A RESTRICTED INHOMOGENEOUS MINIMUM FOR FORMS

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1. Introduction

Let us suppose that $f(x, y)$ is an indefinite binary quadratic form that does not represent zero. If P is the real point (x_0, y_0) then we may define

$$(1.1)\qquad M(f;\, P) = \inf |f(x+x_0,\, y+y_0)|,$$

where the infimum is taken over all integral x, y. The *inhomogeneous minimum* of the form f is defined

$$(1.2)\qquad M(f) = \sup_{P} M(f;\, P),$$

where the supremum taken over all real points P, need only extend over some complete set of points, incongruent mod 1.

We may write f in the following way:

$$(1.3)\qquad f(x, y) = \frac{\pm\Delta}{\theta\varphi-1}\,(\theta x+y)(x+\varphi y),$$

where θ, φ are irrational, and $\Delta = \sqrt{D}$, where D is the discriminant of f. The form f is said to be *I-reduced* (inhomogeneously reduced) if it can be written in the form (1.3) with

$$(1.4)\qquad |\theta| > 1,\ |\varphi| > 1.$$

Thus (1.3) is a unique representation of such an f. If in addition we have $\theta\varphi < 0$, then f is said to be *Gauss-reduced.*

Let f be any I-reduced form, and P the point (x_0, y_0), then if

$$(1.5)\qquad \begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix} = \begin{pmatrix}\theta & 1\\ 1 & \varphi\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix},$$

we may define $M^+(f;\, P)$ and $M^-(f;\, P)$ as follows:

$$M^+(f;\, P) = \inf_{\theta x+y+\xi_0>0} \Delta(\theta x+y+\xi_0)\left|\frac{x+\varphi y+\eta_0}{\theta\varphi-1}\right|,$$

where the infimum is taken over all integers x, y, such that $\theta x+y+\xi_0 > 0$, and

$$M^-(f; P) = \inf_{\theta x+y+\xi_0<0} \frac{\Delta|(\theta x+y+\xi_0)(x+\varphi y+\eta_0)|}{|\theta\varphi-1|},$$

where the infimum extends over all integers x, y such that $\theta x+y+\xi_0 < 0$.

Suppose we write an integral unimodular substitution as the matrix

$$T = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \tag{1.6}$$

where p, q, r, s are integers such that $ps-qr = \pm 1$. If the substitution

$$\begin{aligned} x &= pX+qY \\ y &= rX+sY \end{aligned}$$

gives

$$f(x, y) = F(X, Y),$$

then we will write

$$F = fT = f\begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{1.7}$$

Define the form $g(x, y)$ by

$$g(x, y) = f\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\pm\Delta}{\theta\varphi-1}(\varphi x+y)(x+\theta y), \tag{1.8}$$

and let Q be the point

$$Q = Q(y_0, x_0). \tag{1.9}$$

We readily see that

$$M^+(g; Q) = \inf_{\varphi x+y+\eta_0>0} \Delta(\varphi x+y+\eta_0)\left|\frac{x+\theta y+\xi_0}{\theta\varphi-1}\right|,$$

and

$$M^-(g; Q) = \inf_{\varphi x+y+\eta_0<0} \frac{\Delta|(\varphi x+y+\eta_0)(x+\theta y+\xi_0)|}{|\theta\varphi-1|}.$$

We wish to define a function which is independent of the particular I-reduced form chosen from an equivalence class of forms, and so after (1.8) and (1.9) we put

$$M^*(f; P) = M^*(g; Q) = \max\{M^\pm(f; P), M^\pm(g; Q)\}, \tag{1.10}$$

and

$$M^*(f) = M^*(g) = \sup_P M^*(f; P), \tag{1.11}$$

where the supremum is taken over all real points P, such that, after the definition (1.5), neither $\theta x+y+\xi_0$, nor $x+\varphi y+\eta_0$ represent zero in integers x, y.

It is readily seen that $M^*(f) = M^*(h)$ where f and h are equivalent I-reduced forms. Thus if q is any form which does not represent zero, we may define $M^*(q) = M^*(f)$, where f is any equivalent I-reduced form.

The purpose of this paper is to investigate the supremum of values taken by the function $M^*(f)$, where f does not represent zero, and to evaluate this function for a certain sequence of equivalence classes of forms. We will deduce these results from a related problem investigated by Cassels [6] and Descombes [7], [8].

The major result that we prove is Theorem 7.3, which may be summarized as follows:

A. *We have for all forms f that do not represent zero*

$$M^*(f) \leqq \frac{27\Delta}{28\sqrt{7}}.$$

B. *Furthermore, except for an equivalence class of forms for which equality holds in A, we have*

$$M^*(f) \leqq \frac{359\Delta}{45\sqrt{510}}.$$

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2. The divided cell method

The results of this paper will be obtained by modifications of the divided cell method, discussions of which may be found in [1], [3], [5] and [10]. The reader is particularly referred to [4], where the notation that we will need is introduced.

The grid $\mathscr{L}$, in the ξ, η-plane is again given by

$$\mathscr{L}: \begin{aligned} \xi &= \beta(\theta x+y+\xi_0) \\ \eta &= \gamma(x+\varphi y+\eta_0), \end{aligned} \tag{2.1}$$

for integral x and y. The vertices of the divided cell S_n are denoted A_n, B_n, C_n and D_n, labelled in a clockwise direction. It is essential for applications in §§ 5, 7, to note that if A_nD_n has positive slope then A_{n+1} is in the

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opposite quadrant to A_n, while if A_nD_n has negative slope then A_n and A_{n+1} are in the same quadrant.

