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Invariant manifolds for near identity differentiable maps and splitting of separatrices

E. FONTICH†
Dept. de Matemàtica Aplicada, E.T.S.E.I.B., University Politècnica de Catalunya, Barcelona, Spain

C. SIMÓ
Dept. de Matemàtica Aplicada i Anàlisi, University de Barcelona, Barcelona, Spain

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Abstract. We consider families of differentiable diffeomorphisms with hyperbolic points, close to the identity, which tend to it when the parameter goes to zero.

We study the asymptotic behaviour of the invariant manifolds. Then we consider the case when there are homo-heteroclinic points and we find that the maximum separation between the invariant manifolds is of the order of some power of the parameter which is related to the degree of differentiability.

Finally the analogous case for flows is considered.

1. *Introduction and results*

It is known that for two dimensional diffeomorphisms, the existence of transversal homoclinic or heteroclinic points implies a very complicated dynamics in a neighbourhood of the invariant manifolds which is usually described as chaotic, stochastic, etc. In a given domain, the measure of this neighbourhood depends on the distance between the invariant manifolds. In this work we study the behaviour of the invariant manifolds for families of near identity diffeomorphisms with hyperbolic points. We find that the invariant manifolds tend, in a certain sense, when the diffeomorphisms tend to the identity, to the invariant manifolds of a critical point of a vector field which is constructed in association with the family.

The study of the behaviour of the invariant manifolds for families of diffeomorphisms when the eigenvalues at the hyperbolic point tend to 1 can be reduced to the above because, in such a case, after changes of variables and scalings, the family can be put as a near identity one.

Then we consider near identity families of two dimensional diffeomorphisms with a hyperbolic point and homoclinic points associated with it. In such a case the vector field associated with the family has a homoclinic orbit to which tend the invariant manifolds of the diffeomorphisms. We can prove that the separation

† Permanent address: Dept. de Matemàtica Aplicada i Anàlisi, University de Barcelona, Barcelona, Spain.

between the invariant manifolds in a neighbourhood of a given point is of the order of a power of the parameter related to the degree of differentiability.

Examples of such families are the Poincaré maps of two degrees of freedom Hamiltonian systems taking the energy as the parameter. Concrete examples are provided by the Hénon–Heiles Hamiltonian [**9**] and the restricted three body problem [**10**]. They also appear in the study of the behaviour of a diffeomorphism in a neighbourhood of an elliptic fixed point, near the invariant manifolds of the hyperbolic points given by the Poincaré–Birkhoff theorem [**3, 13**].

In a forthcoming paper [**4**] we shall consider the case of two dimensional analytic diffeomorphisms. The work was inspired by previous work by Lazutkin [**8**] on the standard map and uses some results of the present paper. It also provides several examples.

Now we give the main results.

Let $F_\varepsilon : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a family of diffeomorphisms, $\varepsilon \in (0, \varepsilon_0)$, U an open set, with $F_\varepsilon \in C^{r+1}(U)$ and $r \geq 1$. Let $h : U \subset \mathbb{R}^n \to \mathbb{R}^n$, $h \in C^{r+1}(U)$ and consider the equation

$$\dot{x} = h(x). \tag{1.1}$$

Let φ be the flow of (1.1). We define the family G_ε by $G_\varepsilon(x) = \varphi(a\varepsilon^\alpha, x)$ with $a > 0$ and $\alpha > 0$.

THEOREM A. *Suppose that for* $\varepsilon \in (0, \varepsilon_0)$ *we have*

(i) $p \in U$ *is a hyperbolic fixed point of* F_ε *and* G_ε.

(ii) *If* Spec $DF_\varepsilon(p) = \{\lambda_1, \ldots, \lambda_n\}$ *and* Spec $DG_\varepsilon(p) = \{\mu_1, \ldots, \mu_n\}$, *then* $\lambda_i = 1 + b_i\varepsilon^\alpha + o(\varepsilon^\alpha)$ *and* $\mu_i = 1 + b_i\varepsilon^\alpha + o(\varepsilon^\alpha)$ *with* $b_i < 0$ *for* $1 \leq i \leq l$ *and* $b_i > 0$ *for* $l < i \leq n$.

(iii) $\|F_\varepsilon - I\|_{r+1,U} \leq M\varepsilon^\alpha$,
$\|F_\varepsilon - G_\varepsilon\|_{r,U} \leq N\varepsilon^{\alpha+\beta}$, *with* $\beta > 0$.

(iv) *Let* $\delta > 0$ *and* $q \in W^s_{G_\varepsilon} \cap (U - \delta)$. *In a neighbourhood of q, the stable manifold of p,* $W^s_{G_\varepsilon}$, *can be represented as the graph of a function, g, from an open set of a subspace of* $\mathbb{R}^n$ *which contains l coordinate lines to another subspace which contains the remaining* $n - l$ *coordinate lines.*

Then there exist $\varepsilon_1 > 0$, $C > 0$ *and a neighbourhood V independent of* ε, *such that for* $\varepsilon \in (0, \varepsilon_1)$, $W^s_{F_\varepsilon}$ *can be represented locally, near q, as the graph of an* ε*-dependent function, f, of the same kind as g. Furthermore,*

$$\|f - g\|_{r,V} \leq C\varepsilon^\beta \quad \text{for } \varepsilon \in (0, \varepsilon_1).$$

THEOREM A′. *Under the hypothesis of Theorem A we have the same conclusions for the unstable invariant manifolds.*

As a consequence of these results we obtain the theorem which gives asymptotic bounds of the distance between the invariant manifolds.

THEOREM B. *Let* F_ε *and* G_ε *be as before and* $U \subset \mathbb{R}^2$. *Suppose that for* $\varepsilon \in (0, \varepsilon_0)$.

(i) p_1 *and* p_2 *are two hyperbolic fixed points of both* F_ε *and* G_ε.

(ii) *The eigenvalues* $\lambda_i^{(j)}$ *of* $DF_\varepsilon(p_j)$ *and* $\mu_i^{(j)}$ *of* $DG_\varepsilon(p_j)$, $j = 1, 2$, *are of the form*

$$\lambda_i^{(j)} = 1 + a_i^{(j)}\varepsilon^\alpha + o(\varepsilon^\alpha) \quad \text{and} \quad \mu_i^{(j)} - \lambda_i^{(j)} = o(\varepsilon^\alpha).$$

(iii) $\|F_\varepsilon - I\|_{r+1,U} \le M\varepsilon^\alpha$,
$\|F_\varepsilon - G_\varepsilon\|_{r,U} \le N\varepsilon^{\alpha+\beta}$, $\beta > 0$.

(iv) *F_ε has a heteroclinic point belonging to $W_1 \cap W_2$, where $W_1 = W^u(p_1)$, $W_2 = W^s(p_2)$ (homoclinic if $p_1 = p_2$). Suppose there exists a compact set $B \subset U$ (independent of ε) which contains the pieces of W_i from the fixed points up until the 'clinic' one.*

Then

(1) *The distance between W_1 and W_2 in a given domain is $O(\varepsilon^{\alpha r+\beta'})$ for all $0 < \beta' < \beta$.*

(2) *If $r = \infty$, the distance is $O(\varepsilon^k)$ for all $k \in Z^+$.*

On the other hand we consider the equations of the form $\dot{x} = \varepsilon f(x) + \varepsilon^2 g(x, t, \varepsilon)$, with slow dynamics, which appear in averaging theory [**5**]. Equivalently they can be written in the form

$$\dot{x} = f(x) + \varepsilon g(x, t/\varepsilon, \varepsilon). \tag{1.2}$$

We consider the case when

$$\dot{x} = f(x) \tag{1.3}$$

has a homoclinic orbit. Melnikov's method to evaluate the distance between the perturbed invariant manifolds, equivalent to the study of the first variational equations, does not work, in principle, in that case [**12, 5**]. In this case we obtain the

THEOREM C. *Consider the equations* (1.2) *and* (1.3) *satisfying*

(i) *f and g are of class C^{r+1} with respect to x on U, open set of $\mathbb{R}^2$ which contains the origin, and $D_x^k g$ is continuous on $U^* = U \times \mathbb{R} \times [0, \varepsilon_0)$ for $1 \le k \le r+1$.*

(ii) *$f(0) = 0$ and $Df(0)$ is hyperbolic.*

(iii) $\operatorname{tr} Df = \operatorname{tr} D_x g = 0$ *on U^*.*

(iv) *g is T-periodic with respect to the second variable and*

$$\int_0^T g(x, t, \varepsilon)\, dt = 0 \quad \text{on } U \times [0, \varepsilon_0).$$

(v) (1.3) *has a homoclinic orbit contained in U.*

Then

(1) *There exists $\varepsilon_1 > 0$ such that if $\varepsilon \in (0, \varepsilon_1)$ the equation* (1.2) *has a hyperbolic periodic orbit γ near the origin.*

(2) *The distance between the invariant manifolds of γ in a given domain is $O(\varepsilon^r)$. If $r = \infty$, it is $O(\varepsilon^k)$ for all $k \in Z^+$.*

In § 2 we give some definitions and previous results for the proofs of the next sections. In §§ 3, 4 and 5 we prove Theorems A, B and C.

2. *Definitions and previous results*

We begin by giving some standard definitions. Let F be a diffeomorphism from an open set U of $\mathbb{R}^n$ into its image. A fixed point p of F is called hyperbolic if $DF(p)$ is hyperbolic, that is, all its eigenvalues have a modulus different from 1. To any fixed point there are associated the so-called stable and unstable invariant manifolds which will be denoted by $W_F^s(p)$ and $W_F^u(p)$, respectively [**6, 7**]. When no confusion is possible we shall write W_F^s or simply W^s. If p_1 and p_2 are two hyperbolic points of F, a point $q \in W_F^s(p_1) \cap W_F^u(p_2)$ is called heteroclinic if $p_1 \neq p_2$ and homoclinic if $p_1 = p_2$ and $q \neq p_1$. The images of the homo-heteroclinic points are also homo-heteroclinic.

If $\dot{x} = f(x)$ is a differential equation, a critical point p is called hyperbolic if the real parts of the eigenvalues of $Df(p)$ are different from zero. In that case p has stable and unstable invariant manifolds. A solution is called a heteroclinic orbit if it tends, both for positive and negative time, to hyperbolic points. If the two hyperbolic points coincide it is called homoclinic.

If $\dot{x} = f(x, t)$ is non autonomous, a periodic orbit is called hyperbolic if the corresponding fixed point of the associated Poincaré map is hyperbolic. A hyperbolic periodic orbit also has invariant manifolds.

A heteroclinic orbit is a solution which tends both for positive and negative time to hyperbolic periodic orbits. If they coincide it is called homoclinic.

If φ is the solution of the equation $\dot{x} = f(x, t)$, that is, $d/dt\, \varphi(t, t_0, x) = f(\varphi(t, t_0, x), t)$ and $\varphi(t_0, t_0, x) = x$, we shall denote by $D_1\varphi$ the derivative of φ with respect to t and by $D_2\varphi$ the derivative of φ with respect to the initial conditions.

We shall use the following notations [1]. Let E_1, E_2, E_3 be Banach spaces and $L^j(E_1, E_2)$ be the space of the continuous j-linear maps from E_1 to E_2. We define, for $i \geq 1$ and $j_1, \ldots, j_i \geq 1$ with $j_1 + \cdots + j_i = k$,

$$\lambda^{j_1,\ldots,j_i} : L^i(E_2, E_3) \times L^{j_1}(E_1, E_2) \times \cdots \times L^{j_i}(E_1, E_2) \to L^k(E_1, E_3)$$

by

$$\lambda^{j_1,\ldots,j_i}(A, B_1, \ldots, B_i)(e_1, \ldots, e_k) = A(B_1(e_1, \ldots, e_{j_1}), \ldots, B_i(e_l, \ldots, e_k)),$$

where $l = j_1 + \cdots + j_{i-1} + 1$. $\lambda^{j_1,\ldots,j_i}$ is multilinear and $\|\lambda^{j_1,\ldots,j_i}\| \leq 1$.

We define

$$\mathrm{Sym}^k : L^k(E_1, E_2) \to L^k(E_1, E_2)$$

by

$$\mathrm{Sym}^k A = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma(A),$$

where S_k is the symmetric group of k elements and

$$\sigma(A)(e_1, \ldots, e_k) = A(e_{\sigma(1)}, \ldots, e_{\sigma(k)}).$$

Sym^k is linear and $\|\mathrm{Sym}^k\| \leq 1$.

We define

$$\alpha^{k+1} : L(E_2, E_1) \times \overset{k+1}{\cdots} \times L(E_2, E_1) \to L^k(L(E_1, E_2), L(E_2, E_1))$$

by

$$\alpha^{k+1}(X_1, \ldots, X_{k+1})(\psi_1, \ldots, \psi_k) = (-1)^k X_1 \circ \psi_1 \circ X_2 \circ \psi_2 \circ \cdots \circ X_k \circ \psi_k \circ X_{k+1}.$$

Finally we define

$$\mathrm{Inv} : GL(E_1, E_2) \to GL(E_2, E_1)$$

by $\mathrm{Inv}(\varphi) = \varphi^{-1}$.

Then if f, g are functions of class C^k

$$D^k(g\circ f)=\mathrm{Sym}^k\circ\sum_{i=1}^{k}\sum_{*}C_k(j_1,\ldots,j_i)\lambda^{j_1,\ldots,j_i}\circ(D^ig\circ f, D^{j_1}f,\ldots,D^{j_i}f),\quad(2.1)$$

where $\sum_*$ means, here and from now on, the sum with indices $j_1,\ldots,j_i\geq 1$ such that $j_1+\cdots+j_i=k$ and $C_k(j_1,\ldots,j_i)=C_k$ is an integer number which only depends on the indices $j_1,\ldots,j_i$. Also

$$D^k\mathrm{Inv}=k!\,\mathrm{Sym}^k\circ\alpha^{k+1}\circ(\mathrm{Inv},\overset{k+1}{\ldots},\mathrm{Inv}).$$

The following properties will be used

If $A_1\in GL(E_1,E_2)$ and $A_2\in L(E_1,E_2)$ then

$$\|A_1^{-1}\|^{-1}\|A_2\|\leq\|A_1A_2\|\leq\|A_1\|\cdot\|A_2\|.\quad(2.2)$$

If $A\in GL(E_1,E_2)$ and $\|A-I\|<1$ then

$$\|A^{-1}\|\leq\frac{1}{1-\|A-I\|}.\quad(2.3)$$

If A_1, $A_2\in GL(E_1,E_2)$ then

$$\|A_1^{-1}-A_2^{-1}\|\leq\|A_1^{-1}\|\cdot\|A_2^{-1}\|\cdot\|A_1-A_2\|.\quad(2.4)$$

Furthermore if $\|A_1-A_2\|\leq\|A_1^{-1}\|^{-1}$ then

$$\|A_2^{-1}\|\leq 1/(\|A_1^{-1}\|^{-1}-\|A_1-A_2\|).$$

If F is a C^k map on U we define $\|D^kF\|_U=\sup_{x\in U}\|D^kF(x)\|$ where $\|D^kF(x)\|$ is the norm of $D^kF(x)$ as a multilinear map, and

$$\|F\|_{k,U}=\max(\|F\|_U,\|DF\|_U,\ldots,\|D^kF\|_U).$$

$B(\rho,x)$ and $\bar{B}(\rho,x)$ will be the open and closed balls of radius ρ centered at x. If $x=0$ we shall write $B(\rho)$ and $\bar{B}(\rho)$.

