeta = \infty$. Under this mapping, any upper half plane $\eta > \eta_0 > 0$ corresponds to a neighborhood of the logarithmic branch point. Hence if we consider $\eta(\tilde{P})$ as a function on $\mathfrak{R}^*$, then the complete element $\mathfrak{L}^*$ of harmonic measure zero determined by this function is projected into $\underline{\mathfrak{L}}$. Thus the theorem is proved in this case.

We are next concerned with the case in which $\underline{\mathfrak{R}}$ is conformally equivalent to a domain of a torus $\underline{\mathfrak{R}}_0$. We may suppose that $\underline{\mathfrak{R}}$ is this domain itself. Any associated function is prolongable to a function $V(\underline{P})$ superharmonic everywhere on $\underline{\mathfrak{R}}_0$ except at one point $\underline{P}'$. In virtue of Lemma 9, $f(z)$ tends to a point $\underline{P}_0 \neq \underline{P}'$ of $\underline{\mathfrak{R}}_0$ along the positive real axis and the convergence of a sequence of points of $\underline{\mathfrak{R}}$ to $\underline{P}_0$ implies the convergence to $\underline{\mathfrak{L}}$. By the hypothesis, there exists a point $\underline{P}_1$ such that every point of $\mathfrak{R}$ situated above $\underline{P}_1$ has multiplicity divisible by $\mu_0 \geqq 2$ with at most a finite number of exceptions, or there are at most finitely many points of $\mathfrak{R}$ above $\underline{P}_1$. The part above $\underline{\mathfrak{R}}$ of the regularly ramified simply-connected covering surface of $\underline{\mathfrak{R}}_0$ with branch points of multiplicity $\mu_0$ or with logarithmic branch points above $\underline{P}_1$ will play the role of $\mathfrak{R}^*$.

In case the genus of $\underline{\mathfrak{R}}$ is greater than 1 but finite, $\underline{\mathfrak{R}}$ may be regarded as a part of a closed Riemann surface $\underline{\mathfrak{R}}_0$. The part above $\underline{\mathfrak{R}}$ of the universal covering surface of $\underline{\mathfrak{R}}_0$ may be taken for $\mathfrak{R}^*$.

Finally we consider the case that $\underline{\mathfrak{R}}$ is of infinite genus. We see, by Lemma

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9, that $f(x)$ tends to an inner point or to a boundary component of $\underline{\mathfrak{R}}$ as $x \to +\infty$. The former case can be treated easily by taking the universal covering surface of $\underline{\mathfrak{R}}$ as $\mathfrak{R}^*$. In the latter case we choose two disjoint loops which do not separate $\underline{\mathfrak{R}}$ and form a Riemann surface $\mathfrak{R}^{(\infty)}$ in such a way as we did in Lemma 8. This has a positive boundary by Lemma 8. We insert $\mathfrak{R}^{(\infty)}$ between $\mathfrak{R}$ and $\underline{\mathfrak{R}}$ in any way and take it for $\mathfrak{R}^*$. The image of the positive $x$-axis then lies in a replica of $\underline{\mathfrak{R}}$ and converges to a boundary component $P_C^*$ of $\mathfrak{R}^{(\infty)} = \mathfrak{R}^*$. If we transform an associated function of $\underline{\mathfrak{L}}$ to the function $v_0(P^*)$ in the replica, then this is superharmonic everywhere in it except at one point. We draw a closed analytic curve $c^*$ in the replica such that it separates $P_C^*$ from the images of the loop cuts of $\underline{\mathfrak{R}}$, and draw another closed analytic curve $c_1^*$ near $c^*$ so that they enclose a neat annulus and $c^*$ is separated by $c_1^*$ from $P_C^*$. By adding a constant, if necessary, we may suppose that $v_0(P^*)$ is positive on this annulus. We replace it by the solution of the Dirichlet problem with boundary value $v_0(P^*)$ on $c_1^*$ and 0 on $c^*$ and denote the function thus obtained again by $v_0(P^*)$. This is superharmonic on $c_1^*$. Since $\mathfrak{R}^* = \mathfrak{R}^{(\infty)}$ has a positive boundary, the harmonic measure $u(P^*)$ of $c^*$ with respect to the domain $D^*$ not containing $c_1^*$ is not a constant. If $\alpha$ is taken sufficiently large, the function equal to $\alpha(u(P^*) - 1)$ in $D^*$ and to $v_0(P^*)$ in $\mathfrak{R}^* - D^*$ is superharmonic on $\mathfrak{R}^*$ and defines a complete element $\mathfrak{L}^*$ of harmonic measure zero to add to $\mathfrak{R}^*$. The image of the positive $x$-axis by $f(z)$ converges to $\mathfrak{L}^*$ and $\mathfrak{L}^*$ is obviously projected into $\underline{\mathfrak{L}}$. Thus the proof of our theorem is completed.