The algorithm applied to a grid $\mathscr{L}$ with no points on either axis generates a doubly infinite chain of integer pairs $\{a_{n+1}, \varepsilon_n\}$ which satisfies the set of conditions given by equation (2.6) of [4]. This enables an evaluation of $M(f; P)$ in terms of the variables φ_n, θ_n, μ_n, λ_n, of equation (2.8) in [4].

THEOREM 2.1. *If*

$$(2.2)\qquad \begin{cases} M_n^{(1)} = \dfrac{\Delta|(\theta_n+1-\lambda_n)(\varphi_n+1-\mu_n)|}{4|\theta_n\varphi_n-1|} \\[2ex] M_n^{(2)} = \dfrac{\Delta|(\theta_n-1-\lambda_n)(\varphi_n-1+\mu_n)|}{4|\theta_n\varphi_n-1|} \\[2ex] M_n^{(3)} = \dfrac{\Delta|(\theta_n-1+\lambda_n)(\varphi_n-1-\mu_n)|}{4|\theta_n\varphi_n-1|} \\[2ex] M_n^{(4)} = \dfrac{\Delta|(\theta_n+1+\lambda_n)(\varphi_n+1+\mu_n)|}{4|\theta_n\varphi_n-1|} \end{cases}$$

then

$$M(f; P) = \inf_n M_n,$$

where

$$M_n = M_n(f; P) = \inf_{1\leqq k\leqq 4} M_n^{(k)},$$

We also have

$$|\lambda_n| < |\theta_n|-1$$
$$|\mu_n| < |\varphi_n|-1.$$

Examination of the proof of this theorem indicates that the $M_n^{(k)}$ arise from the points C_n, D_n, B_n and A_n, respectively.

Now corresponding to an indefinite form f, and point P, which gives rise to a grid with no point on an axis, the integer sequence $\{a_{n+1}, \varepsilon_n\}$ generates a sequence of I-reduced forms $\{f_n\}$ given by

$$f_n(x, y) = \frac{\Delta(\theta_n x+y)(x+\varphi_n y)}{\theta_n\varphi_n-1}.$$

In the above context we call $\{a_n\}$ an a-chain *of the form f, from the form f_0.*

Pitman [11] has shown by producing counter-examples that it is not always possible to obtain all I-reduced forms equivalent to f by taking all forms that occur in the chains from f. Nor is it always possible to obtain all chains of equivalent forms of f, by taking all chains from some one form, even in the case when it is an integral Gauss-reduced form. However, Pitman [11], [12] has shown the following results to be true.

THEOREM 2.2. *Suppose f is any I-reduced form (given, say, by (1.3)), and g is any Gauss-reduced form, properly equivalent to f, say,*

$$g(x, y) = \frac{\pm\Delta}{\omega\omega'-1}(\omega x+y)(x+\omega' y), \tag{2.3}$$

with $\omega < -1$, $\omega' > 1$. Then every form chain from f contains at least one of the three forms

$$g, \quad g\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad g\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}. \tag{2.4}$$

If f and g have integral coefficients, then every form chain from f contains infinitely many occurrences of a form of the triad (2.4), and the distance between these occurrences (not necessarily the same form) is bounded.

We will use this theorem to obtain an analogous result for quadratic irrationals.

LEMMA 2.1. *Any semi-regular expansion of a quadratic irrational α, contains a complete quotient α_n, such that $|\bar{\alpha}_n| < 1$, where the bar denotes the algebraic conjugate.*

PROOF. This result is readily obtained by writing α as a linear fractional form in terms of α_n, then solving for α_n, and taking the algebraic conjugate. We obtain

$$\bar{\alpha}_n = \frac{q_{n-2}\bar{\alpha}-p_{n-2}}{q_{n-1}\bar{\alpha}-p_{n-1}},$$

where the p_r, q_r are the semi-regular convergents of the expansion (see [1] p. 213). Now since $p_{n-2}q_{n-1}-p_{n-1}q_{n-2} = 1$, we have

$$|\bar{\alpha}_n| \leqq \left|\frac{q_{n-2}}{q_{n-1}}\right|\left(1+\frac{1}{\left|q_{n-1}q_{n-2}\left(\bar{\alpha}-\dfrac{p_{n-1}}{q_{n-1}}\right)\right|}\right).$$

By the properties of the convergents p_n/q_n,

$$\lim_{n\to\infty}\frac{p_n}{q_n} = \alpha \neq \bar{\alpha}, \quad \lim_{n\to\infty}|q_n| = \infty, \text{ and } |q_{n-2}| \leqq |q_{n-1}|-1.$$

Thus, provided n is large enough,

$$|\bar{\alpha}_n| \leqq \left|\frac{q_{n-2}}{q_{n-1}}\right|\left(1+\frac{1}{|q_{n-1}|}\right)$$

$$\leqq \left(1-\frac{1}{|q_{n-1}|}\right)\left(1+\frac{1}{|q_{n-1}|}\right) < 1.$$

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THEOREM 2.3. *Suppose* ψ, $|\psi| > 1$, *is a quadratic irrational, and* α *is an equivalent number satisfying*

$$\alpha > 1, \ -1 < \bar{\alpha} < 0 \tag{2.5}$$

then every semi-regular continued fraction expansion of ψ *of the type we are considering, has as a complete quotient one of the numbers*

$$\alpha, \quad \alpha+1, \quad \frac{\alpha}{1-\alpha}, \quad \text{or their negatives.} \tag{2.6}$$

PROOF. The existence of such an α satisfying (2.5) (usually called a reduced quadratic irrational) follows from the well known result that to every binary quadratic form, there exists an equivalent Gauss-reduced form. We can assume that $\alpha > 1$ by making the transformation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, if necessary.