If U is a set and $\delta>0$ we define $U+\delta=\bigcup_{x\in U}B(\delta,x)$ and $U-\delta=\{x\in U,\bar{B}(\delta,x)\subset U\}$.

Now we give some results which will be used later. Most of these results are well known without explicit bounds depending on some parameter ε. However, for the forthcoming sections it is essential to have such bounds. What is new is the consideration of families of maps which are close to the identity and tending to it when ε goes to zero.

LEMMA 2.1. *Let U be an open set of $\mathbb{R}^n$ and $F:U\to\mathbb{R}^n$ a homeomorphism such that $\|F-I\|_U<\varepsilon<1$. Then $F(U)\supset U-\varepsilon$.*

Proof. See the geometrical lemmas of [**2**]. □

LEMMA 2.2. *Let U be an open set of $\mathbb{R}^n$ and $F_\varepsilon:U\to\mathbb{R}^n$, $\varepsilon\in(0,\varepsilon_0)$, a family of diffeomorphisms of class C^1 such that $\|DF_\varepsilon\|_U<1+M\varepsilon^\alpha$ and $\|DF_\varepsilon^{-1}\|_{F_\varepsilon(U)}<1+M'\varepsilon^\alpha$, with $\alpha>0$ and $(1+M\varepsilon_0^\alpha)(1+M'\varepsilon_0^\alpha)<2$. Let $\delta>0$. Then we have*

(1) *If $x,y\in U-\delta$ and $\|x-y\|<\delta$ then*

$$\{(1-t)F_\varepsilon(x)+tF_\varepsilon(y),\,t\in[0,1]\}\subset F_\varepsilon(U)$$

and

$$\|F_\varepsilon(x)-F_\varepsilon(y)\| \geq \|DF_\varepsilon^{-1}\|_{F_\varepsilon(U)}^{-1}\|x-y\|.$$

(2) *If* $x \in U-\delta$ *and* $r<\delta/2$ *then*

$$\bar{B}((1+M'\varepsilon^\alpha)^{-1}r, F_\varepsilon(x)) \subset F_\varepsilon(B(r,x)).$$

Proof. We define

$$s=\sup\{t\in[0,1]:(1-u)F_\varepsilon(x)+uF_\varepsilon(y)\in F_\varepsilon(U) \quad \text{for } u\in[0,t)\}.$$

Since $F_\varepsilon(U)$ is open, $s>0$. We suppose that $s<1$. In that case we can suppose that $s\leq 1/2$. Let (s_k) be a sequence of elements of $[0,s)$ such that $\lim s_k=s$. Let $z_k=F_\varepsilon^{-1}((1-s_k)F_\varepsilon(x)+s_kF_\varepsilon(y))$. Since F_ε is a diffeomorphism and has its derivatives bounded, (z_k) tends to a point in FrU, where Fr denotes the boundary, so that $\sup_{k\geq 1}\|z_k-x\|\geq\delta$. On the other hand

$$\begin{aligned}\|z_k-x\| &< \|F_\varepsilon^{-1}((1-s_k)F_\varepsilon(x)+s_kF_\varepsilon(y))-F_\varepsilon^{-1}F_\varepsilon(x)\|\\ &\leq (1+M'\varepsilon^\alpha)\|s_k(F_\varepsilon(x)-F_\varepsilon(y))\|\\ &\leq (1+M'\varepsilon^\alpha)(1+M\varepsilon^\alpha)s_k\|x-y\|<\delta\end{aligned}$$

which produces a contradiction. So we have $s=1$. Now, applying the mean value theorem

$$\|x-y\|=\|F_\varepsilon^{-1}F_\varepsilon(x)-F_\varepsilon^{-1}F_\varepsilon(y)\|\leq\|DF_\varepsilon^{-1}\|_{F_\varepsilon(U)}\|F_\varepsilon(x)-F_\varepsilon(y)\|$$

so that (1) follows directly.

If $x\in U-\delta$ and $r\leq\|x-y\|<\delta/2$ then x, $y\in U-\delta/2$ and

$$r\leq\|x-y\|\leq\|DF_\varepsilon^{-1}\|_{F_\varepsilon(U)}\|F_\varepsilon(x)-F_\varepsilon(y)\|.$$

Hence if $y\in B(\delta/2,x)-B(r,x)$ then

$$F_\varepsilon(y)\notin\bar{B}((1+M'\varepsilon^\alpha)^{-1}r, F_\varepsilon(x))$$

which implies $\bar{B}((1+M'\varepsilon^\alpha)^{-1}r, F_\varepsilon(x))\subset F_\varepsilon(B(r,x))$. □

Proposition 2.3. *Let E and E' be Banach spaces, U an open set of E and F_ε, $G_\varepsilon: U\to E'$ two families of diffeomorphisms from U onto its image, of class C^{r+1}, $r\geq 1$. Let α, $\beta>0$.*

If there exist constants M, N such that, for $0<\varepsilon<\varepsilon_0$

$$\|F_\varepsilon-I\|_{r+1,U}<M\varepsilon^\alpha,$$
$$\|G_\varepsilon-I\|_{r+1,U}<M\varepsilon^\alpha,$$
$$\|F_\varepsilon-G_\varepsilon\|_{r,U}<N\varepsilon^{\alpha+\beta},$$

then given any $\delta>0$ there exist ε_1, M', N' such that for $0<\varepsilon<\varepsilon_1$ we have

$$\|F_\varepsilon^{-1}-I\|_{r+1,F_\varepsilon(U)}<M'\varepsilon^\alpha, \tag{2.5}$$
$$\|G_\varepsilon^{-1}-I\|_{r+1,G_\varepsilon(U)}<M'\varepsilon^\alpha, \tag{2.6}$$
$$\|F_\varepsilon^{-1}-G_\varepsilon^{-1}\|_{r,V}<N'\varepsilon^{\alpha+\beta}, \tag{2.7}$$

where $V=F_\varepsilon(U-\delta)\cap G_\varepsilon(U-\delta)$.

Proof. We shall not write the index ε corresponding to F_ε and G_ε. Let ε_1 be such that $0<\varepsilon_1^\alpha<\min(\varepsilon_0^\alpha,(2M)^{-1},\delta/(2M))$.

For $r=0$ (2.5) is obtained from

$$\|F^{-1}-I\|_{F(U)}=\|(I-F)\circ F^{-1}\|_{F(U)}=\|I-F\|_U<M\varepsilon^\alpha$$

and

$$\begin{aligned}\|DF^{-1}-I\|_{F(U)}&=\|(DF\circ F^{-1})^{-1}-I\|_{F(U)}\\&\le\|(DF\circ F^{-1})^{-1}\|_{F(U)}\|I-DF\circ F^{-1}\|_{F(U)}\\&\le(1-M\varepsilon^\alpha)^{-1}\|I-DF\|_U<2M\varepsilon^\alpha.\end{aligned}$$

For $r>0$ we proceed by induction. For $r=1$ suppose $\|F^{-1}-I\|_{1,F(U)}<\bar{M}'\varepsilon^\alpha$ and let $e\in E'$

$$\begin{aligned}D(DF^{-1})(e)&=D(\mathrm{Inv}\circ DF\circ F^{-1})(e)\\&=-(DF\circ F^{-1})^{-1}(D^2F\circ F^{-1}(DF\circ F^{-1})^{-1}(e))(DF\circ F^{-1})^{-1}\end{aligned}$$

and so $\|D^2F^{-1}\|_{F(U)}<(1+\bar{M}'\varepsilon^\alpha)^3M\varepsilon^\alpha$. We take $M'=\max(\bar{M}',(1+\bar{M}'\varepsilon_1^\alpha)^3M)$.

Now suppose $\|F^{-1}-I\|_{r,F(U)}<\bar{M}'\varepsilon^\alpha$. First, if $j_s\le r$, using (2.1)

$$\begin{aligned}\|D^{j_s}(DF\circ F^{-1})\|_{F(U)}&\le\sum_{i=1}^{j_s}\sum_{*}C_{j_s}\|D^{i+1}F\circ F^{-1}\|_{F(U)}\|D^{l_1}F^{-1}\|_{F(U)}\cdots\|D^{l_i}F^{-1}\|_{F(U)}\\&<K_{j_s}\varepsilon^\alpha,\quad l_1+\cdots+l_i=j_s,\ l_k\ge1\quad\text{for }1\le k\le i,\end{aligned}$$

where K_{j_s} depends on j_s and M'. Let $e_1,\ldots,e_r\in E'$. Then

$$\begin{aligned}&D^r(DF^{-1})(e_1,\ldots,e_r)\\&\quad=D^r(\mathrm{Inv}\circ DF\circ F^{-1})(e_1,\ldots,e_r)\\&\quad=\mathrm{Sym}^r\sum_{i=1}^{r}\sum_{*}C_k i!\,\lambda^{j_1,\ldots,j_i}\\&\qquad\circ[-(DF\circ F^{-1})^{-1}(D^{j_1}(DF\circ F^{-1})(e_1,\ldots,e_{j_1}))(DF\circ F^{-1})^{-1},\ldots,\\&\qquad-(DF\circ F^{-1})^{-1}(D^{j_i}(DF\circ F^{-1})(e_l,\ldots,e_r))(DF\circ F^{-1})^{-1}],\end{aligned}$$

with $l=j_1+\cdots+j_{i-1}+1$, and hence

$$\begin{aligned}\|D^{r+1}F^{-1}\|_{F(U)}&=\|D^{r+1}(F^{-1}-I)\|_{F(U)}\\&\le\sum_{i=1}^{r}\sum_{*}C_k i!\,(1+M'\varepsilon^\alpha)^{i+1}K_{j_1}K_{j_2}\cdots K_{j_i}\varepsilon^{i\alpha}<M''\varepsilon^\alpha.\end{aligned}$$

We take $M'=\max(\bar{M}',M'')$. The proof of (2.6) is identical to the one of (2.5). To prove (2.7) first consider

$$\begin{aligned}\|F\circ F^{-1}-G\circ G^{-1}\|&\ge\|F^{-1}-G^{-1}\|_V-\|(F-I)\circ F^{-1}\|_V-\|(G-I)\circ G^{-1}\|_V\\&\ge\|F^{-1}-G^{-1}\|_V-2M\varepsilon^\alpha\end{aligned}$$

so that $\|F^{-1}-G^{-1}\|_V<2M\varepsilon^\alpha$. If $x\in V$, since $F^{-1}(x),G^{-1}(x)\in U-\delta$ and $\|F^{-1}(x)-G^{-1}(x)\|<\delta$, by Lemmas 2.1 and 2.2 we have

$$\begin{aligned}0&=\|F(F^{-1}(x))-F(G^{-1}(x))\|-\|F(G^{-1}(x))-G(G^{-1}(x))\|\\&\ge\|DF^{-1}\|_V^{-1}\|F^{-1}(x)-G^{-1}(x)\|-\|F-G\|_U.\end{aligned}$$

Hence

$$\|F^{-1}-G^{-1}\|_V\le\|DF^{-1}\|_V\|F-G\|_U\le(1+M'\varepsilon^\alpha)N\varepsilon^{\alpha+\beta}.$$

Furthermore, using (2.2)

$$0=\|(DG^{-1})^{-1}DF^{-1}-(DG^{-1})^{-1}DG^{-1}\|_V-\|(DF^{-1})^{-1}DF^{-1}-(DG^{-1})^{-1}DF^{-1}\|_V$$
$$\geq \|DG^{-1}\|_V^{-1}\|DF^{-1}-DG^{-1}\|_V-\|DF\circ F^{-1}-DG\circ G^{-1}\|_V\|DF^{-1}\|_V.$$

Also, by Lemma 2.1

$$\|DF\circ F^{-1}-DG\circ G^{-1}\|_V\leq \|DF\circ F^{-1}-DF\circ G^{-1}\|_V+\|DF\circ G^{-1}-DG\circ G^{-1}\|_V$$
$$\leq \|D^2F\|_U\|F^{-1}-G^{-1}\|_V+\|DF-DG\|_U$$

and we obtain

$$\|DF^{-1}-DG^{-1}\|_V\leq (1+M'\varepsilon^\alpha)^2[M\varepsilon^\alpha N'\varepsilon^{\alpha+\beta}+N\varepsilon^{\alpha+\beta}].$$

This proves (2.7) for $r=1$. For $r>1$ we proceed by induction.

For $r=2$ we write $D^2F^{-1}-D^2G^{-1}$ in terms of the derivatives of F and G as in (2.5). Using (2.5), (2.6), (2.7) for $r=1$ and

$$\|D^2F\circ F^{-1}-D^2G\circ G^{-1}\|_V\leq \|D^2F\circ F^{-1}-D^2F\circ G^{-1}\|_V$$
$$+\|D^2F\circ G^{-1}-D^2G\circ G^{-1}\|_V$$
$$\leq \|D^3F\|_U\|F^{-1}-G^{-1}\|_V+\|D^2F-D^2G\|_U$$
$$\leq M\varepsilon^\alpha N'\varepsilon^{\alpha+\beta}+N\varepsilon^{\alpha+\beta}$$

we obtain $\|D^2F^{-1}-D^2G^{-1}\|_V<N''\varepsilon^{\alpha+\beta}$ for some N''. Now suppose $\|F^{-1}-G^{-1}\|_{r-1,V}<\bar{N}'\varepsilon^{\alpha+\beta}$.