*Remark.* The beginning part of the proof suggests the possibility to extend further the theorem.

**6.** The final theorem will show that the condition in Theorem 6 that $f(x) \to \underline{\mathfrak{L}}$ as $x \to +\infty$ can not be replaced by the condition that $f(x) \to \underline{\mathfrak{L}}$ as $x \to +\infty$ along a *part* $F$ of the $x$-axis however large $F$ may be metrically (with regard to linear measure). Actually we shall prove

THEOREM 7. *Let $\underline{\mathfrak{R}}$ be any Riemann surface with null boundary, and $\underline{P}_0$ any point or boundary component of $\underline{\mathfrak{R}}$. Then there exists an analytic mapping of $B$ into $\underline{\mathfrak{R}}$ such that it is continuous at the positive $x$-axis outside a closed set of linear measure zero and tends to $\underline{P}_0$ as $x \to +\infty$ outside of the set, but has no definite limit as $z \to \infty$ along any curve in $B$.*

*Proof.* If the universal covering surface of $\mathfrak{R}-\{P_0\}$ is not conformally equivalent to a disc, then we exclude one or two points from $\mathfrak{R}$ so that this condition is fulfilled and denote still by $\mathfrak{R}$ the remaining surface.

First we consider the case that $P_0$ is an inner point of $\mathfrak{R}$, and fix a parameter circle $|W|<1$ of $P_0$. We map the universal covering surface of $\mathfrak{R}-\{P_0\}$ onto $U_\zeta : |\zeta|<1$ and denote by $\mathfrak{G}$ the corresponding Fuchsian or Fuchsoid group. Let $U_n \subset U_\zeta$ be any connected component of the image of the outside of the part of $\mathfrak{R}$ that corresponds to $|W|=1/n$ $(n \geqq 2)$. We take them so that $U_1 \subset U_2 \subset \ldots$ . The part $\gamma_n$ in $U_\zeta$ of the boundary of $U_n$ corresponds to the circle $|W|=1/n$ and consists of a countable number of curves starting from and terminating at the parabolic fixed points, which are defined with respect to $\mathfrak{G}$ and correspond to $P_0$. Since $\mathfrak{R}$ has a null boundary, the harmonic measure of $\gamma_n$ with respect to $U_n$ is the constant 1. In other words, $|\zeta|=1$ is of harmonic measure zero with respect to $U_n$. If we exclude the inside of the part $\gamma_n(A)$ of $\gamma_n$ having end points on a closed arc $A$ on $|\zeta|=1$ and denote the remaining domain in $U_\zeta$ by $U_A^{(n)}$, then the harmonic measure of $A$ with respect to $U_A^{(n)}$ is zero. To prove this, it is sufficient to show that every closed subarc $A'$ of $A$ is of harmonic measure zero with respect to $U_A^{(n)}$. Suppose, to the contrary, that the harmonic measure $\omega(\zeta)$ of $A'$ with respect to $U_A^{(n)}$ were positive. We denote by $m$ the supremum of $\omega(\zeta)$ on $\gamma_n$. Then $0<m<1$. The function $\omega(\zeta)-m$ would be $<1$ and positive somewhere in $U_n$, and would not exceed 0 as $\zeta$ approaches $\gamma_n$. This contradicts the vanishing of the harmonic measure of $|\zeta|=1$ with respect to $U_n$.