Now, by the previous lemma, every semi-regular expansion of ψ leads forward to a complete quotient φ, say, with the property that $|\bar{\varphi}| < 1$. Putting $\theta = 1/\bar{\varphi}$, $\omega = 1/\bar{\alpha}$, and $\omega' = \alpha$, the forms (1.3) and (2.3) are equivalent, and multiples of integral forms.

Suppose f is properly equivalent to g, then since g is Gauss reduced, we may apply Theorem 2.2. Hence every form chain of f in fact leads forward to one of the forms (2.4), and so any semi-regular expansion of φ (and thus ψ) leads forward to one of ω', $\omega'+1$, or $\omega'/(1-\omega')$ (e.g. by [9] p. 99). This gives the required result.

If f is improperly equivalent to g, then $f\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is properly equivalent to g, and the same argument implies that every expansion of $-\varphi$ leads forward to one of the triad (2.6).

For the purpose of approximating the M_n of Theorem 2.1, we will require in later sections Theorems 2.4 and 17.1 of [4]. Modifications of the following result quoted from [1] will enable much chain exclusion to be done without additional case splitting.

THEOREM 2.4. *The value of* $M(f; P)$ *remains unchanged after the following elementary chain operations have been applied to the chain pair* $\{a_{n+1}, \varepsilon_n\}$;

(i) *reversing the chain about some point,*

(ii) *negating the* $\{\varepsilon_n\}$ *chain,*

(iii) *negating the* $\{a_n\}$ *chain, and alternate members of the* $\{\varepsilon_n\}$ *chain.*

3. Continued fractions to the integer above

If α is irrational and $\alpha > 1$, then among the infinitely many semi-regular expansions of α, there is that unique expansion for which $a_n \geqq 2$ for all n. We will call this the *A-expansion* of α. Note that $a_n > 2$ for infi-

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nitely many n. The A-expansion enjoys many analogous properties to the ordinary continued fractions expansion, but we will not explicitly use many of these. However, we will require a transformation which will convert one kind of expansion into the other. We will use the usual notation for a repeated chain segment (for any type of expansion) by enclosing it in brackets and subscripting with the number of repetitions. For example

$$[a_1, a_2, \cdots, a_n, (b_1, b_2, \cdots, b_r)_s, a_{n+rs+1}, \cdots].$$

In this notation we will use the convention that $s = 0$ will imply that the chain segment is deleted altogether from the expansion.

We will consistently distinguish ordinary from semi-regular expansions by using round and square external brackets, respectively.

THEOREM 3.1. *If in ordinary continued fractions*

$$\alpha = (a, r+1, x),$$

where

$$r \geqq 0, x > 1, a \geqq 0,$$

then we have the following A-expansion for $\alpha+1$;

$$\alpha+1 = [a+2, (2)_r, x+1].$$

PROOF.

$$\alpha = a + \cfrac{1}{(r+1) + \cfrac{1}{x}}$$

$$= a+1 - \cfrac{1}{\cfrac{(r+1)x+1}{rx+1}}$$

$$= a+1 - \cfrac{1}{\cfrac{(r+1)(x+1)-r}{r(x+1)-(r-1)}}.$$

It is readily seen by an inductive argument (or by using the convergents p_n, q_n as in [11]) that

$$[(2)_r, x+1] = \frac{(r+1)(x+1)-r}{r(x+1)-(r-1)},$$

and the result follows.

Using the convention mentioned above, and inserting an appropriate $(2)_0$ into the A-expansion, if necessary, we obtain the following result as a corollary.

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COROLLARY. $\alpha = (a_1, a_2, a_3, a_4, \cdots)$ *if and only if*

$$\alpha+1 = [a_1+2, (2)_{a_2-1}, a_3+2, (2)_{a_4-1}, \cdots]. \tag{3.1}$$

This relationship enables many of the properties of the A-expansions to be deduced from the corresponding property of the ordinary continued fraction.

4. The inhomogeneous approximation problem

Let φ be an irrational number, and α real such that $\varphi x+y+\alpha$ does not represent zero in integers x, y. If $||\theta||$ denotes the distance from θ to the nearest integer, then put

$$k^+(\varphi, \alpha) = \liminf_{x\to+\infty} x||\varphi x+\alpha||. \tag{4.1}$$

Cassels [6] and Descombes [7] showed that there is a decreasing sequence of isolated values, taken by $k^+(\varphi, \alpha)$, which approach the limit

$$\frac{1}{\gamma} = \frac{773868-28547\sqrt{510}}{366795} = 0\cdot 352\cdots \tag{4.2}$$

Descombes used the algorithm originally described by Cassels to find this sequence. The method involved the ordinary continued fraction expansion of φ, together with an associated sequence of integers which arise from the inhomogeneity of the problem. By means of a modification of the divided cell method described in § 2, we will reformulate the problem in terms of semi-regular continued fractions, and then convert Descombes' critical chains into this context. We will then connect the approximation problem and the restricted form problem of § 1.

The couples (φ, α) and (φ', α') are said to be *equivalent* if there exist integers p, q, r, s, a, b with $ps-qr = \pm 1$, such that

$$\varphi' = \frac{p\varphi+q}{r\varphi+s},\ \alpha' = \frac{(ps-qr)\alpha}{r\varphi+s}+\frac{a\varphi+b}{r\varphi+s},\ r\varphi+s > 0. \tag{4.3}$$

LEMMA 4.1. *If* (φ, α) *and* (φ', α') *are equivalent then*

$$k^+(\varphi, \alpha) = k^+(\varphi', \alpha').$$

The proof may be found in [7].