First we consider

$$D^j(DF\circ F^{-1})-D^j(DG\circ G^{-1}),\quad j\leq r-1 \tag{2.8}$$

whose norm is, in V, less than

$$\sum_{i=1}^{j}\sum_{*}C_j\|D^{i+1}F\circ F^{-1}\times D^{j_1}F^{-1}$$
$$\times\cdots\times D^{j_i}F^{-1}-D^{i+1}G\circ G^{-1}\times D^{j_1}G^{-1}\times\cdots\times D^{j_i}G^{-1}\|_V.$$

Developing each term in telescopic form and noticing that $j_s\leq j\leq r-1$ we deduce by analogous reasonings as before that the norm of (2.8) is less than $K_j\varepsilon^{\alpha+\beta}$.

Finally, let $e_1,\ldots,e_{r-1}\in E'$,

$$[D^{r-1}(DF^{-1})-D^{r-1}(DG^{-1})](e_1,\ldots,e_{r-1})$$
$$=[D^{r-1}(\mathrm{Inv}\circ(DF\circ F^{-1}))-D^{r-1}(\mathrm{Inv}\circ(DG\circ G^{-1}))](e_1,\ldots,e_{r-1}).$$

Writing the differences in telescopic form and using (2.1), (2.5), (2.6) and (2.7) for $j\leq r-1$ we get $\|D^kF^{-1}-D^kG^{-1}\|_V<N''\varepsilon^{\alpha+\beta}$. We take $N'=\max(\bar{N}',N'')$. □

Consider the equation $\dot{x}=f(x)$ and let φ be its flow. We define two families of maps by

$$G_\varepsilon(x)=\varphi(a\varepsilon^\alpha,x),$$
$$F_\varepsilon(x)=x+a\varepsilon^\alpha f(x)+\varepsilon^{\alpha+\beta}g(x,\varepsilon),$$

with a, α, $\beta>0$.

PROPOSITION 2.4. *Let f be of class C^{r+1} in U (open set of $\mathbb{R}^n$), g of class C^{r+1} with respect to x in $U^* = U\times[0, \varepsilon_0)$, and $D_x^k g$ continuous on U^* for $0\le k\le r+1$. Then for all compact set $B\subset U$ there exist an open set U_1, $B\subset U_1\subset U$, and $\varepsilon_1>0$ such that for $\varepsilon\in(0, \varepsilon_1)$ we have*

(1) *G_ε is well defined on U_1,*
(2) $\|G_\varepsilon - I\|_{r+1,U_1} < M\varepsilon^\alpha$,
(3) *$F_{\varepsilon|U_1}$ is a diffeomorphism,*
(4) $\|F_\varepsilon - I\|_{r+1,U_1} < M'\varepsilon^\alpha$,
(5) $\|F_\varepsilon - G_\varepsilon\|_{r,U_1} < N'\varepsilon^{\alpha+\gamma}$ *where* $\gamma=\min(\alpha,\beta)$.

Proof. It is not restrictive to suppose that $a=1$. Since B is compact there exists $\delta>0$ such that $B_1=\overline{B+\delta}\subset U$. Let

$$U_1 = B+\delta/2,\ K_i=\|D^if\|_{B_1},\quad K_i'=\|D_x^ig\|_{B_1}$$

and ε_1 be such that

$$\varepsilon_1^\alpha < \min(\varepsilon_0^\alpha, \delta/(4K_0), \delta/(4K_0'), 1/(2K_1), 1/(2K_1')).$$

To prove (1) we recall that if $\varepsilon_1^\alpha<\delta/(4K_0)$ and $x\in U_1$, by the existence theorem for ordinary differential equations, the solution φ of $\dot x = f(x)$ with $\varphi(0,x)=x$ is defined for $|t|\le\varepsilon_1^\alpha$ and furthermore $\varphi(\varepsilon^\alpha, x)\in B_1$, for $0<\varepsilon<\varepsilon_1$.

(2) φ verifies $\varphi(t,x)=\varphi(0,x)+\int_0^t f(\varphi(s,x))\,ds$ and so

$$\|\varphi(\varepsilon^\alpha, x)-x\|\le\int_0^{\varepsilon^\alpha} K_0\,ds\le K_0\varepsilon^\alpha.$$

It is clear that $D^kG_\varepsilon(x)=D_2^k\varphi(\varepsilon^\alpha,x)$ and that $D_2^k\varphi$ satisfies the equation

$$\begin{aligned} D_1D_2^k\varphi(t,x) &= D_2^kD_1\varphi(t,x)=D_2^k(f\circ\varphi)(t,x) \\ &= \Lambda(t)D_2^k\varphi(t,x)+b_k(t),\quad k\ge1, \end{aligned} \tag{2.9}$$

with $D_2\varphi(0,x)=I$ and $D_2^k\varphi(0,x)=0$ for $k>1$, where $\Lambda(t): E^k\to E^k$ with $E^k = L^k(\mathbb{R}^n,\mathbb{R}^n)$, is defined by $\Lambda(t)\cdot A=\lambda^k\circ(Df(\varphi(t,x),A)$ and

$$\begin{aligned} b_k(t) = \mathrm{Sym}^k\circ\sum_{i=2}^{k}\sum_{*} C_k(j_1,\dots,j_i)\lambda^{j_1,\dots,j_i} \\ \circ(D^if\circ\varphi(t,x)\times D_2^{j_1}\varphi(t,x)\times\cdots\times D_2^{j_i}\varphi(t,x)) \end{aligned}$$

if $k>1$ and $b_1(t)=0$. We notice that b_k only contains derivatives of order less than k. Let $\varepsilon\in(0,\varepsilon_1)$ and $t\in[0,\varepsilon^\alpha]\cdot\Lambda(t)$ is linear and from

$$\|\Lambda(t)A\|\le\|Df(\varphi(t,x))\|\cdot\|A\|\le K_1\|A\|\quad\text{we get }\|\Lambda(t)\|\le K_1.$$

Furthermore

$$\|b_k(t)\|\le\sum_{i=2}^{k}\sum_{*} C_k\|D^if(\varphi(t,x))\|\cdot\|D_2^{j_1}\varphi(t,x)\|\cdots\|D_2^{j_i}\varphi(t,x)\|.$$

First, we consider the homogeneous linear equation

$$A'=\Lambda(t)\circ A \tag{2.10}$$

with $A(t)\in L(E^k,E^k)$ and $A(0)=I_{E^k}$. From (2.10) in integral form we have

$$\|A(t)\|\le 1+\int_0^t\|\Lambda\circ A(s)\|\,ds\le 1+K_1\int_0^t\|A(s)\|\,ds$$

and by Gronwall's lemma we get $\|A(t)\| \leq \exp(K_1 t)$. Then, for $k=1$, by (2.9)

$$\begin{aligned}\|D_2\varphi(t,x)-I\| &\leq \int_0^t \|Df(\varphi(s,x))\|\,ds \\ &\quad + \int_0^t \|Df(\varphi(s,x))\|\cdot\|D_2\varphi(s,x)-I\|\,ds \\ &\leq K_1 t + K_1\int_0^t \|D_2\varphi(s,x)-I\|\,ds\end{aligned}$$

and again by Gronwall's lemma

$$\|D_2\varphi(t,x)-I\| \leq (K_1 t)\exp(K_1 t) < K_1 e\varepsilon^\alpha.$$

In particular $\|D_2\varphi(t,x)\| \leq 1+e/2$.

For $k \geq 2$ we have

$$\begin{aligned}\|D_2^k\varphi(t,x)\| &\leq \int_0^t \|A(t-s)\|\cdot\|b_k(s)\|\,ds \\ &\leq \exp(K_1 t)\cdot\int_0^t \|b_k(s)\|\,ds. \qquad (2.11)\end{aligned}$$

Until now we have proved (2) for $r=0$. For $r>0$ we proceed by induction. We have

$$\|b_2(t)\| \leq \|D^2 f(\varphi(t,x))\|\cdot\|D_2\varphi(t,x)\|\cdot\|D_2\varphi(t,x)\| \leq K_2(1+e/2)^2.$$

Then by (2.11) $\|D_2^2\varphi(t,x)\| \leq M_2 t$ with $M_2 = e(1+e/2)^2 K_2$. If $\|G-I\|_{r,B} \leq \tilde{M}\varepsilon^\alpha$ for $r \geq 1$, it is clear that $\|b_{r+1}(t)\| < C_{r+1}$ with C_{r+1} independent of ε. Again by (2.11) we obtain $\|D_2^{r+1}\varphi(t,x)\| \leq M_{r+1}t$ with $M_{r+1} = eC_{r+1}$. We take $M = \max(\tilde{M}, M_{r+1})$.

(3) We want to see that $F_{\varepsilon|U_1}$ is injective. Let $x, y \in U_1$. $F_\varepsilon(x) = F_\varepsilon(y)$ implies that

$$\|x-y\| \leq \varepsilon^\alpha(\|f(x)\|+\|f(y)\|) + \varepsilon^{\alpha+\beta}(\|g(x,\varepsilon)\|+\|g(y,\varepsilon)\|).$$

By the definition of ε_1 we have $\|x-y\| < \delta$. Furthermore the segment $\overline{xy}$ is contained in $\bar{B}(\delta/2, x) \cup \bar{B}(\delta/2, y) \subset B_1$. Then if $x \neq y$,

$$\|x-y\| \leq \varepsilon^\alpha \|Df\|_{B_1}\|x-y\| + \varepsilon^{\alpha+\beta}\|D_x g\|_{B_1}\|x-y\| < \|x-y\|,$$

which gives a contradiction. Finally

$$\|DF_\varepsilon(x)-I\| = \|\varepsilon^\alpha Df(x) + \varepsilon^{\alpha+\beta}D_x g(x,\varepsilon)\| < 1 \qquad (2.12)$$

proves that $F_{\varepsilon|_{U_1}}$ is a diffeomorphism.

(4) It is a consequence of the fact that the derivatives of f and g are bounded on U_1.

(5) From

$$F_\varepsilon(x) - G_\varepsilon(x) = \int_0^{\varepsilon^\alpha} (f(x) - f(\varphi(s,x)))\,ds + \varepsilon^{\alpha+\beta}g(x,\varepsilon) \qquad (2.13)$$

and

$$\begin{aligned}\|f(\varphi(s,x)) - f(x)\| &= \|f(\varphi(s,x)) - f(\varphi(0,x))\| \\ &\leq \|Df\|_{B_1}\|\varphi(s,x)-\varphi(0,x)\| \leq K_1 K_0 s,\end{aligned}$$

we have

$$\|F_\varepsilon(x)-G_\varepsilon(x)\| < \int_0^{\varepsilon^\alpha} K_1K_0 s\,ds + \varepsilon^{\alpha+\beta}\|g(x,\varepsilon)\| \le (1/2)K_0K_1\varepsilon^{2\alpha} + K_0'\varepsilon^{\alpha+\beta}.$$

For $1\le k\le r$, by derivation of (2.13)

$$D^kF_\varepsilon(x)-D^kG_\varepsilon(x) = \int_0^{\varepsilon^\alpha} [D^kf(x)-D_2^k(f\circ\varphi)(s,x)]\,ds + \varepsilon^{\alpha+\beta}D_x^k g(x,\varepsilon).$$

By the bounds of the derivatives of g we only need to study the first term

$$D_2^k(f\circ\varphi)(s,x)-D^kf(x) = \{\lambda^{1,\overset{k}{\dots},1}\circ[D^kf(\varphi(s,x)), D_2\varphi(s,x), \dots, D_2\varphi(s,x)] - D^kf(x)\}$$

$$+\operatorname{Sym}^k\circ\sum_{i=1}^{k-1}\sum_{*} C_k\lambda^{j_1,\dots,j_i}\circ(D^if(\varphi(s,x)), D_2^{j_1}\varphi(s,x),\dots, D_2^{j_i}\varphi(s,x)).$$

We call Q_1 the first term and Q_2 the second one. We notice that we can write

$$D^kf(x) = \lambda^{1,\overset{k}{\dots},1}\circ(D^kf(\varphi(0,x)), D_2\varphi(0,x),\dots, D_2\varphi(0,x)).$$

Writing Q_1 in telescopic form we get $k+1$ terms; k of them have norm less than

$$K_k(1+e/2)^{k+1}\|D_2\varphi(s,x)-D_2\varphi(0,x)\|.$$

The other one has norm less than

$$\|D^{k+1}f\|_{B_1}\|\varphi(s,x)-\varphi(0,x)\| \le K_{k+1}K_0 s.$$

On the other hand, since all the terms of Q_2 contain at least one derivative of φ of order bigger than 1, each one is of order ε^α. Since the number of terms is independent of ε, Q_2 is of order ε^α. Hence there exists $C>0$ such that $\|Q_1+Q_2\|_{B_1} < C\varepsilon^\alpha$ and so $\|D^kF_\varepsilon - D^kG_\varepsilon\|_{U_1} \le C\varepsilon^{2\alpha} + K_k'\varepsilon^{\alpha+\beta}$. □

We will need a version of the unstable (and stable) manifold theorem valid uniformly for a family of diffeomorphisms near the identity. The following results whose proof can be found in [7] will be used.

THEOREM 2.5 (Lipschitz inverse function theorem). *Let E, E' be Banach spaces, $U\subset E$ and $V\subset E'$ open sets and $F: U\to V$ a homeomorphism such that F^{-1} is Lipschitz. Let $G: U\to E'$ be such that* $\operatorname{Lip}(G-F)\cdot\operatorname{Lip} F^{-1}<1$. *Then G is a homeomorphism onto an open set, and* $\operatorname{Lip} G^{-1} \le [(\operatorname{Lip} F^{-1})^{-1} - \operatorname{Lip}(G-F)]^{-1}$.

PROPOSITION 2.6. (Size estimate.) *Let X, Y be metric spaces and $F: X\to Y$ a bijective map such that* $(\operatorname{Lip} F^{-1})^{-1}\ge\lambda$. *Then $B(\lambda\rho, F(x))\subset F(B(\rho,x))$ for all $\rho>0$ and $x\in X$.*

We define the graph transform: Given $F: B(\rho)\subset E\to E$ with $E=E_1\times E_2$, $f_1, f_2: B_1(\rho)\to B_2(\rho)$ where $B_1(\rho)$ and $B_2(\rho)$ are balls of radius ρ centered at zero in E_1 and E_2 respectively, we write $\Gamma(f_1)=f_2$ if

$$F(\operatorname{graph}(f_1))\cap B(r) = \operatorname{graph}(f_2).$$

Putting $F_i=\pi_i\circ F$, $i=1,2$, where $\pi_i: E\to E_i$ is the canonical projection, we can write this condition as $\Gamma(f)\circ F_1\circ(I,f) = F_2\circ(I,f)$ where I is the identity on $B_1(\rho)$. If $F_1\circ(I,f)$ is invertible

$$\Gamma(f) = F_2\circ(I,f)\circ(F_1\circ(I,f))^{-1}|_{B_1(r)}.$$

If $\Gamma(f)=f$, graph f is an invariant set for F. The proof of the following proposition is parallel to that of the unstable manifold theorem in [7].