Now let $\zeta_0=e^{i\theta_0}$ be a hyperbolic fixed point on $|\zeta|=1$ with respect to $\mathfrak{G}$. We take a sequence of points $\{e^{i\theta_n}\}$, $\theta_1<\theta_2<\ldots<\theta_n \to \theta_0$, on $|\zeta|=1$ which are not parabolic fixed points. We denote the arc between $e^{i\theta_{n-1}}$ and $e^{i\theta_n}$ by $A_n$ and consider $\gamma_n(A_n)$ for $n \geqq 2$. The domain bounded by $\{\gamma_n(A_n)\}$ and $|\zeta|=1$ in $U_\zeta$ will be denoted by $U_0$. As we have seen, the harmonic measure of the arc $\widehat{e^{i\theta_1}e^{i\theta_0}}$ with respect to $U_0$ is zero. We then map $U_0$ onto $B$ so that the point at infinity corresponds to $\zeta_0=e^{i\theta_0}$ and $z=0$ corresponds to $e^{i\theta_1}$. Under this mapping, the image of the arc $\widehat{e^{i\theta_1}e^{i\theta_0}}$ is a closed set of linear measure zero on the positive $x$-axis. If we consider the composition of the inverse of this mapping and the mapping $U_\zeta \to \mathfrak{R}$, then it is obvious that it is the function required in the theorem.

Next we consider the case that $\underline{P}_0$ is a boundary component of $\mathfrak{R}$. Let $\{\underline{c}_n\}$ be a sequence of closed curves in $\mathfrak{R}$ shrinking to $\underline{P}_0$ such that the outside of $\underline{c}_1$ is at least of triply-connected, and take $\underline{c}_n$ in stead of the image on $\mathfrak{R}$ of $|W|=1/n$ in the first case. A component arc of the boundary $\gamma_n$ in $U_\zeta$ of $U_n$ may terminate at hyperbolic fixed points but the harmonic measure of $\gamma_n$ is again 1 with respect to $U_n$. Let $\zeta_0=e^{i\theta_0}$ be a hyperbolic fixed point which is on the boundary of $U_1$. We shall show that an arc of $\gamma_1$ may terminate at $\zeta_0$ but an infinite number of arcs of $\gamma_1$ cluster to $\zeta_0$ at least from one side. First we notice that $U_1$ is the image of the universal covering surface $\underline{G}_1^\infty$ of the domain $\underline{G}_1$ outside $\underline{c}_1$ on $\mathfrak{R}$. We map $U_1$ onto $|Z|<1$ in a one-to-one conformal manner, and denote by $\mathfrak{G}_Z$ the Fuchsian or Fuchsoid group corresponding to the mapping of $\underline{G}_1^\infty$ onto $|Z|<1$. The image $Z_0$ of $\zeta_0$ is a hyperbolic fixed point with respect to $\mathfrak{G}_Z$. Therefore, at least from one side, an infinite number of images of arcs of $\gamma_1$ cluster to $Z_0$. Thus an infinite number of arcs of $\gamma_1$ cluster to $\zeta_0$ at least from one side, say, in the counter-clockwise. We take $\theta_1<\theta_2<\ldots<\theta_n\to\theta_0$ such that the points $e^{i\theta_n}$ are on the boundary of $U_1$, and denote by $A_n$ the arc $\widehat{e^{i\theta_{n-1}}e^{i\theta_n}}$ as before. The rest of the proof will be the same as in the first case and the proof will be completed.

Thus it is really necessary to distinguish the case where $\mathfrak{R}$ has a null boundary from the case where $\mathfrak{R}$ has a positive boundary.

We shall close this paper with a remark to the case of pseudo-analytic functions (with bounded dilatation) in Pfluger-Ahlfors's sense. We refer to Mori [11] for this class of functions (cf. [26], too). If we take into account the fact that a quasi-conformal mapping with bounded dilatation in Pfluger-Ahlfors's sense of a strip $B$ onto another strip $B'$ can be extended so that it is topological between the closures of $B$ and $B'$ and that the image of a strictly narrower substrip of $B$ is contained in some strictly narrower substrip of $B'$ (see [11]), then it follows that Theorems 4 and 6, in case $F$ is identical with the whole positive $x$-axis, are valid also for pseudo-analytic functions in the present sense.

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