By the means of a set of eight integer sequences, Descombes defines three sequences of real numbers which we shall denote by

$$\{\varphi_r\}, \{\alpha_r\} \text{ and } \{\gamma_r\}, \text{ for } r \geqq -2,$$

where

$$\alpha_{-2} = -\tfrac{1}{14}, \quad \alpha_{-1} = -\tfrac{1}{90}, \quad \alpha_0 = -\tfrac{1}{10},$$
$$\psi_{-2} = \frac{7-\sqrt{7}}{14}, \quad \psi_{-1} = \frac{225-\sqrt{510}}{2340}, \quad \psi_0 = \frac{15-\sqrt{110}}{10},$$
$$\gamma_{-2} = \frac{28\sqrt{7}}{27}, \quad \gamma_{-1} = \frac{45\sqrt{510}}{359}, \quad \gamma_0 = \frac{10\sqrt{110}}{37}.$$

The whole of the paper [7] is devoted to the proof of the following theorem.

THEOREM 4.1.

(i) *$\{\gamma_r\}$ is an increasing sequence, and if γ is given by* (4.2), *then* $\lim_{r\to\infty} \gamma_r = \gamma$.

(ii) *For all $r \geqq -2$, we have*

$$k^+(\psi_r, \alpha_r) = \frac{1}{\gamma_r}.$$

(iii) *If we exclude all couples equivalent* (*in the sense of* (4.3)) *to one of* (ψ_r, α_r), *for* $-2 \leqq r \leqq n$, *then*

$$k^+(\psi, \alpha) < k^+(\psi_n, \alpha_n) = \frac{1}{\gamma_n}.$$

(iv) *Furthermore, if* (ψ, α) *is not equivalent to* (ψ_r, α_r) *for some* r, *then*

$$k^+(\psi, \alpha) \leqq \frac{1}{\gamma},$$

and equality holds for uncountably many couples (ψ, α).

This result provides a parallel to the classical results of Markov on the approximation of irrationals by rationals. The explicit values of the sequence pair (ψ_r, α_r) will be of less interest to us for the purpose of this paper, than the algorithmic development of the pair. As with the homogeneous case, since we are dealing with a lim inf problem, only the tail of this development will be relevant.

Let β'_r be the tail of the ordinary continued fraction development of ψ_r. Define the following ordinary continued fraction blocks

$$(4.4) \qquad A' = (4, 1, 1, 1), \quad B' = (4, 1, 1, 1, 1, 1), \quad C' = (3, 1, 1, 1).$$

Then from [7] (pp. 324, 327—330, 351) we may suppose in the notation introduced above

$$(4.5) \qquad \begin{aligned} &\beta'_{-2} = (A'_\infty), \quad \beta'_{-1} = ((B'C')_\infty), \quad \beta'_0 = (B'_\infty) \\ &\beta'_r = ((A'(B'C')_r)_\infty), \text{ for } r \geqq 1. \end{aligned}$$

It follows from a well-known result (see [5]) that the β'_r are reduced for all $r \geqq -2$. We put $\beta_r = \beta'_r+1$.

Define the following semi-regular blocks.

$$A = [6, 3], \quad B = [6, 3, 3], \quad C = [5, 3]. \tag{4.6}$$

Now it is apparent by the transformation (3.1) that the equations (4.5) hold when the prime is removed, and the external round brackets replaced by square brackets. Hence

$$\begin{aligned} &\beta_{-2} = [A_\infty], \quad \beta_{-1} = [(BC)_\infty], \quad \beta_0 = [B_\infty], \\ &\beta_r = [(A(BC)_r)_\infty], \text{ for } r \geqq 1. \end{aligned} \tag{4.7}$$

If $\{m_k\}$ is an arbitrary increasing sequence of positive integers, then any irrational ψ, whose ordinary continued fraction tail is given by

$$(A'(B'C')_{m_1} A'(B'C')_{m_2} \cdots) = (A'(B'C')_{m_k})_{k=1}^{\infty}, \tag{4.8}$$

together with some corresponding α has

$$k^+(\psi, \alpha) = \frac{1}{\gamma}.$$

Clearly there are uncountably many such ψ. The proof of this result may be found in [7] (p. 349). The corresponding A-expansion in semi-regular continued fractions is given by (4.8) with the primes removed from the blocks.

5. Alternative method for obtaining $k^+(\varphi, \alpha)$

Barnes [2] used a degenerate case of the divided cell method in order to evaluate $k^+(\varphi, \alpha)$. He considered the grid

$$\begin{aligned} \xi &= x \\ \eta &= \varphi x + y + \alpha, \end{aligned} \tag{5.1}$$

where x and y take integral values. Commencing from a particular 'divided cell', he constructed a one-sided chain of divided cells, and a corresponding one-sided chain of integer pairs $\{a_{n+1}, \varepsilon_n\}$, satisfying a similar set of conditions to equation (2.6) in [4]. A detailed discussion of this formulation may be found in [5]. The main result that we require is analogous to Theorem 2.1. Note, however, that in this case θ_n and λ_n are rational numbers.

THEOREM 5.1. *Using the notations* (2.9),

$$k^+(\varphi, \alpha) = \liminf_{n\to+\infty} M_n^+ \tag{5.2}$$

where

$$(5.3) \qquad M_n^+ = \begin{cases} \min\{M_n^{(1)}, M_n^{(2)}\}, & \text{if } (-1)^n a_1 a_2 \cdots a_n < 0 \\ \min\{M_n^{(3)}, M_n^{(4)}\}, & \text{if } (-1)^n a_1 a_2 \cdots a_n > 0. \end{cases}$$

In (5.3) we will denote the occurrence of the upper alternative by X_n and the lower alternative by Y_n.