PROPOSITION 2.7. *Let E be a Banach space, $U\subset E$ an open set and $F_\varepsilon: U\to E$, $\varepsilon\in(0,\varepsilon_0)$, a family of homeomorphisms onto $F_\varepsilon(U)$ such that F_ε is Lipschitz and $F_\varepsilon(0)=0$. We suppose there exist a linear map $T_\varepsilon: E\to E$ with an invariant splitting $E=E_1\times E_2$, $T_{\varepsilon,1}=T_{\varepsilon|E_1}$ and $T_{\varepsilon,2}=T_{\varepsilon|E_2}$ verifying* $\max(\|T_{\varepsilon,1}^{-1}\|, \|T_{\varepsilon,2}\|)\le 1-c\varepsilon$ *and* $\mathrm{Lip}\,(F_\varepsilon-T_\varepsilon)<N\varepsilon$ *on $B(\rho)\subset U$.*

If $c\varepsilon_0<1$ and $N<c/2$, for all $\varepsilon\in(0,\varepsilon_0)$ there exists a unique Lipschitz map $f_\varepsilon: B_1(\rho)\to B_2(\rho)$ with $\mathrm{Lip}\,f_\varepsilon\le 1$ such that $\{(x,f_\varepsilon(x)), x\in B_1(x)\}\subset W^u_{F_\varepsilon}$.

Proof. Let $\varepsilon\in(0,\varepsilon_0)$. For the sake of simplicity we shall not write explicitly the dependence of F, T, T_1, T_2 and f on ε. We shall use the norm $\|x\|=\max(\|x_1\|,\|x_2\|)$ if $x_1=(x_1,x_2)\in E_1\times E_2$. We define

$$\Omega=\{f: B_1(\rho)\to B_2(\rho), f(0)=0, \mathrm{Lip}\,f\le 1\}.$$

With $\|f\|=\sup_{x\in B_1(\rho)}\|f(x)\|$, Ω is a complete metric space. Given $f\in\Omega$ we define $\psi f: B_1(\rho)\to E_1$ by $\psi f=F_1\circ(I,f)$, $\Phi f: B_1(\rho)\to E_2$ by $\Phi f=F_2\circ(I,f)$ and $\Gamma:\Omega\to\Omega$ by $\Gamma(f)=\Phi f\circ(\psi f)|^{-1}_{B_1(\rho)}$.

First we prove that $(\psi f)^{-1}$ exists and Γ is well defined. By Theorem 2.5, from $\mathrm{Lip}\,(\psi f-T_1)\le\mathrm{Lip}\,(F-T)<N\varepsilon<(c/2)\varepsilon_0<1/2$ we have that ψf is a homeomorphism onto an open set and

$$\mathrm{Lip}\,(\psi f)^{-1}\le((1-c\varepsilon)^{-1}-N\varepsilon)^{-1}<(1+(c/2)\varepsilon)^{-1}.$$

By Proposition 2.6 $\psi f(B_1(\rho))\supset B_1(\rho)$ and hence $\Phi f\cdot(\psi f)^{-1}$ is well defined. Furthermore

$$\mathrm{Lip}\,\Gamma(f)\le\mathrm{Lip}\,\Phi f\,\mathrm{Lip}\,(\psi f)^{-1}<(1-c\varepsilon+N\varepsilon)(1+(c/2)\varepsilon)^{-1}<1$$

and $\Gamma(f)(0)=0$ show that Γ is well defined.

Now we prove that Γ is a contraction on Ω. Let $f_1, f_2\in\Omega$. We observe that

$$\psi f_1-\psi f_2=(F_1-T_1)(I,f_1)-(F_1-T_1)(I,f_2)+T_1(I,f_1)-T_1(I,f_2)$$

and since the last difference is zero,

$$\|\psi f_1-\psi f_2\|\le\mathrm{Lip}\,(F_1-T_1)\|f_1-f_2\|<N\varepsilon\|f_1-f_2\|.$$

Also

$$\Phi f_1-\Phi f_2=(F_2-T_2)(I,f_1)-(F_2-T_2)(I,f_2)+T_2(I,f_1)-T_2(I,f_2)$$

and from that

$$\|\Phi f_1-\Phi f_2\|\le N\varepsilon\|f_1-f_2\|+(1-c\varepsilon)\|f_1-f_2\|.$$

We evaluate

$$\begin{aligned}\|\Gamma(f_1)-\Gamma(f_2)\| &\le \|\Phi f_1(\psi f_1)^{-1}-\Phi f_1(\psi f_2)^{-1}\|+\|\Phi f_1(\psi f_2)^{-1}-\Phi f_2(\psi f_2)^{-1}\|\\ &\le \mathrm{Lip}\,(\Phi f_1)\|(\psi f_1)^{-1}-(\psi f_2)^{-1}\|+\|\Phi f_1-\Phi f_2\|\\ &\le \mathrm{Lip}\,(\Phi f_1)\,\mathrm{Lip}\,(\psi f_1)^{-1}\|\psi f_1-\psi f_2\|+(1-c\varepsilon+N\varepsilon)\|f_1-f_2\|\\ &\le[(1-c\varepsilon+N\varepsilon)(1+(c/2)\varepsilon)^{-1}N\varepsilon+(1-(c/2)\varepsilon)]\|f_1-f_2\|\\ &<(1-c^2\varepsilon^2/(2+c\varepsilon))\|f_1-f_2\|.\end{aligned}$$

The unique fixed point of Γ is the function we looked for. □

PROPOSITION 2.8. *Under the hypothesis of Proposition* 2.7, *with* $\max(\|T_1\|, \|T_2^{-1}\|) < 1 - c\varepsilon$, F_ε^{-1} *Lipschitz and* $\mathrm{Lip}(F_\varepsilon^{-1} - T^{-1}) < N\varepsilon$ *on* $B(\rho) \subset B \subset F_\varepsilon(U)$, *we have that for all* $\varepsilon \in (0, \varepsilon_0)$ *there exists a unique Lipschitz function* $g: B_1(\rho) \to B_2(\rho)$, *with* $\mathrm{Lip}\, g \le 1$ *such that* $\{(x, g(x)), x \in B_1(\rho)\} \subset W^s_{F_\varepsilon}$.

Proof. We apply Proposition 2.7 to F_ε^{-1} changing the role of E_1 and E_2. We obtain the result using that

$$W^u_{F_\varepsilon^{-1}|B(\rho)} \subset W^s_{F_\varepsilon}. \qquad \square$$

Remarks. (1) The functions such that their graphs are the local invariant manifolds of F_ε are well defined on a ball $B_1(\rho)$ of radius independent of ε.

(2) By the usual invariant manifold theorems, if F_ε is of class C^r then f and g are of class C^r. In such a case $\|Df\|_{B_1(\rho)} \le 1$ and $\|Dg\|_{B_1(\rho)} \le 1$.

3. *Proof of Theorem A*

By Proposition 2.4 we can suppose that G_ε satisfies condition (iii) as F_ε does. It is not restrictive to suppose that $p = 0$ and $a = 1$. The proof has three main steps. The first one deals with the local case for $r = 0$ and $r = 1$. The second one deals with the global case and the third one with the case $r > 1$. To simplify the line of the proof we introduce some lemmas whose proof will appear at the end of the main proof. To simplify the notation we shall not write the dependence of F and G with respect to ε.

Step 1. We write $\mathbb{R}^n = E_1 \times E_2$ where $E_1 = \mathbb{R}^l$ and $E_2 = \mathbb{R}^{n-l}$.

LEMMA 3.1. *There exists a linear transformation S, independent of* ε, *such that* $\tilde{G} = S^{-1}GS$ *has the form*

$$\tilde{G}(x, y) = (B_1(x) + Q_1(x, y), B_2(y) + Q_2(x, y)), \quad x \in E_1, y \in E_2,$$

where B_1 *and* B_2 *are linear and* $Q_1(0,0) = Q_2(0,0) = 0$, $DQ_1(0,0) = DQ_2(0,0) = 0$ *and if* ε *is small enough there exists* $c > 0$ *such that* $\max(\|B_1\|, \|B_2^{-1}\|) < 1 - c\varepsilon^\alpha$. (*In fact* B_1, B_2, Q_1 *and* Q_2 *do depend on* ε.)

In $\mathbb{R}^n$ we shall use the norm $\|(x, y)\| = \max(\|x\|, \|y\|)$ for $x \in E_1$, $y \in E_2$. In E_1 and E_2 we shall use the euclidean norm. We define $\tilde{F} = S^{-1}FS$. $\tilde{F}$ has the form

$$\tilde{F}(x, y) = (B_1(x) + C_1(x, y) + P_1(x, y), B_2(y) + C_2(x, y) + P_2(x, y))$$

where C_1 and C_2 are linear and $P_1(0,0) = P_2(0,0) = 0$ and $DP_1(0,0) = DP_2(0,0) = 0$.

LEMMA 3.2. *There exist constants* M'' *and* N'' *such that*

$$\|\tilde{F} - I\|_{r+1, S^{-1}(U)} \le M''\varepsilon^\alpha,$$
$$\|\tilde{G} - I\|_{r+1, S^{-1}(U)} \le M''\varepsilon^\alpha,$$
$$\|\tilde{F} - \tilde{G}\|_{r, S^{-1}(U)} \le N''\varepsilon^{\alpha+\beta}.$$

LEMMA 3.3. *There exists* $\delta_1 > 0$ *such that* $S^{-1}(U - \delta) \subset S^{-1}(U) - 3\delta_1$.

From now on we shall write F and G instead of $\tilde{F}$ and $\tilde{G}$, U instead of $S^{-1}(U)$ and M and N instead of M'' and N''. We shall also write h instead of $S^{-1}hS$ and φ to denote the flow of $\dot{x} = S^{-1}hS(x)$.

From Lemma 3.2 it is easy to see that

$$\|C\| = \|(C_1, C_2)\| = \|DF(0,0) - DG(0,0)\| < N\varepsilon^{\alpha+\beta},$$
$$\|D^k(P_1, P_2)\|_U < M\varepsilon^\alpha \quad \text{for } 2 \le k \le r+1,$$
$$\|D^k(Q_1, Q_2)\|_U < M\varepsilon^\alpha \quad \text{for } 2 \le k \le r+1,$$
$$\|D(P_1, P_2) - D(Q_1, Q_2)\|_U < 2N\varepsilon^{\alpha+\beta},$$
$$\|D^k(P_1, P_2) - D^k(Q_1, Q_2)\|_U < N\varepsilon^{\alpha+\beta} \quad \text{for } 2 \le k \le r, \tag{3.1}$$

and

$$\|(P_1, P_2) - (Q_1, Q_2)\|_{B(\rho)} \le \|F - G\|_{B(\rho)} + \sup_{x \in B(\rho)} \|C(x)\| \le N(1+\rho)\varepsilon^{\alpha+\beta}.$$

Also, if $(x, y) \in B(\rho)$ with ρ small enough

$$\begin{aligned} \|D(P_1, P_2)(x, y)\| &\le \|D^2(P_1, P_2)\|_{B(\rho)} \|(x, y)\| \le M\varepsilon^\alpha \|(x, y)\|, \\ \|D(Q_1, Q_2)(x, y)\| &\le \|D^2(Q_1, Q_2)\|_{B(\rho)} \|(x, y)\| \le M\varepsilon^\alpha \|(x, y)\|. \end{aligned} \tag{3.2}$$

We define U_i as $U - i\delta_1$, $i = 1,2,3$.

LEMMA 3.4. *For ε small enough $U_2 \subset F(U_1) \cap G(U_1)$.*

By Proposition 2.3, F^{-1} and G^{-1} satisfy a condition like (iii) on $F(U_1) \cap G(U_1)$ with constants M' and N'.

LEMMA 3.5. *There exists $\rho_1 > 0$, independent of ε, such that W^s_F is, locally, the graph of a C^{r+1} function $f: B_1(\rho_1) \to E_2$ with $\|Df\|_{B_1(\rho_1)} \le 1$. In fact $\rho_1 < c/(4M)$.*

Since W^s_G does not depend on ε, there exists $\rho_2 > 0$ such that W^s_G is, locally, the graph of a C^{r+1} function $g: B_1(\rho_2) \to E_2$ with $\|Dg\|_{B_1(\rho_2)} \le 1$. Let ρ be $\min(\rho_1, \rho_2)$ and $d_k = \sup_{x \in B_1(\rho)} \|D^k g(x)\|$. We define

$$W^s_{F,l} = \{(x, f(x)), x \in B_1(\rho)\} \quad \text{and} \quad W^u_{G,l} = \{(x, g(x)), x \in B_1(\rho)\}.$$

Let R and R' be defined by $R(x, y) = (x, y + f(x))$ and $R'(x, y) = (x, y + g(x))$ in $B(\rho)$. They transform the stable invariant manifolds to the subspace E_1. We define $\tilde{F} = R^{-1}FR$ and $\tilde{G} = R'^{-1}GR'$. We have,

$$\begin{aligned} \tilde{F}(x, y) = (&B_1(x) + C_1(x, y + f(x)) + P_1(x, y + f(x)), B_2(y + f(x)) \\ &+ C_2(x, y + f(x)) + P_2(x, y + f(x)) \\ &- f(B_1(x) + C_1(x, y + f(x)) + P_1(x, y + f(x)))), \\ \tilde{G}(x, y) = (&B_1(x) + Q_1(x, y + g(x)), B_2(y + g(x)) + Q_2(x, y + g(x)) \\ &- g(B_1(x) + Q_1(x, y + g(x)))). \end{aligned}$$

The condition of E_1 being invariant can be expressed as the second components of $\tilde{F}$ and $\tilde{G}$ to be zero on $(x, 0)$. From that we have

$$\begin{aligned} f(x) &= B_2^{-1}[f(B_1(x) + C_1(x, f(x)) \\ &\quad + P_1(x, f(x))) - C_2(x, f(x)) - P_2(x, f(x))], \\ g(x) &= B_2^{-1}[g(B_1(x) + Q_1(x, g(x))) - Q_2(x, g(x))]. \end{aligned} \tag{3.3}$$

Now our objective is to find upper bounds of $\|f - g\|_{B_1(\rho)}$ and $\|Df - Dg\|_{B_1(\rho)}$.

