A Version of Hagen's Proof of the "Law of Error."

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Hagen's proof (1837), as described in the 8th edition of Mansfield Merriman's "Method of Least Squares," is based on the assumption that the error may be supposed to consist of the algebraic sum of an infinite number of infinitesimal errors of equal amount ϵ, each one of which is equally likely to be positive or negative. Thus if $2m$ is the number of the infinitesimal errors, the probability of the error $x \equiv 2p\,\epsilon$ occurring is

$$P \equiv \frac{(2m)!}{(m-p)!\,(m+p)!} \cdot \frac{1}{2^{2m}},$$

and the maximum value of P occurs when $p = 0$, and is

$$P_0 \equiv \frac{(2m)!}{m!\,m!}\,\frac{1}{2^{2m}}.$$

Thus the error zero has maximum probability, which however, like that of any other error of the series 0, $\pm 2\epsilon$, $\pm 4\epsilon$, $\pm 6\epsilon \ldots$, is infinitesimal.

Applying Stirling's formula for the approximate value of $n!$ when n is great, we get

$$\frac{\sqrt{2\pi . 2m}\left(\frac{2m}{e}\right)^{2m}}{\sqrt{2\pi m}\;\sqrt{2\pi m}\left(\frac{m}{e}\right)^{m}\left(\frac{m}{e}\right) . \, 2^{2m}}$$

as the value differing from P_0 by an infinitesimal of higher order.

This simplifies to $\frac{1}{\sqrt{\pi m}}$ which is the limit to which P_0 approaches when m tends to infinity.

Since the various errors of the series form an arithmetical progression with common difference 2ϵ, we may take it that $\frac{P_0}{2\epsilon}$ is finite.* Thus if $2\epsilon\sqrt{m} = \frac{1}{h}$, h is finite.

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* This is equivalent to the assumption that the probability that the error lies between two finite values differing by a finite amount, is finite.

Again, the error $x = 2p\epsilon$; $\therefore\ p = \dfrac{x}{2\epsilon}$ which tends to infinity as ϵ tends to 0. But $\dfrac{p}{m} = 2\epsilon x h^2$, which is infinitesimal for finite values of x.

Now

$$\frac{P}{P_0} = \frac{m!\ m!}{(m-p)!\ (m+p)!} = \sqrt{\frac{m\ .\ m}{(m-p)(m+p)}}\ \frac{m^{2m}}{(m-p)^{m-p}(m+p)^{m+p}}$$

by Stirling's Formula.

Thus

$$\frac{P}{P_0} = \frac{1}{\sqrt{1-\dfrac{p^2}{m^2}}} \cdot \frac{1}{\left(1-\dfrac{p^2}{m^2}\right)^m} \frac{\left(1+\dfrac{p}{m}\right)^p}{\left(1+\dfrac{p}{m}\right)^p}.$$

Since $\dfrac{p}{m}$ is infinitesimal, the limit to which this tends is

$$\frac{P}{P_0} = 1 \times \frac{1}{e^{-\frac{p^2}{m}}} \times \frac{e^{-\frac{p^2}{m}}}{e^{\frac{p^2}{m}}} = e^{-\frac{p^2}{m}}$$

Thus

$$\frac{P}{P_0} = e^{-h^2x^2},\ \text{or}\ P = P_0\, e^{-h^2x^2}$$

Here, of course, both P and P_0 are infinitesimal.

If $y\,dx$ be the probability that the error lies between x and $x + dx$, then $y dx = \dfrac{P\,dx}{2\epsilon}$ and $y_0\,dx = \dfrac{P_0\,dx}{2\epsilon}$.

Hence

$$y = y_0 e^{-x^2h^2},\ \text{where}\ y_0 = \frac{P_0}{2\epsilon} = \frac{1}{2\epsilon\sqrt{\pi m}} = \frac{h}{\sqrt{\pi}}.$$

This is the equation to the "curve of error" or of "facility of error."

The foregoing proof, it will be observed, has been worked out without involving differential equations or definite integrals, and without using any theorem of higher mathematics beyond the formula of Stirling for the approximate value of $n!$ when n is great, and the theorem $\lim\limits_{n \to \infty}\left(1+\dfrac{1}{n}\right)^n = e$.

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The assumption on which Hagen's proof of the Law of Error is based is no doubt open to criticism, but the same is true of all the alternatives that have been proposed. I do not enter into that question, but merely offer the present version of the proof as being somewhat clearer and simpler than what I have found in the text books.

Hagen's work on "Probabilities," *Grundzüge der Wahrscheinlichkeitsrechnung* is inaccessible to me, and my information as to his treatment of the question is derived solely from the account given in Mansfield Merriman's text-book.

In the 24th Volume of the *T.R.S.E.* there is a paper by Professor Tait "On the Law of Frequency of Error" in which, apparently without knowledge of Hagen's previous work, he makes a similar fundamental assumption, and makes use of Stirling's Theorem somewhat as I have done in working out the result. Had I seen Tait's paper before trying my hand at improving the details of the proof, it is probable that I should have been satisfied with his version; but as mine seems to me simpler in some details I have ventured to bring it before the Society, while disclaiming (like Professor Tait) any special qualification as an expert on the general subject of "Laws of Error."