Since the lim inf is required in (5.2), then any behaviour of the chain which occurs only a finite number of times, will not effect the value of $k^+(\varphi, \alpha)$, provided that the correct alternative is maintained.

If we reverse the rules for deciding which alternative to take in (5.3) (define this value to be M_n^-) then we are evaluating $|\xi\eta|$ in the left hand plane.

Put

$$(5.4) \qquad \begin{aligned} k^-(\varphi, \alpha) = \liminf_{n\to+\infty} M_n^- &= \liminf_{x\to-\infty} |x| \, ||\varphi x+\alpha|| \\ &= \liminf_{x\to+\infty} x \, ||\varphi x-\alpha|| \\ &= k^+(\varphi, -\alpha). \end{aligned}$$

Suppose that we have two one-sided chains which are identical from some point onwards. Let the chain for (φ, α) be $\{a_{n+1}, \varepsilon_n\}$, and the chain for (φ', α') be $\{a'_{n+1}, \varepsilon'_n\}$, where

$$\begin{aligned} a_{n+r+1} &= a'_{m+r+1}, \\ \varepsilon_{n+r} &= \varepsilon'_{m+r}, \end{aligned}$$

for some m, n and all $r \geqq 0$. Then it follows that

$$(5.5) \qquad k^+(\varphi, \alpha) = \begin{cases} k^+(\varphi', \alpha'), & \text{if } (-1)^{m+n} a_1 \cdots a_n a'_1 \cdots a'_m > 0 \\ k^+(\varphi', -\alpha'), & \text{if } (-1)^{m+n} a_1 \cdots a_n a'_1 \cdots a'_m < 0. \end{cases}$$

We will now discuss the application of two of the elementary chain operations mentioned in Theorem 2.4, and their effect on the value of $k^+(\varphi, \alpha)$. A prime attached to a variable will signify its new value after the operation has been applied.

THEOREM 5.2.

(i) *If the sign of the* $\{\varepsilon_n\}$ *chain is reversed, then*

$$k^+(\varphi', \alpha') = k^+(\varphi, -\alpha) = k^-(\varphi, \alpha).$$

(ii) *If the signs of the* $\{a_n\}$ *chain, and alternate members of the* $\{\varepsilon_n\}$ *chain are reversed, then*

$$k^+(\varphi', \alpha') = k^+(\varphi, \alpha) \text{ or } k^+(\varphi, -\alpha).$$

PROOF.

(i) We have $\theta_n = \theta'_n$, $\varphi_n = \varphi'_n$, $\lambda_n = -\lambda'_n$, $\mu_n = -\mu'_n$. Hence for all $n \geqq 0$, the application of this operation interchanges the values of $M_n^{(1)}$

and $M_n^{(4)}$, and also $M_n^{(2)}$ and $M_n^{(3)}$. Consequently, although the pairing in (5.3) is maintained, the alternatives X_n and Y_n are interchanged. The result follows by (5.3) and (5.4).

(ii) We have $\theta_n = -\theta'_n$, $\varphi_n = -\varphi'_n$. Suppose that we have $\varepsilon_n = \varepsilon'_n$ (*either* $\varepsilon_r = (-1)^r\varepsilon'_r$ with n even, *or* $\varepsilon_r = (-1)^{r-1}\varepsilon'_r$ with n odd). Then $\mu_n = \mu'_n$, $\lambda_n = -\lambda'_n$, and it is easily checked as in (i), that the products within the alternatives X_n and Y_n are interchanged. If, however, $\varepsilon_n = -\varepsilon'_n$, then we can show that the products within the alternatives X_n and Y_n are preserved. Now since

$$\operatorname{sgn}((-1)^n a'_1 a'_2 \cdots a'_n) = (-1)^n \operatorname{sgn}((-1)^n a_1 a_2 \cdots a_n),$$

we can readily check that after the application of the operation, the rules (5.3) constantly give either the same alternatives, or the opposite alternatives for the two chains. The result follows from this.

REMARK. As a consequence of this theorem, if we are investigating the possibilities arising from each of the two alternatives (e.g. the value of $\max\{k^+(\varphi, \pm\alpha)\}$), then we may arbitrarily choose the sign of some a_n and ε_r.

6. The critical semi-regular chains

In this section we will determine the critical semi-regular chains corresponding to those of Descombes in § 4. Now β'_r is a reduced quadratic irrational (for each $r \geqq -2$), and so Theorem 2.3 implies that any semi-regular rth critical chain must belong to the set of semi-regular expansions that lead forward to one of the numbers

$$\beta'_r = \beta_r - 1, \quad \beta'_r + 1 = \beta_r, \quad \frac{\beta'_r}{1-\beta'_r} = -\frac{\beta_r - 1}{\beta_r - 2}, \tag{6.1}$$

or their negatives. We will later show that the appropriate semi-regular expansions for the critical chains are those A-expansions of the β_r (or their negatives) indicated in (4.6), (4.7). In order to obtain this result we will require the following lemma which considerably restricts the sequence $\{a_n\}$ for any critical chain.

LEMMA 6.1. *If any one of the following three situations arises in a chain* $\{a_n\}$,

(i) $|a_n| = |a_{n+1}| = 2$

(ii) $|a_n| = 2, \quad |a_{n+1}| \leqq 6, \quad a_n a_{n+1} < 0$ *or*

(iii) $|a_n| > 100$,

then for at least one r, *such that* $n-2 \leqq r \leqq n+1$, *we have*

$$\max \{M_r^\pm\} < \frac{1}{\gamma} = 0{\cdot}352\cdots$$

PROOF. By the remark following Theorem 5.2 we may suppose that $a_n > 0$ and $\lambda_n \geqq 0$.