We introduce the following notation

$$u=(x,f(x)), \qquad u'=(x,g(x)),$$
$$v=B_1(x)+C_1(u)+P_1(u), \qquad v'=B_1(x)+Q_1(u').$$

If $\|x\|<\rho$ then $\|u\|=\max(\|x\|,\|f(x)\|)<\|x\|<\rho$ and analogously $\|u'\|<\rho$. If we suppose $\varepsilon_1^\beta<c/4N$ then by Lemma 3.1, (3.1) and

$$\|P_1(u)\|=\|P_1(u)-P_1(0)\|\le\|DP_1\|_{B_1(\rho)}\|u\|$$
$$<M\varepsilon^\alpha\rho\|x\|<(c/4)\varepsilon^\alpha\|x\|$$

we have

$$\|v\|\le\|B_1(x)\|+\|C_1(u)\|+\|P_1(u)\|<(1-(c/2)\varepsilon^\alpha)\|x\|$$

and analogously $\|v'\|<(1-(3c/4)\varepsilon^\alpha)\|x\|$.

On the other hand we have

$$\|P_1(u)-Q_1(u')\|\le\|P_1(u)-Q_1(u)\|+\|Q_1(u)-Q_1(u')\|$$
$$\le N(1+\rho)\varepsilon^{\alpha+\beta}+(c/4)\varepsilon^\alpha\|f-g\|_{B_1(\rho)} \tag{3.4}$$

and the same bound for $\|P_2(u)-Q_2(u')\|$. Furthermore

$$\|v-v'\|\le\|P_1(u)-Q_1(u')\|+\|C_1(u)\|$$
$$\le N(1+2\rho)\varepsilon^{\alpha+\beta}+(c/4)\varepsilon^\alpha\|f-g\|_{B_1(\rho)}. \tag{3.5}$$

From (3.3), (3.4) and (3.5)

$$\|f(x)-g(x)\|\le\|B_2^{-1}\|[\|f(v)-g(v')\|+\|P_2(u)-Q_2(u')\|+\|C_2(u)\|]$$
$$\le\|B_2^{-1}\|[\|f-g\|_{B_1(\rho)}+\|g(v)-g(v')\|+N(1+2\rho)\varepsilon^{\alpha+\beta}$$
$$+(c/4)\varepsilon^\alpha\|f-g\|_{B_1(\rho)}]$$
$$\le(1-c\varepsilon^\alpha)[(1+(c/2)\varepsilon^\alpha)\|f-g\|_{B_1(\rho)}+2N(1+2\rho)\varepsilon^{\alpha+\beta}]$$
$$\le(1-(c/2)\varepsilon^\alpha)\|f-g\|_{B_1(\rho)}+K_1\varepsilon^{\alpha+\beta}.$$

Hence there exists $C_0>0$ such that $\|f-g\|_{B_1(\rho)}<C_0\varepsilon^\beta$.

For $\|x\|<\rho$ we also have

$$\|D_xP_1(u)-D_xQ_1(u')\|$$
$$\le\|D_xP_1(u)-D_xQ_1(u)\|+\|D_xQ_1(u)-D_xQ_1(u')\|$$
$$\le 2N\varepsilon^{\alpha+\beta}+M\varepsilon^\alpha\|f-g\|\le K_2\varepsilon^{\alpha+\beta},$$

$$\|D_yP_1(u)Df(x)-D_yQ_1(u')Dg(x)\|$$
$$\le\|D_yP_1(u)Df(x)-D_yQ_1(u)Df(x)\|$$
$$+\|D_yQ_1(u)Df(x)-D_yQ_1(u')Df(x)\|$$
$$+\|D_yQ_1(u')Df(x)-D_yQ_1(u')Dg(x)\|$$
$$\le\|DP_1-DQ_1\|_U+\|DQ_1(u)-DQ_1(u')\|+\|DQ_1(u')\|\,\|Df(x)-Dg(x)\|$$
$$\le(c/4)\varepsilon^\alpha\|Df-Dg\|_{B_1(\rho)}+K_3\varepsilon^{\alpha+\beta},$$

and the same bounds for $\|D_xP_2(u)-D_xQ_2(u')\|$ and

$$\|D_yP_2(u)Df(x)-D_yQ_2(u')Dg(x)\|.$$

Now we evaluate

$$\|Df(x)-Dg(x)\|$$
$$\leq \|B_2^{-1}\{Df(v)[B_1+D_xP_1(u)+D_yP_1(u)Df(x)+D_xC_1(u)+D_yC_1(u)Df(x)]$$
$$-D_xP_2(u)-D_yP_2(u)Df(x)-D_xC_2(u)-D_yC_2(u)Df(x)$$
$$-[Dg(v')(B_1+D_xQ_1(u')+D_yQ_1(u')Dg(x))-D_xQ_2(u')$$
$$-D_yQ_2(u')Dg(x)]\}\|.$$

We introduce more notation

$$e_1=B_1+D_xP_1(u)+D_yP_1(u)Df(x)+D_xC_1(u)+D_yC_1(u)Df(x),$$
$$e_1'=B_1+D_xQ_1(u')+D_yQ_1(u')Dg(x),$$
$$e_2=Df(v)e_1-Dg(v')e_1',$$
$$e_3=D_xP_2(u)-D_xQ_2(u'),$$
$$e_4=D_yP_2(u)Df(x)-D_yQ_2(u')Dg(x),$$
$$e_5=D_xC_2(u)+D_yC_2(u)Df(x),$$

so that

$$\|Df(x)-Dg(x)\|\leq\|B_2^{-1}\|[\|e_2\|+\|e_3\|+\|e_4\|+\|e_5\|].$$

We have

$$\|e_1'\|\leq\|B_1\|+\|DQ_1\|+\|DQ_1\|\cdot\|Dg\|\leq 1-(c/2)\varepsilon^{\alpha},$$
$$\|e_1-e_1'\|\leq\|DP_1-DQ_1\|+(c/4)\varepsilon^{\alpha}\|Df-Dg\|+K_3\varepsilon^{\alpha+\beta}+\|DC_1\|+\|DC_1\|\cdot\|Df\|$$
$$\leq(c/4)\varepsilon^{\alpha}\|Df-Dg\|+K_4\varepsilon^{\alpha+\beta},$$
$$\|e_2\|\leq\|Df(v)e_1-Df(v)e_1'\|+\|Df(v)e_1'-Dg(v)e_1'\|+\|Dg(v)e_1'-Dg(v')e_1'\|$$
$$\leq(1-(c/4)\varepsilon^{\alpha})\|Df-Dg\|+K_5\varepsilon^{\alpha+\beta}.$$

And the following is easily obtained

$$\|e_3\|<K_6\varepsilon^{\alpha+\beta},$$
$$\|e_4\|<(c/4)\varepsilon^{\alpha}\|Df-Dg\|+K_7\varepsilon^{\alpha+\beta},$$
$$\|e_5\|<K_8\varepsilon^{\alpha+\beta}.$$

Hence

$$\|Df(x)-Dg(x)\|\leq(1-c\varepsilon^{\alpha})[\|Df-Dg\|_{B_1(\rho)}+K_9\varepsilon^{\alpha+\beta}],$$

from which we find $\|Df-Dg\|_{B_1(\rho)}\leq C_1\varepsilon^{\beta}$.

Step 2. Now we want to obtain bounds for the separation of the invariant manifolds while they are in U_3.

Let $p\in W_G^s\cap U_3$. We note that W_G^s is the stable invariant manifold of the origin of $\dot{x}=h(x)$ and so does not depend on ε. There exists a neighbourhood V of p such that the connex component of $V\cap W_G^s$ which contains p is the graph of a function from an open set of E_3 to E_4, where E_3 and E_4 are vectorial subspaces of $\mathbb{R}^n$ of dimensions l an $n-l$ and contain l and $n-l$ coordinate lines respectively. Note that they may not coincide with E_1 and E_2. Let π_3 and π_4 be the projection operators onto E_3 and E_4. We suppose that $\pi_3(V)$ is convex.

Since $\lim_{t\to\infty}\varphi(t,p)=0$ there exists t_0 such that if $t>t_0$, $\varphi(t,p)\in B(\rho)$. Let $T>t_0$. By continuity there exists an open set V_1 of $W_G^s\cap B(\rho)$ such that $\pi_1(V_1)$ is convex and $p\in\varphi(-T, V_1)\subset V$. Let $V_1' = V_1 - L\varepsilon_1^\alpha$ where $L=\sup_{U_1}\|h\|$ and $V_2=\varphi(-T, V_1')$. We are going to study the proximity of the invariant manifolds as graphs of functions defined on $\pi_3(V_2)$. We suppose that the piece of W_G^s from the origin to V_2 is contained in U_3. Let $\eta=(N'+N'M'e^{TM'})Te^{TM'}$ and suppose that $\varepsilon_1^\beta<\delta_1/2\eta$. Now let N be the integer part of T/ε^α and $V_3=G^N(V_2)$ (which depends on ε). It is clear that $V_3\subset V_1$.

We define $\Phi_i=\pi_i\circ G^{-N}\circ(I,g)$ and $\psi_i=\pi_i\circ F^{-N}\circ(I,f)$, $i=3, 4$.

V_2 is the graph of

$$g_e=\Phi_4\circ\Phi_3^{-1}|_{\pi_3(V_2)},$$

where $\pi_3\circ G^{-N}\circ(I,g)(x)=\pi_3\varphi(-N\varepsilon^\alpha,(x,g(x)))$ is invertible because we have supposed that W_G^s is the graph of a function from E_3 to E_4 and its inverse is defined on $\pi_3(V)$. Let $c_1\ge\|D\Phi_i\|_{\pi_1(V_1)}$ for $i=3, 4$, $c_2\ge\|D^2\Phi_3\|_{\pi_1(V_1)}$ and $c_3\ge\|D\Phi_3^{-1}\|_{\pi_3(V)}$. They can be chosen independently of ε. Now we are going to see that

$$f_e=\psi_4\circ\psi_3^{-1}$$

is well defined in a suitable domain.

LEMMA 3.6. *If* $G^{-k}(x)\in U_2-2\eta\varepsilon^\beta$ *for* $0\le k\le N$ *and* $\|x-y\|<C_0\varepsilon^\beta$, *then for* $0\le k\le N$

$$G^{-k}(y)\in U_1-\eta\varepsilon^\beta, \tag{3.6}$$

$$\|DG^{-k}(y)\|<(1+M'\varepsilon^\alpha)^k<e^{TM'}, \tag{3.7}$$

$$F^{-k}(y)\in U_1, \tag{3.8}$$

$$\|F^{-k}(y)-G^{-k}(y)\|<k(1+M'\varepsilon^\alpha)^kN'\varepsilon^{\alpha+\beta}<Te^{TM'}N'\varepsilon^\beta<\eta\varepsilon^\beta, \tag{3.9}$$

$$\|D^2G^{-k}(y)\|<k(1+M'\varepsilon^\alpha)^{2k}M'\varepsilon^\alpha<Te^{2TM'}M', \tag{3.10}$$

$$\begin{aligned}\|DF^{-k}(y)-DG^{-k}(y)\|&<[k(1+M'\varepsilon^\alpha)^kN'+k^2(1+M'\varepsilon^\alpha)^{2k}N'M'\varepsilon^\alpha]\varepsilon^{\alpha+\beta}\\&<\eta\varepsilon^\beta.\end{aligned} \tag{3.11}$$

We have

$$\begin{aligned}\|\psi_i-\Phi_i\|_{\pi_1(V_1)}&\le\|\pi_1\circ F^{-N}\circ(I,f)-\pi_i\circ G^{-N}\circ(I,f)\|_{\pi_1(V_1)}\\&\quad+\|\pi_i\circ G^{-N}\circ(I,f)-\pi_i\circ G^{-N}\circ(I,g)\|_{\pi_1(V_1)}\\&\le\|F^{-N}\circ(I,f)-G^{-N}\circ(I,f)\|_{\pi_1(V_1)}\\&\quad+\|G^{-N}\circ(I,f)-G^{-N}(I,g)\|_{\pi_1(V_1)}.\end{aligned}$$

By (3.9) the first term is less than $\eta\varepsilon^\beta$ since for $0\le k\le N$, $G^{-k}(x, g(x))\in U_2-2\eta\varepsilon^\beta$ and $\|(x,f(x))-(x,g(x))\|\le C_0\varepsilon^\beta$ for $x\in B_1(\rho)$. By (3.7) the second one is less than $e^{TM'}\|f-g\|_{B_1(\rho)}<\eta\varepsilon^\beta$ and so we get $\|\psi_i-\Phi_i\|_{\pi_1(V_1)}<2\eta\varepsilon^\beta$.

Also we have

$$\begin{aligned}&\|D\psi_i-D\Phi_i\|_{\pi_1(V_1)}\\&\quad\le\|DF^{-N}\circ(I,f)(I,Df)-DG^{-N}\circ(I,g)(I,Df)\|_{\pi_1(V_1)}\\&\qquad+\|DG^{-N}\circ(I,g)(I,Df)-DG^{-N}\circ(I,g)(I,Dg)\|_{\pi_1(V_1)}\\&\quad\le[\|DF^{-N}\circ(I,f)-DG^{-N}\circ(I,f)\|_{\pi_1(V_1)}\\&\qquad+\|DG^{-N}\circ(I,f)-DG^{-N}\circ(I,g)\|_{\pi_1(V_1)}]\|(I,Df)\|_{\pi_1(V_1)}\\&\qquad+\|DG^{-N}\circ(I,g)\|_{\pi_1(V_1)}\|Df-Dg\|_{B_1(\rho)}.\end{aligned}$$

As before, since $G^{-k}(x, g(x)) \in U_2 - 2\eta\varepsilon^{\beta}$ for $0 \le k \le N$ and $\|(x, f(x)) - (x, g(x))\| \le C_0\varepsilon^{\beta}$ for $x \in B_1(\rho)$, by (3.11), (3.10) and (3.7) we obtain

$$\|D\psi_i - D\Phi_i\|_{\pi_1(V)} \le (N' + N'M'e^{TM'})Te^{TM'}\varepsilon^{\beta} + Te^{2TM'}M'\|f - g\|_{B_1(\rho)}$$
$$+ e^{TM'}\|Df - Dg\|_{B_1(\rho)} \le K_{10}\varepsilon^{\beta}.$$

LEMMA 3.7. *If ε_1 is small enough*
(1) *ψ_3 is a diffeomorphism from $\pi_1(V_3)$ onto its image and $\|D\psi_3^{-1}\|_{\psi_3(\pi_1(V_3))} \le K_{11}$.*
(2) *There exists $\delta' > 0$ such that $\pi_3(V_2) - \delta'\varepsilon^{\beta} \subset \psi(\pi_1(V_3))$.*

Let $\eta' > 0$. We define $V_4 = \pi_3(V_2) - \eta'$. (We suppose that η' is such that $V_4 \ne \emptyset$.) We suppose that ε_1 is such that $\delta'\varepsilon_1^{\beta} < \eta'$. By Lemma 3.7, ψ_3^{-1} is well defined on V_4 and $\psi_3^{-1}(V_4) \subset \pi_1(V_3)$. So we have that f_e is well defined on V_4. Now we are going to bound $\|f_e - g_e\|_{V_4}$ and $\|Df_e - Dg_e\|_{V_4}$.