(i) *Suppose* $a_{n+1} = 2$, then since $\lambda_n < 1$, $|\mu_n| < 1$, we consider the following cases of (2.2) and (5.3).

(a) If $\theta_n < 2$, $\varphi_n < 2$, then since $|\lambda_n| < |\theta_n|-1$, $|\mu_n| < |\varphi_n|-1$ we have

$$M_n^\pm \leqq \frac{(\theta_n-1+\lambda_n)(\varphi_n-1+|\mu_n|)}{4(\theta_n\varphi_n-1)} < \frac{(\theta_n-1)(\varphi_n-1)}{\theta_n\varphi_n-1} < \tfrac{1}{3}.$$

(b) If $\theta_n > 2$, $\varphi_n > 2$, we have

$$M_n^\pm < \frac{\theta_n\varphi_n}{4(\theta_n\varphi_n-1)} < \tfrac{1}{3}.$$

(c) The other two cases follow readily by a combination of the methods of (a) and (b).

Consequently in any critical chain we cannot have two consecutive two's of the same sign, infinitely often, and so $|\theta_n| > \frac{3}{2}$, $|\varphi_n| > \frac{3}{2}$ for all large n.

Suppose $a_{n+1} = -2$, then since we may suppose

$$\tfrac{3}{2} < \theta_n < 3,\ \tfrac{3}{2} < |\varphi_n| < 3,$$

and $|\mu_n| < |\varphi_n|-1$, we have from (2.2) and (5.3) at X_n:

$$\begin{aligned}\min \{M_n^{(1)}, M_n^{(2)}\} &\leqq \frac{(\theta_n-1)(|\varphi_n|+1+|\mu_n|)}{4(\theta_n|\varphi_n|+1)}\\ &< \frac{(\theta_n-1)|\varphi_n|}{2(\theta_n|\varphi_n|+1)} < \frac{\theta_n-1}{2\theta_n} < \tfrac{1}{3};\end{aligned}$$

at Y_n:

$$\begin{aligned}&\min \{M_n^{(3)}, M_n^{(4)}\}\\ &\leqq \max\left(\frac{(\theta_n-1+\lambda_n)(|\varphi_n|+1-|\mu_n|)}{4(\theta_n|\varphi_n|+1)}, \frac{(\theta_n+1+\lambda_n)(|\varphi_n|-1-|\mu_n|)}{4(\theta_n|\varphi_n|+1)}\right)\\ &< \max\left(\frac{\theta_n(|\varphi_n|+1)}{4(\theta_n|\varphi_n|+1)}, \frac{\theta_n(|\varphi_n|-1)}{2(\theta_n|\varphi_n|+1)}\right)\\ &< \max\left(\frac{3(\frac{3}{2}+1)}{4\{3(\frac{3}{2})+1\}}, \frac{3}{10}\right) = \tfrac{15}{44}.\end{aligned}$$

(ii) By part (i), we may suppose that $-6 \leqq a_{n+1} \leqq -3$, and $0 \leqq \lambda_n < 1$. Now at X_n: examination of the proof of Theorem 2.4 in [4] reveals, in fact that

$$\min \{M_n^{(1)}, M_n^{(2)}\} < \frac{(\theta_n-\lambda_n)|\varphi_n|}{4(\theta_n|\varphi_n|+1)} < \tfrac{1}{4};$$

at Y_n we consider the two cases.

When $\mu_n \leqq 0$, then since $\theta_n < 3$, $|\varphi_n| > 2$, we have

$$M_n^{(3)} = \frac{(\theta_n-1+\lambda_n)(|\varphi_n|+1-|\mu_n|)}{4(\theta_n|\varphi_n|+1)} \leqq \frac{\theta_n(|\varphi_n|+1)}{4(\theta_n|\varphi_n|+1)} < \tfrac{9}{28}.$$

When $\mu_n > 0$, we consider the following three subcases.

(a) If $|\varphi_n| < 4$, then

$$M_n^{(4)} = \frac{(\theta_n+1+\lambda_n)(|\varphi_n|-1-|\mu_n|)}{4(\theta_n|\varphi_n|+1)} < \frac{\theta_n(|\varphi_n|-1)}{2(\theta_n|\varphi_n|+1)} < \tfrac{9}{26}.$$

(b) If $|\varphi_n| > 4$, and $0 < \mu_n \leqq 1$, then

$$M_n^{(3)} \leqq \frac{\theta_n(|\varphi_n|+2)}{4(\theta_n|\varphi_n|+1)} < \tfrac{9}{26}.$$

(c) If $4 < |\varphi_n| < 7$, and $\mu_n > 1$, then as in (a)

$$M_n^{(4)} < \frac{\theta_n(|\varphi_n|-2)}{2(\theta_n|\varphi_n|+1)} < \tfrac{15}{44}.$$

This concludes the case (ii).

(iii) The bound 100 in the enunciation of this lemma is just a convenient number, which could be reduced to 6 with considerably more effort.

When $0 \leqq \lambda_n \leqq 0{\cdot}39\theta_n$, since we may suppose $|\varphi_n| > \frac{3}{2}$, then $|\theta_n\varphi_n| > 150$, and so we have, again as in the proof of Theorem 2.4 in [4],

$$M_n^{\pm} < \frac{(\theta_n+\lambda_n)|\varphi_n|}{4(\theta_n|\varphi_n|-1)} < \frac{(1{\cdot}39)(150)}{4(149)} < \frac{1}{\gamma}.$$

When $\lambda_n > 0{\cdot}39\theta_n$, we have since $\mu_{n-1} > \lambda_n-2$,

$$\begin{aligned}\frac{\mu_{n-1}}{\varphi_{n-1}} &> \frac{\lambda_n\theta_n}{\theta_n\varphi_{n-1}} - \frac{2}{\varphi_{n-1}}\\ &> \frac{(0{\cdot}39)(a_n-1)-2}{a_n+1}\\ &\geqq \tfrac{37}{102} > 0{\cdot}36.\end{aligned}$$

Now since $|\theta_{n-1}| > \frac{3}{2}$, and $\varphi_{n-1} > 100$, we may assert in all cases

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$$M^{\pm}_{n-1} < \frac{(|\theta_{n-1}|+1+|\lambda_{n-1}|)(\varphi_{n-1}+1-\mu_{n-1})}{4(|\theta_{n-1}|\varphi_{n-1}-1)}$$

$$< \frac{|\theta_{n-1}|\{(0\cdot 64)\varphi_{n-1}+1\}}{2(|\theta_{n-1}|\varphi_{n-1}-1)}\cdot$$

$$< \frac{(1\cdot 5)(65)}{2(149)} < \frac{1}{\gamma}.$$

We have now concluded the proof of the lemma. As a consequence, the situations (i), (ii) and (iii) cannot occur infinitely often in any semi-regular critical chain.

THEOREM 6.1. *The tail of the critical $\{a_n\}$ chains consists of the A-expansions of the β_r, as given by* (4.6) *and* (4.7).

PROOF. We have already noted that the critical chains $\{a_n\}$ are among those semi-regular expansions which lead forward to one of

$$\beta'_r,\ \beta'_r+1,\ \frac{\beta'_r}{1-\beta'_r},$$

or their negatives.

Suppose that $\{a_n\}$ is an arbitrary semi-regular chain which leads forward to $\beta'_r/(1-\beta'_r)$, for some r, $r \geqq -2$. Now by (4.4) and (4.5), we have $\beta'_r > 4$, and so

$$\frac{\beta'_r}{\beta'_r-1} < \tfrac{4}{3} = [2,\ 2,\ 2].$$

Thus any such chain contains consecutive twos, and so, by the previous lemma, the complete quotients $\pm\beta'_r/(1-\beta'_r)$ may occur only a finite number of times.

Suppose then that we have a chain which leads forward to either β'_r or β_r, then we show that only their A-expansions from this point on can be in the tail of any critical chain. Suppose that we have expanded β'_r or β_r in continued fraction to the integer above, so that it equals, for some $k > 0$,

$$[a_1, a_2, \cdots, a_k, \alpha],$$

where the a_r will be 3, 5 or 6.

We will investigate the effect of changing the a_k to a_k-1, to give

$$\left[a_1, a_2, \cdots, a_k-1, \frac{-\alpha}{\alpha-1}\right].$$

Consider the following cases.

(i) $a_k = 3$. Formulae (4.6) and (4.7) imply that $\alpha > 2$, and hence

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$$\frac{\alpha}{\alpha-1} < 2.$$

Since $a_k - 1 = 2$, then Lemma 6.1 implies that this change cannot be made infinitely often in a critical chain.

(ii) $a_k = 5$. Again by (4.6) and (4.7), $\alpha = [3, 6, \cdots]$, and using the notation described in § 3, and the transformation (3.1) twice, we obtain

$$\alpha = [3, (2)_0, 6, \cdots]$$

and

$$\alpha - 1 = (1, 1, 4, \cdots),$$

whereby

$$\frac{1}{\alpha-1} = (0, 1, 1, 4, \cdots),$$

and

$$\frac{\alpha}{\alpha-1} = 1 + \frac{1}{\alpha-1} = [2, 3, (2)_3, \cdots].$$

Since the 3 leads to consecutive twos, then this chain segment cannot be part, infinitely often, of a critical chain. However, if we choose the other alternative, and change the 3 to 2, then we will again violate Lemma 6.1.

(iii) $a_k = 6$. If $\alpha = [3, 6, \cdots]$, then the result follows exactly as in part (ii). If not, then from (4.6) and (4.7), we readily see that $\alpha = [3, 3, a, 3, \cdots]$, where a is either 5 or 6. Following the method of part (ii), we obtain

$$\frac{1}{\alpha-1} = (0, 1, 1, 1, 1, a-2, 1, \cdots),$$

implying

$$\frac{\alpha}{\alpha-1} = [2, 3, 3, (2)_{a-3}, 3, \cdots].$$

Clearly we cannot leave (infinitely often) the consecutive threes, since they lead to consecutive twos, nor can we change the first 3 to a 2, without violating Lemma 6.1. Also, using the same method, we find that

$$[3, (2)_{a-3}, 3, \cdots] = [2, -(a-1), \cdots],$$

which contravenes Lemma 6.1, for a critical chain.

Thus we have shown that we cannot deviate from the A-expansion of β_r (or β'_r) infinitely often, without implying, for the corresponding φ, and any α (such that $\varphi x + y + \alpha$ does not represent zero),

$$k^+(\varphi, \alpha) < \frac{1}{\gamma}.$$

Hence, as a consequence of Theorem 2.3, the tail of any critical chain must be given by those semi-regular expansions (4.7) (or their negatives). Associated with each of these a-chains will be a corresponding ε-chain which we will now determine.