LEMMA 3.8. *If ε_1 is small enough,*

$$\|\psi_3^{-1} - \Phi_3^{-1}\|_{V_4} \le K_{12}\varepsilon^{\beta},$$
$$\|D\psi_3^{-1} - D\Phi_3^{-1}\|_{V_4} \le K_{13}\varepsilon^{\beta}.$$

Finally we have

$$\|f_e - g_e\|_{V_4} = \|\psi_4 \circ \psi_3^{-1} - \Phi_4 \circ \Phi_3^{-1}\|_{V_4} \le \|\psi_4 \circ \psi_3^{-1} - \Phi_4 \circ \psi_3^{-1}\|_{V_4}$$
$$+ \|\Phi_4 \circ \psi_3^{-1} - \Phi_4 \circ \Phi_3^{-1}\|_{V_4} \le \|\psi_4 - \Phi_4\|_{\pi_1(V_1)}$$
$$+ \|D\Phi_4\|_{\pi_1(V_1)}\|\psi_3^{-1} - \Phi_3^{-1}\|_{V_4} \le K_{14}\varepsilon^{\beta},$$

and

$$\|Df_e - Dg_e\|_{V_4} = \|D\psi_4 \circ \psi_3^{-1}D\psi_3^{-1} - D\Phi_4 \circ \Phi_3^{-1}D\Phi_3^{-1}\|_{V_4}$$
$$\le \|D\psi_4 \circ \psi_3^{-1}D\psi_3^{-1} - D\psi_4 \circ \psi_3^{-1}D\Phi_3^{-1}\|_{V_4}$$
$$+ \|D\psi_4 \circ \psi_3^{-1}D\Phi_3^{-1} - D\Phi_4 \circ \psi_3^{-1}D\Phi_3^{-1}\|_{V_4}$$
$$+ \|D\Phi_4 \circ \psi_3^{-1}D\Phi_3^{-1} - D\Phi_4 \circ \Phi_3^{-1}D\Phi_3^{-1}\|_{V_4}$$
$$\le \|D\psi_4\|_{\pi_1(V_1)}\|D\psi_3^{-1} - D\Phi_3^{-1}\|_{V_4}$$
$$+ \|D\psi_4 - D\Phi_4\|_{\pi_1(V_1)}\|D\Phi_3^{-1}\|_{V_4}$$
$$+ \|D\Phi_4\|_{\pi_1(V_1)}\|\psi_3^{-1} - \Phi_3^{-1}\|_{V_4}\|D\Phi_3^{-1}\|_{V_4} \le K_{15}\varepsilon^{\beta}.$$

Since now we have proved the theorem for $r = 1$. The next step proves the general case.

Step 3. We are going to continue by induction. Suppose the theorem is true for $r - 1$. We define $\Delta F: U \times \mathbb{R}^n \to F(U) \times \mathbb{R}^n$ and $\Delta G: U \times \mathbb{R}^n \to G(U) \times \mathbb{R}^n$ by $\Delta F(y, v) = (F(x), DF(x) \cdot v)$ and $\Delta G(x, v) = (G(x), DG(x) \cdot v)$. In fact they are families of diffeomorphisms depending on ε. It is obvious that

$$\Delta F^{-1}(y, w) = (F^{-1}(y), DF^{-1}(y) \cdot w)$$

and

$$\Delta G^{-1}(y, w) = (G^{-1}(y), DG^{-1}(y) \cdot w).$$

It is clear that $(0, 0)$ is a fixed point of ΔF and furthermore it is hyperbolic because

$$D(\Delta F)(0,0) = \begin{pmatrix} DF(0) & 0 \\ 0 & DF(0) \end{pmatrix}.$$

Now we consider $W^s_{\Delta F}$.

LEMMA 3.9. *If W^s_F can be represented as the graph of a function $f: U \subset E_3 \to E_4$ in a neighbourhood of p, then $W^s_{\Delta F}$ can be represented as the graph of a function $U \times \mathbb{R}^l \to E_4 \times \mathbb{R}^l$ defined by $(x, v) \to (f(x), Df(x).v)$ in a neighbourhood of (p, v_p).*

It is clear that ΔF is a family of C^r diffeomorphisms and ΔG is the flow time ε^α of the vector field (h, Dh) defined by $(h, Dh)(x, v) = (h(x), Dh(x)v)$ which is also of class C^r. We are going to verify the hypotheses (i), (ii), (iii) and (iv) for ΔF and ΔG in $U \times B(\rho)$ where $B(\rho) \subset \mathbb{R}^n$. In fact (i) and (ii) are immediate. To prove (iii) and (iv), let $(x, v) \in U \times B(\rho)$ and $u = (u_1, u_2) \in \mathbb{R}^{2n}$ with $\|u\| \le 1$. We have

$$\begin{aligned} \|\Delta F(x, v) - (x, v)\| &= \|(F(x) - x, DF(x) \cdot v - v)\| \\ &\le M\varepsilon^\alpha + M\varepsilon^\alpha \|v\| < M(1+\rho)\varepsilon^\alpha, \\ \|(D(\Delta F)(x, v) - I)u\| &= \|((DF(x) - I)u_1, D^2F(x)(v, u_1) + (DF(x) - I)u_2)\| \\ &\le M\varepsilon^\alpha \|v\| + M\varepsilon^\alpha < M(1+\rho)\varepsilon^\alpha, \\ \|D^k(\Delta F)(x, v)\| &= \|(D^kF(x), D^{k+1}F(x)(v, \cdot))\| \\ &\le M(1+\rho)\varepsilon^\alpha, \quad \text{if } 2 \le k \le r, \end{aligned}$$

$$\begin{aligned} &\|D^k(\Delta F)(x, v) - D^k(\Delta G)(x, v)\| \\ &= \|(D^kF(x) - D^kG(x), (D^{k+1}F(x) - D^{k+1}G(x))(v, \cdot))\| \\ &\le N(1+\rho)\varepsilon^{\alpha+\beta}, \quad \text{if } 0 \le k \le r-1. \end{aligned}$$

Then if ε_1 is small enough and $\tilde{f}$ and $\tilde{g}$ are the functions such that their graphs are $W^s_{\Delta F}$ and $W^s_{\Delta G}$ in a neighbourhood of a point $(p, v_p) \in W^s_G$ we have that $\|\tilde{f} - \tilde{g}\|_{r-1} < C'\varepsilon^\beta$ which implies that $\|D^{r-1}\tilde{f} - D^{r-1}\tilde{g}\| < C'\varepsilon^\beta$ and $\|D^rf - D^rg\| < C''\varepsilon^\beta$. Then $\|f - g\|_r < C\varepsilon^\beta$ with $C = \text{Max}(C', C'')$. □

Proof of Lemma 3.1. By the definition of G we have that $DG(0,0) = D_2\varphi(\varepsilon^\alpha, 0, 0)$. $D_2\varphi$ verifies $D_1D_2\varphi = Dh \circ \varphi D_2\varphi$ with the condition $D_2\varphi(0, x, y) = I$. By (i) $G(0,0) = (0,0)$ and so $\varphi(t, 0, 0) = (0, 0)$ is a solution of $\dot{x} = h(x)$. Then $D_2\varphi(t, 0, 0) = \exp(Dh(0,0)t)$. Let $b = \min(-b_1, \ldots, -b_l, b_{l+1}, \ldots, b_n)$. The linear transformation S which transforms the matrix $Dh(0,0)$ into its modified Jordan normal form in such a way that the non-diagonal terms are $b/2$ instead of 1 and the boxes corresponding to negative eigenvalues are located in the first term satisfies the conditions of the lemma. Indeed, first we note that

$$D\tilde{G}(0,0) = S^{-1}DG(0,0)S = S^{-1}(\exp(Dh(0,0)\varepsilon^\alpha))S = \exp(S^{-1}Dh(0,0)S\varepsilon^\alpha).$$

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A box of the normal form of $Dh(0,0)$ associated with the eigenvalue ν corresponds to a box

$$B_\nu = e^{\nu\varepsilon^\alpha}\begin{pmatrix} 1 & & & \\ \frac{b}{2}\varepsilon^\alpha & 1 & & \\ \left(\frac{b}{2}\right)^2\frac{\varepsilon^{2\alpha}}{2} & \frac{b}{2}\varepsilon^\alpha & 1 & \\ \left(\frac{b}{2}\right)^{k-1}\frac{\varepsilon^{(k-1)\alpha}}{(k-1)!} & \cdots & & 1 \end{pmatrix}$$

in the matrix of $D\tilde{G}(0,0)$. The boxes of the matrix of $B'_\nu = (D\tilde{G}(0,0))^{-1}$ are of a similar form, changing the sign in ν and b.

It is clear that $\|B_\nu\| = e^{\nu\varepsilon^\alpha}(1+(b/2)\varepsilon^\alpha + O(\varepsilon^{2\alpha}))$ and

$$\|B'_\nu\| = e^{-\nu\varepsilon^\alpha}(1+(b/2)\varepsilon^\alpha + O(\varepsilon^{2\alpha})).$$

We take $B_1 = D\tilde{G}(0,0)|_{E_1}$ and $B_2 = D\tilde{G}(0,0)|_{E_2}$. Since $e^{\nu\varepsilon^\alpha}$ is an eigenvalue of $DG(0,0)$, if $\nu<0$ then $\nu<-b$ and $\|B_\nu\|<1-(b/2)\varepsilon^\alpha+O(\varepsilon^{2\alpha})$ and if $\nu>0$ then $\nu>b$ and $\|B'_\nu\|<1-(b/2)\varepsilon^\alpha+O(\varepsilon^{2\alpha})$. Since (using the euclidean norm) $\|B_1\|$ is less than the maximum of the norms of $\|B_\nu\|$ with $\nu<0$ and analogously for $\|B_2^{-1}\|$ with B'_ν with $\nu>0$ we have that $\|B_1\|$, $\|B_2^{-1}\|<1-(b/2)\varepsilon^\alpha+O(\varepsilon^{2\alpha})$. Then if ε_1 is small enough there exists $c>0$ satisfying the lemma. □

Proof of Lemma 3.2. From $\tilde{F}-I = S^{-1}\circ F\circ S - I = S^{-1}\circ(F-I)\circ S$ we have

$$\|\tilde{F}-I\|_{S^{-1}(U)} \le \|S^{-1}\|\cdot\|F-I\|_U < \|S^{-1}\|M\varepsilon^\alpha.$$

From $D\tilde{F}-I = D(S^{-1}FS)-I = S^{-1}\cdot(DF\circ S - I)\cdot S$ we have

$$\|D\tilde{F}-I\|_{S^{-1}(U)} \le \|S\|\cdot\|S^{-1}\|\cdot\|DF-I\|_U \le \|S\|\cdot\|S^{-1}\|M\varepsilon^\alpha.$$

We have $D^2\tilde{F} = S^{-1}\cdot(DF\circ S)\cdot S\cdot S$ and by induction $D^k\tilde{F} = S^{-1}\cdot(D^kF\circ S)\cdot S^k$ so that

$$\|D^k\tilde{F}\|_{S^{-1}(U)} \le \|S\|^k\|S^{-1}\|\cdot\|D^kF\|_U \le \|S\|^k\|S^{-1}\|M\varepsilon^\alpha.$$

The same happens for $\tilde{G}$. The existence of M'' is obvious. Finally, from

$$\|D^k\tilde{F}-D^k\tilde{G}\|_{S^{-1}(U)} = \|S^{-1}\cdot(D^kF\circ S - D^kG\circ S)\cdot S^k\|_{S^{-1}(U)} \le \|S\|^k\|S^{-1}\|N\varepsilon^{\alpha+\beta}$$

we get the existence of N''. □

Proof of Lemma 3.3. Let $z\in S^{-1}(U-\delta)$ and $x=S(z)$. Then $\bar{B}(\delta,x)\subset U$. By Proposition 2.6

$$\bar{B}(\|S\|^{-1}\delta, S^{-1}(x)) \subset S^{-1}(\bar{B}(\delta,x)) \subset S^{-1}(U)$$

and so $S^{-1}(x) = z\in S^{-1}(U)-\|S\|^{-1}\delta$.

We take $\delta_1 = (1/3)\|S\|^{-1}\delta$. □

Proof of Lemma 3.4. From (iii) and the Proposition 2.1 if $\varepsilon_1^\alpha<\delta/M$ we have that $U-2\delta\subset U-\delta-M\varepsilon^\alpha\subset F(U-\delta)$ and in the same way $U-2\delta\subset G(U-\delta)$. □

Proof of Lemma 3.5. Let $\rho>0$ such that $B(\rho,0)\subset U_2$. By Lemma 3.4, $B(\rho,0)\subset F(U)$.

It is clear that $F|_{B(\rho,0)}^{-1}$ is Lipschitz. On the other hand

$$DF^{-1}(0,0)=\begin{pmatrix} B_1^{-1} & 0 \\ 0 & B_2^{-1}\end{pmatrix}$$

and by Lemma 3.1, $\max(\|B_1\|, \|B_2^{-1}\|)<1-c\varepsilon^{\alpha}$.

$$\begin{aligned}\operatorname{Lip}(F|_{B(\rho)}^{-1}-DF^{-1}(0,0))&\le \|DF^{-1}-DF^{-1}(0,0)\|_{B(\rho)}\\ &\le \|D^2F^{-1}\|_{B(\rho)}\cdot\rho\le M\rho\varepsilon^{\alpha}.\end{aligned}$$

Let ρ_1 be such that $M\rho_1<c/4$. Then by Proposition 2.8 there exists f satisfying the Lemma. f is of class C^{r+1} and since $\operatorname{Lip} f\le 1$ in $B_1(\rho_1)$ we have $\|Df\|_{B_1(\rho_1)}\le 1$. □

Proof of Lemma 3.6. The second inequality of (3.7) is a consequence of $k\le N\le T/\varepsilon^{\alpha}$. Now we prove (3.6) and (3.7) by induction. If $x\in U_2-2\eta\varepsilon^{\beta}$, then $y\in U_2-\eta\varepsilon^{\beta}$ and $\|DG^{-1}(y)\|<1+M'\varepsilon^{\alpha}$. Also

$$\|G^{-1}(x)-G^{-1}(y)\|\le(1+M'\varepsilon^{\alpha})\|x-y\|<(1+M'\varepsilon^{\alpha})C_0\varepsilon^{\beta}<\eta\varepsilon^{\beta}$$

which implies $G^{-1}(y)\in U_2-\eta\varepsilon^{\beta}$.

If they are true for $0\le k-1\le N$,

$$\begin{aligned}\|DG^{-k}(y)\|&\le\|DG^{-1}(G^{-(k-1)}(y))\|\cdot\|DG^{-(k-1)}(y)\|\\ &\le(1+M'\varepsilon^{\alpha})(1+M'\varepsilon^{\alpha})^{k-1}\end{aligned}$$

and again by the mean value theorem we get $G^{-k}(y)\in U_1-\eta\varepsilon^{\beta}$.

We also prove (3.8) and (3.9) by induction. For $k=1$,

$$\|F^{-1}(y)-G^{-1}(y)\|<N'\varepsilon^{\alpha+\beta}<\eta\varepsilon^{\beta}\quad\text{and so } F^{-1}(y)\in U_1.$$

If they are true for $0\le k-1\le N$,

$$\begin{aligned}\|F^{-k}(y)-G^{-k}(y)\|&\le\|F^{-1}F^{-(k-1)}(y)-G^{-1}F^{-(k-1)}(y)\|\\ &\quad+\|G^{-1}F^{-(k-1)}(y)-G^{-1}G^{-(k-1)}(y)\|\\ &\le N'\varepsilon^{\alpha+\beta}+\|DG^{-1}\|_{U_1}\|F^{-(k-1)}(y)-G^{-(k-1)}(y)\|\\ &\le N'\varepsilon^{\alpha+\beta}+(1+M'\varepsilon^{\alpha})(k-1)(1+M'\varepsilon^{\alpha})^{k-1}N'\varepsilon^{\alpha+\beta}\\ &<k(1+M'\varepsilon^{\alpha})^{k}N'\varepsilon^{\alpha+\beta}<\eta\varepsilon^{\beta},\end{aligned}$$

and hence $F^{-k}(y)\in U_1$.

(3.10) is obvious for $k=1$. If it is true for $0\le k-1\le N$, using the formula

$$\begin{aligned}&D^2(g\circ f)(x)(e_1,e_2)\\ &\quad=Dg\circ f(x)(D^2f(x)(e_1,e_2))+D^2g\circ f(x)(Df(x)(e_1),Df(x)(e_2))\end{aligned}$$

we obtain, for $\|e_1\|, \|e_2\|\le 1$,

$$\begin{aligned}\|D^2G^{-k}(y)(e_1,e_2)\|&\le\|DG^{-1}(G^{-(k-1)}(y))\|\cdot\|D^2G^{-(k-1)}(y)\|\\ &\quad+\|D^2G^{-1}(G^{-(k-1)}(y))\|\cdot\|DG^{-(k-1)}(y)\|\cdot\|DG^{-(k-1)}(y)\|\\ &\le(1+M'\varepsilon^{\alpha})(k-1)(1+M'\varepsilon^{\alpha})^{2k-2}M'\varepsilon^{\alpha}\\ &\quad+M'\varepsilon^{\alpha}(1+M'\varepsilon^{\alpha})^{2(k-1)}<k(1+M'\varepsilon^{\alpha})^{2k}M'\varepsilon^{\alpha}.\end{aligned}$$

Finally (3.11) is obvious for $k=1$. Supposing that it is true for $0\le k-1\le N$

$$\begin{aligned}
&\|DF^{-k}(y)-DG^{-k}(y)\|\\
&\quad\le \|DF^{-1}(F^{-(k-1)}(y))DF^{-(k-1)}(y)-DF^{-1}(DF^{-(k-1)}(y))DG^{-(k-1)}(y)\|\\
&\qquad+\|DF^{-1}(F^{-(k-1)}(y))DG^{-(k-1)}(y)-DG^{-1}(F^{-(k-1)}(y))DG^{-(k-1)}(y)\|\\
&\qquad+\|DG^{-1}(F^{-(k-1)}(y))DG^{-(k-1)}(y)-DG^{-1}(G^{-(k-1)}(y))DG^{-(k-1)}(y)\|\\
&\quad\le(1+M'\varepsilon^\alpha)[(k-1)(1+M'\varepsilon^\alpha)^{k-1}N'+(k-1)^2(1+M'\varepsilon^\alpha)^{2k-2}N'M'\varepsilon^\alpha]\varepsilon^{\alpha+\beta}\\
&\qquad+N'\varepsilon^{\alpha+\beta}(1+M'\varepsilon^\alpha)^{k-1}+M'\varepsilon^\alpha k(1+M'\varepsilon^\alpha)^kN'\varepsilon^{\alpha+\beta}(1+M'\varepsilon^\alpha)^{k-1}\\
&\quad\le[k(1+M'\varepsilon^\alpha)^kN'+k^2(1+M'\varepsilon^\alpha)^{2k}N'M'\varepsilon^\alpha]\varepsilon^{\alpha+\beta}. \qquad\square
\end{aligned}$$

Proof of Lemma 3.7. (1) Since $\pi_3(V)$ is convex, Φ_3^{-1} is Lipschitz with Lip $\Phi_3^{-1}\le C_3$. By Theorem 2.5, if ε_1 is small enough $(\varepsilon_1^\beta<(2\eta C_3)^{-1})$, ψ_3 is a homeomorphism and since it is of class C^{r+1} it is a diffeomorphism. Furthermore

$$\text{Lip }\psi_3^{-1}\le((\text{Lip }\Phi_3^{-1})^{-1}-\text{Lip }(\Phi_3-\psi_3))^{-1}\le(C_3^{-1}-2\eta\varepsilon_1^\beta)^{-1}=K_{11}.$$

(2) Let $\delta'=2\eta K_{11}C_1$. If $x\in\pi_3(V_2)-\delta'\varepsilon^\beta$ there exists $z\in\pi_1(V_3)$ such that $\Phi_3(z)=x$. By Proposition 2.6,

$$B(C_1^{-1}\delta'\varepsilon^\beta,z)\subset\Phi_3^{-1}(B(\delta'\varepsilon^\beta,x))\subset\Phi_3^{-1}(\pi_3(V_2))=\pi_1(V_3).$$

Using again Proposition 2.6,

$$B(2\eta\varepsilon^\beta,\psi_3(z))\subset\psi_3(B(C_1^{-1}\delta'\varepsilon^\beta,z))\subset\psi_3(\pi_1(V_3)).$$

Since

$$\|x-\psi_3(z)\|=\|\Phi_3\Phi_3^{-1}(x)-\psi_3\Phi_3^{-1}(x)\|<2\eta\varepsilon^\beta$$

we get $x\in\psi_3(\pi_1(V_3))$. $\square$

Proof of Lemma 3.8. Since Φ_3^{-1} is Lipschitz in $\pi_3(V)$ we have

$$\begin{aligned}
0&=\|\Phi_3\Phi_3^{-1}-\Phi_3\psi_3^{-1}\|_{V_4}-\|\Phi_3\psi_3^{-1}-\psi_3\psi_3^{-1}\|_{V_4}\\
&\ge(\text{Lip }\Phi_3^{-1})^{-1}\|\Phi_3^{-1}-\psi_3^{-1}\|_{V_4}-\|\Phi_3-\psi_3\|_{\pi_1(V_1)}.
\end{aligned}$$

So we get

$$\|\Phi_3^{-1}-\psi_3^{-1}\|_{V_4}\le(1/C_3)\|\Phi_3-\psi_3\|\le2\eta\varepsilon^\beta/C_3=K_{12}\varepsilon^\beta.$$

If ε_1 is small enough we have

$$\begin{aligned}
&\|D\Phi_3^{-1}-D\psi_3^{-1}\|_{V_4}\le\|D\Phi_3^{-1}\|_{\pi_3(V)}\|D\psi_3^{-1}\|_{V_4}[\|D^2\Phi_3\|_{\pi_1(V_1)}\\
&\|\Phi_3^{-1}-\psi_3^{-1}\|_{V_4}+\|D\Phi_3-D\psi_3\|_{\pi_1(V)}]\le C_3K_{11}[(2\eta C_2/C_3)\varepsilon^\beta+K_{10}\varepsilon^\beta]=K_{13}\varepsilon^\beta. \quad\square
\end{aligned}$$

Proof of Lemma 3.9. It is easily seen by induction that

$$(\Delta F)^k(x,f(x),v,Df(x)v)=(F^k(x,f(x)),D(F^k\circ(I,f))(x)v).$$

To see that $z=(x,f(x),v,Df(x)v)\in W_F^s$ we must prove that $\lim_{n\to\infty}(\Delta F)^nz=0$. The first component tends to zero because $(x,f(x))\in W_{\Delta F}^s$. The second one because it is the transport by the derivative of the vector $(v,Df(x)v)$ tangent to W_F^s. $\square$

To end this section we give the proof of Theorem A′.

Proof. The unstable invariant manifolds of F_ε and G_ε are the stable invariant manifolds of F_ε^{-1} and G_ε^{-1}. G_ε^{-1} is the flow time ε^α of $\dot{x}=-h(x)$. By Proposition 2.3 we can apply Theorem A to obtain the result. □

4. *The distance between split separatrices for diffeomorphisms*

In this section we prove Theorem B and two corollaries.

Proof of Theorem B. We shall prove the homoclinic case since the heteroclinic one is analogous. By Theorems A and A′ the distance between the invariant manifolds of F_ε and G_ε (as defined in § 1) is of order ε. As the invariant manifolds of G_ε do not depend on ε, near a homoclinic point of F_ε, they must intersect which implies that they intersect becoming a homoclinic orbit, σ, of $\dot{x}=h(x)$. It is clear that σ is contained in B. Let $\rho>0$ be such that $B\subset B(\rho)$. Let P be a point of σ and $g: V\subset \mathbb{R}\to\mathbb{R}$ such that its graph represents σ locally in a neighbourhood of P. For ε_0 small enough, by theorems A and A′, W_1 and W_2 can be represented by the graphs of functions f_1 and f_2 defined on $V_1\subset V$ (independent of ε) and there exists a constant C such that $\|f_1-g\|_{r,V_1}<C\varepsilon^\beta$ and $\|f_2-g\|_{r,V_1}<C\varepsilon^\beta$. We define $f=f_1-f_2$ in V_1. Clearly $\|f\|_{r,V_1}=O(\varepsilon^\beta)$. Let z be a homoclinic point of F_ε such that $\pi(z)\in V_1$ where π is the appropriate projection operator. Of course $f(\pi(z))=0$, and also $f(\pi(F^i(z)))=0$ while $\pi(F^i(z))\in V_1$. By hypothesis (iii) $\|\pi(F(z))-\pi(z)\|\le M\varepsilon^\alpha$ so that if ε_0 is small enough there are r zeroes of f in V_1.

We call generically u_i the zeroes of D^if. Between two zeros of D^if there is a zero of $D^{i+1}f$. It is easily seen by induction that the distance between two consecutive zeroes of D^if is less than $M\varepsilon^\alpha$.

Now we prove (1). Suppose it is not true that $\|f\|=O(\varepsilon^{r\alpha+\beta'})$ on V_1. Let $\varepsilon\in(0,\varepsilon_0)$ and $C>0$. There exist $0<\varepsilon_1<\varepsilon$ and V_0 such that $\|f(v_0)\|>C\varepsilon_1^{r\alpha+\beta'}$. By the mean value theorem there exists v_1 such that

$$\|f(v_0)\|=\|f(v_0)-f(u_0)\|<\|Df(v_1)\|M\varepsilon_1^\alpha$$

and hence $\|Df(v_1)\|>C\varepsilon_1^{r\alpha+\beta'}/M\varepsilon_1^\alpha$. Applying the mean value theorem again there exists v_2 such that

$$\|Df(v_1)\|=\|Df(v_1)-Df(u_1)\|<\|D^2f(v_2)\|2M\varepsilon_1^\alpha$$

and hence $\|D^2f(v_2)\|>C\varepsilon^{r\alpha+\beta'}/2(M\varepsilon_1^\alpha)^2$. By induction there exists v_r such that $\|D^rf(v_r)\|>C\varepsilon_1^{r\alpha+\beta'}/r!\,(M\varepsilon_1^\alpha)^r$ which contradicts the fact that $\|f\|_{r,V_1}=O(\varepsilon^\beta)$.

To prove (2) we suppose there exists $k\in\mathbb{Z}^+$ such that $d(W_1,W_2)$ is not $O(\varepsilon^k)$. Since $F\in C^{k+1}$ and $\alpha>0$, from (1) we get a contradiction.

COROLLARY 4.1. *Let F_ε and h be as in theorem B and verifying* (i), (ii) *and* (iii) *with* $p_1=p_2$. *Furthermore we suppose*

(iv) *F_ε is a family of conservative diffeomorphisms.*

(v) *$\dot{x}=h(x)$ has a homoclinic orbit σ.*

Then we have the same conclusions as in Theorem B.

Proof. The hypotheses (i), (ii) and (iii) let us to apply Theorems A and A′. From them we have that the distance between the invariant manifolds of F_ε and σ is of order of ε so that the distance between them is also of order of ε. By (iv) they must

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intersect [11] and we have a homoclinic point. Furthermore, if ε_0 is small enough there exists a compact set contained in U which contains the pieces of invariant manifolds from p_1 to the homoclinic point and finally we can apply Theorem B. □

COROLLARY 4.2. *Let F_ε be as in Theorem B, of the form*

$$F_\varepsilon(x, y) = (\lambda x, \mu y) + \varepsilon^\alpha f(x, y) + \varepsilon^{\alpha+\beta} g(x, y, \varepsilon), \quad \alpha \geq 1,$$

with $\lambda = 1 + a_1\varepsilon^\alpha + o(\varepsilon^\alpha)$ *and* $\mu = 1 - a_2\varepsilon^\alpha + o(\varepsilon^\alpha)$, $a_1, a_2 > 0, f(0,0) = g(0, 0, \varepsilon) = 0$, *and* $Df(0,0) = Dg(0, 0, \varepsilon) = 0$.

If the hypothesis (iv) *of Theorem B is satisfied with* $p_1 = p_2 = (0,0)$ *then we have the same conclusions of Theorem B with* $0 < \beta' < \min(\alpha, \beta)$.

Proof. We consider here G_ε defined through the flow φ of

$$\dot{x} = a_1 x + f_1(x, y),$$
$$\dot{y} = -a_2 y + f_2(x, y),$$

by

$$G_\varepsilon(x, y) = \varphi(\varepsilon^\alpha, (x, y)).$$

The hypothesis (i) of Theorem B clearly holds. The hypothesis (ii) comes from

$$DG_\varepsilon(0,0) = \exp(\varepsilon^\alpha \operatorname{diag}(a_1, -a_2))$$

and the hypothesis (iii) is a consequence of Proposition 2.4. □

5. *The case of a flow with a periodic orbit*

First we establish the existence of a periodic orbit in a neighbourhood of the origin and we find bounds of its amplitude for equations of the form

$$\dot{x} = f(x) + \varepsilon g(x, t/\varepsilon, \varepsilon) \tag{5.1}$$

with $f(0) = 0$.

Notice that we shall not require $Df(0)$ to be hyperbolic. This is due to the fact that g is rapidly oscillating.

PROPOSITION 5.1. *Consider the equation* (5.1) *with* $f: U \to \mathbb{R}^n$, $g: U^* \to \mathbb{R}^n$ *where* U *is an open set of* $\mathbb{R}^n$ *containing the origin,* $U^* = U \times \mathbb{R} \times [0, \varepsilon_0)$ *and such that*

(i) $f \in C^{1+L}(U)$, *that is* Df *is Lipschitz in* U,
(ii) $g \in C^0(U^*)$ *and is Lipschitz with respect to the first variable,*
(iii) $f(0) = 0$,
(iv) g *is T-periodic with respect to the second variable and* $\int_0^T g(0, t, \varepsilon)\, dt = 0$.

Then there exist $\varepsilon_1, c > 0$ *such that for* $0 < \varepsilon < \varepsilon_1$ (5.1) *has a unique periodic orbit* γ *of period* εT *such that* $\|\gamma\| < c\varepsilon^2$.

Proof. We can suppose that g is bounded in U^*. Let k_1 be a positive bound. We call k_2 and k_3 the Lipschitz constants of Df and g. We define $A = Df(0)$ and $\Phi(t) = \exp At$.

We take

$$c = 4k_1 T\|A\| \cdot \|A^{-1}\| + 1,$$
$$\varepsilon_1 \leq \min((2T\|A\|)^{-1}, (8k_3 c\|A^{-1}\|)^{-1}, (10k_2 c^2\|A^{-1}\|)^{-1/2}, \varepsilon_0)$$

and we suppose that $B(c\varepsilon_1^2)\subset U$. Given a function $\gamma:\mathbb{R}\to U$ we define

$$\psi(s)=f(\gamma(s))-A\gamma(s)+\varepsilon g(\gamma(s), s/\varepsilon, \varepsilon).$$

We fix $\varepsilon\in(0, \varepsilon_1)$. First we shall prove that γ is an εT-periodic solution of (5.1) if and only if γ is εT-periodic and

$$\gamma(t)=(I-\Phi(\varepsilon T))^{-1}\int_0^{\varepsilon T}\Phi(\varepsilon T-s)\psi(s+t)\,ds. \tag{5.2}$$

Indeed, if γ is an εT-periodic solution of (5.1) we have

$$\gamma(t)=\Phi(t)\left[\gamma(0)+\int_0^t\Phi^{-1}(s)\psi(s)\,ds\right]. \tag{5.3}$$

Then

$$\Phi(t)\left[\gamma(0)+\int_0^t\Phi^{-1}(s)\psi(s)\,ds\right]$$
$$=\Phi(t+\varepsilon T)\left[\gamma(0)+\int_0^t\Phi^{-1}(s)\psi(s)\,ds+\int_0^{\varepsilon T}\Phi^{-1}(s+t)\psi(s+t)\,ds\right]. \tag{5.4}$$

To find $\gamma(0)$ we must prove that $\Phi(t)-\Phi(t+\varepsilon T)=\Phi(t)[I-\Phi(\varepsilon T)]$ is inversible. We need to consider $I-\Phi(\varepsilon T)$. From

$$I-\Phi(\varepsilon T)=-\varepsilon TA\sum_{k=0}^{\infty}(\varepsilon TA)^k/(k+1)!$$

we only need to consider $\sum_{k=0}^{\infty}(\varepsilon TA)^k/(k+1)!$. We have

$$\left\|I-\sum_{k=0}^{\infty}(\varepsilon TA)^k/(k+1)!\right\|\le\sum_{k=1}^{\infty}(\varepsilon T\|A\|)^k/(k+1)!$$
$$=(e^{\varepsilon T\|A\|}-\varepsilon T\|A\|-1)/\varepsilon T\|A\|\le 1/2,$$

so that $I-\Phi(\varepsilon T)$ is invertible. Furthermore

$$\|(I-\Phi(\varepsilon T))^{-1}\|$$
$$\le\left(1\Big/\left(1-\left\|I-\sum_{k=0}^{\infty}(\varepsilon TA)^k/(k+1)!\right\|\right)\right)\|A^{-1}\|/\varepsilon T\le 2\|A^{-1}\|/\varepsilon T. \tag{5.5}$$

Finding $\gamma(0)$ from (5.4) and putting it into (5.3) we get (5.2). Conversely, we suppose that γ satisfies (5.2) and is εT-periodic. We write (5.2) in the form

$$\gamma(t)=(I-\Phi(\varepsilon T))^{-1}\int_t^{t+\varepsilon T}\Phi(\varepsilon T-s+t)\psi(s)\,ds.$$

Now, taking the derivative, we can immediately verify that it is a solution of (5.1).

To find εT-periodic solutions of (5.1) we define $X=\{\gamma:\mathbb{R}\to U$ continuous, εT-periodic with $\|\gamma\|\le c\varepsilon^2\}$ and $\Lambda:X\to X$ by

$$(\Lambda\gamma)(t)=(I-\Phi(\varepsilon T))^{-1}\int_0^{\varepsilon T}\Phi(\varepsilon T-s)\psi(s+t)\,ds. \tag{5.6}$$

X is a complete metric space. We shall see that Λ is a contraction operator. The unique fixed point of Λ in X will give us the periodic orbit we are looking for.

Making the change $s=\varepsilon u$, from (5.6) we obtain

$$\|(\Lambda\gamma)(t)\|$$
$$\le \|(I-\Phi(\varepsilon T))^{-1}\|\left\{\varepsilon\left\|\int_0^T \Phi(\varepsilon(T-u))(f(\gamma(\varepsilon u+t))-A\gamma(\varepsilon u+t))\,du\right\|\right.$$
$$\left.+\varepsilon^2\left\|\int_0^T \Phi(\varepsilon(T-u))g(\gamma(\varepsilon u+t),(\varepsilon u+t)/\varepsilon,\varepsilon)\,du\right\|\right\}.$$

We call I_1 and I_2 the first and second integrals. If $\|x\|<\delta$ we have

$$\|f(x)-Ax\|=\|f(x)-f(0)-Df(0)x\|\le \sup_{\xi\in B(\delta)}\|Df(\xi)-Df(0)\|\cdot\|x\|\le k_2\delta^2$$

and hence $\|f(\gamma(\varepsilon u+t))-A\gamma(\varepsilon u+t)\|\le k_2\|\gamma\|^2$ so that $\|I_1\|\le e^{\varepsilon T\|A\|}Tk_2c^2\varepsilon^4$.

Integrating by parts

$$I_2=\int_0^T g(\gamma(\varepsilon s+t),(\varepsilon s+t)/\varepsilon,\varepsilon)\,ds$$
$$+\varepsilon A\int_0^T e^{\varepsilon(T-u)A}\left(\int_0^u g(\gamma(\varepsilon s+t),(\varepsilon s+t)/\varepsilon,\varepsilon)\,ds\right)du.$$

We call I_3 and I_4 the last two integrals. By (iv)

$$\|I_3\|\le\int_0^T\|g(\gamma(\varepsilon s+t),(\varepsilon s+t)/\varepsilon,\varepsilon)-g(0,(\varepsilon s+t)/\varepsilon,\varepsilon)\|\,ds$$
$$\le\int_0^T k_3\|\gamma(\varepsilon s+t)\|\,ds\le k_3Tc\varepsilon^2,$$
$$\|I_4\|\le\varepsilon\|A\|\int_0^T e^{\varepsilon T\|A\|}\left(\int_0^u k_1\,ds\right)du\le\varepsilon\|A\|k_1\,e^{\varepsilon T\|A\|}T^2/2.$$

By (5.5) and the definitions of c and ε_1 we get $\|\Lambda\gamma\|<c\varepsilon^2$. On the other hand, if $\gamma,\sigma\in X$

$$\|(\Lambda\gamma)(t)-(\Lambda\sigma)(t)\|$$
$$\le\varepsilon\|(I-\Phi(\varepsilon T))^{-1}\|$$
$$\times\left\{\int_0^T\|\Phi(\varepsilon(T-u))\|\cdot\|f(\gamma(\varepsilon u+t))-A\gamma(\varepsilon u+t)-f(\sigma(\varepsilon u+t))\right.$$
$$+A\sigma(\varepsilon u+t)\|\,du+\varepsilon\int_0^T\|\Phi(\varepsilon(T-u))\|\cdot\|g(\gamma(\varepsilon u+t),(\varepsilon u+t)/\varepsilon,\varepsilon)$$
$$\left.-g(\sigma(\varepsilon u+t),(\varepsilon u+t)/\varepsilon,\varepsilon)\|\,du\right\}.$$

We call I_5 and I_6 the two last integrals. We have

$$\|I_5\|\le\int_0^T e^{\varepsilon T\|A\|}\sup_{\xi\in B(c\varepsilon^2)}\|Df(\xi)-Df(0)\|\cdot\|\gamma-\sigma\|\,du$$
$$\le e^{\varepsilon T\|A\|}Tk_2c\varepsilon^2\|\gamma-\sigma\|,$$
$$\|I_6\|\le\int_0^T e^{\varepsilon T\|A\|}k_3\|\gamma-\sigma\|\,du\le e^{\varepsilon T\|A\|}k_3T\|\gamma-\sigma\|,$$

where in bounding I_5 we have used the mean value theorem for the map $z \to f(z) - Df(0)z$. Again by (5.5) and the definitions of c and ε_1 we get

$$\|\Lambda\gamma - \Lambda\sigma\| < \frac{1}{c}\|\gamma - \sigma\|. \qquad \square$$

The point $x = 0$ is not, in general, a singular point of equation (5.1). Consider instead

$$\dot{x} = f(x + \gamma(t)) + \varepsilon g(x + \gamma(t), t/\varepsilon, \varepsilon) - \dot{\gamma}(t) \tag{5.7}$$

obtained from (5.1) when we translate γ to the origin. Now $x = 0$ is a singular point of (5.7). Consider also

$$\dot{x} = f(x). \tag{5.8}$$

Suppose that $f: U \to \mathbb{R}^n$ and $g: U^* \to \mathbb{R}^n$ are as before.

Let $\varphi_1(t, \tau, x)$ and $\varphi_2(t, \tau, x)$ be the solutions of (5.7) and (5.8) such that $\varphi_1(\tau, \tau, x) = x$ and $\varphi_2(\tau, \tau, x) = x$. We define the families of diffeomorphisms $F_\varepsilon(x) = \varphi_1(\varepsilon T + \tau, \tau, x)$ and $G_\varepsilon(x) = \varphi_2(\varepsilon T + \tau, \tau, x)$.

Proposition 5.2. *Consider the equations* (5.7) *and* (5.8) *with*

(i) $f \in C^{r+1}(U)$, $r \geq 1$,

(ii) *g is of class C^{r+1} with respect to the first variable and $D_x^k g$ is continuous in U^* for $0 \leq k \leq r+1$,*

(iii) *$f(0) = 0$ and $Df(0)$ is hyperbolic,*

(iv) *g is T-periodic with respect to the second variable and $\int_0^T g(x, t, \varepsilon)\, dt = 0$ for all $x \in U$ and $\varepsilon \in [0, \varepsilon_0)$.*

Then given a compact set $B \subset U$ containing the origin there exist ε_1, $M, N > 0$ such that

(1) $\|F_\varepsilon - I\|_{r+1,B} \leq M\varepsilon$, $\|G_\varepsilon - I\|_{r+1,B} \leq M\varepsilon$,

(2) $\|F_\varepsilon - G_\varepsilon\|_{r,B} \leq N\varepsilon^3$, *for* $\varepsilon \in [0, \varepsilon_1)$.

The proof of this proposition is analogous to that of Proposition 2.4 and so we omit it. We notice that the bound in (2) is $N\varepsilon^3$. It is essentially due to hypothesis (iv) and the fact that both F_ε and G_ε come from flows.

With this result we can finally give the following.

Proof of Theorem C. The existence of the periodic orbit is a consequence of Proposition 5.1. From (ii) and (iii) we have that the eigenvalues of $Df(0)$ are $\pm\mu$ with $\mu > 0$. Let $B \subset U$ be a compact set which contains σ and let F_ε and G_ε be defined as in Proposition 5.2. From that proposition $\|DF_\varepsilon(0) - DG_\varepsilon(0)\| \leq N\varepsilon^3$. That implies that the coefficients of the characteristic polynomials of $DF_\varepsilon(0)$ and $DG_\varepsilon(0)$ differ in terms of order ε^3 and that the eigenvalues differ in terms of order ε^2.

From this we see that if ε is small enough, $DF_\varepsilon(0)$ is hyperbolic and hence γ is hyperbolic. That finishes the proof of (1). Corollary 4.1 tells us that F_ε has homoclinic points and the distance between the invariant manifolds is $O(\varepsilon^r)$ ($O(\varepsilon^k)$ for all k if $r = \infty$).

This finishes the proof for the invariant manifolds of $x(t) = 0$ of (5.7). The same holds for the invariant manifolds of γ of (5.1) because the latter are related with the former by a translation. $\square$

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