LEMMA 6.2. *The tail of the ε-chain for critical chains*

(i) *is alternating in sign if $\{a_n\}$ has β_r as its tail,*

(ii) *has constant sign if $\{a_n\}$ has $-\beta_r$ as its tail.*

PROOF. By Theorem 5.2, (ii) follows from (i), and so we may suppose $a_n > 0$ for all $n > N$. From the form of the relevant expansions if n is large enough, we have $\theta_n \varphi_n > 4$, and so, as in the proof of Theorem 2.4 in [4],

$$\min \{M_n^{\pm}\} < \frac{(\theta_n - |\lambda_n|)\varphi_n}{4(\theta_n \varphi_n - 1)} < \tfrac{1}{3}.$$

Since $a_{n+1} > 0$, then by (5.3) the cases X_n and Y_n will alternate with successive values of n; hence so too must the sign of λ_n, in order to maintain the products containing the factors $(\theta_n \pm 1 + |\lambda_n|)$.

COROLLARY. *The appropriate products, for large enough n, are*

$$\tag{6.2} \begin{cases} \dfrac{(|\theta_n|+1+|\lambda_n|)(|\varphi_n|+1-|\mu_n|)}{4(|\theta_n \varphi_n|-1)} & \textit{and} \\[2ex] \dfrac{(|\theta_n|-1+|\lambda_n|)(|\varphi_n|-1+|\mu_n|)}{4(|\theta_n \varphi_n|-1)}. \end{cases}$$

This follows immediately from (2.2) and (5.3), and the previous lemma.

LEMMA 6.3. *In any critical chain, for n large enough, whenever*

$$a_{n+1} = 3,\ 5,\ \text{or}\ 6,$$

then

$$|\varepsilon_n| = 1,\ 1,\ \text{or}\ 2,$$

respectively.

PROOF.

(i) When $a_{n+1} = 3$, the result follows (2.6) in [4], since ε_n must be odd.

(ii) When $a_{n+1} = 5$, by (4.7) and Theorem 6.1, we have a chain segment

$$[\cdots, 6, 3, 3, 5, 3, 6, \cdots].$$

If $|\varepsilon_n| = 3$, then

$$|\mu_n| > 3 + \frac{1}{\varphi_{n+1}} > 3{\cdot}25,$$

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and $|\lambda_n| < 2$, $\theta_n > 2{\cdot}5$, $\varphi_n < 4{\cdot}75$. Thus by (6.2),

$$M^{\pm} < \frac{(\theta_n+3)(\varphi_n-2{\cdot}75)}{4(\theta_n\varphi_n-1)} < \frac{(5{\cdot}5)(2{\cdot}5)}{4\{(2{\cdot}5)(4{\cdot}75)-1\}}$$
$$< \frac{1}{\gamma}.$$

Consequently, $|\varepsilon_n| = 1$.

(iii) When $a_{n+1} = 6$, we have the chain segment

$$[\cdots, 3, 6, 3, \cdots].$$

If $|\varepsilon_n| = 4$, then as in case (ii), $|\mu_n| > 4{\cdot}25$, $|\lambda_n| < 2$, $\varphi_n < 5{\cdot}75$, and $\theta_n > 2{\cdot}5$, implying

$$M_n^{\pm} < \frac{(5{\cdot}5)(2{\cdot}5)}{4\{(2{\cdot}5)(5{\cdot}75)-1\}} < \frac{1}{\gamma}.$$

If $\varepsilon_n = 0$, then by the method of proof of Theorem 2.4 in [4], we have, since $\theta_{n+1} > 5{\cdot}5$, $\varphi_{n+1} > 2{\cdot}5$,

$$M_{n+1}^{\pm} < \frac{(\theta_{n+1}+|\lambda_{n+1}|)\varphi_{n+1}}{4(\theta_{n+1}\varphi_{n+1}-1)} < \frac{(\theta_{n+1}+1)\varphi_{n+1}}{4(\theta_{n+1}\varphi_{n+1}-1)}$$
$$< \frac{(6{\cdot}5)(2{\cdot}5)}{4\{(5{\cdot}5)(2{\cdot}5)-1\}} < \frac{1}{\gamma}.$$

The lemma now follows in full.

Consequently, if n is large enough, the ε_n associated with a_{n+1} in a critical chain is automatically fixed by these two lemmas. We may therefore consider the blocks A, B, and C of (4.6) to be blocks of integer pairs. We may now state the following result which follows Theorem 4.1.

THEOREM 6.2. *Suppose we have a one-sided chain pair which has as its tail the A-expansion of β_r, for some $r \geqq -2$, given by* (4.6) *and* (4.7), (*or any chain obtained from such a chain pair by application of one of the operations of Theorem* 5.2); *then for the corresponding φ and α, exactly one of $k^+(\varphi, \alpha)$ or $k^+(\varphi, -\alpha)$ has the value $1/\gamma_r$ (of* § 4), *while the other has a value not exceeding $\frac{1}{4}$.*

7. Supremum of values taken by $M^*(f)$

If f is any I-reduced form given by (1.3), and P is a point such that the corresponding grid $\mathscr{L}$, given by (1.5) and (2.1), does not have a point on either axis, then we readily see that

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$$
(7.1)\qquad
\begin{aligned}
M^{+}(f;P) &= \inf\{|\xi\eta|;\ (\xi,\eta)\in\mathscr{L},\ \xi>0\}\\
M^{-}(f;P) &= \inf\{|\xi\eta|;\ (\xi,\eta)\in\mathscr{L},\ \xi<0\}\\
M^{+}(g;Q) &= \inf\{|\xi\eta|;\ (\xi,\eta)\in\mathscr{L},\ \eta>0\}\\
M^{-}(g;Q) &= \inf\{|\xi\eta|;\ (\xi,\eta)\in\mathscr{L},\ \eta<0\}.
\end{aligned}
$$

Because the rules for moving from cell to cell by the algorithm are the same for the modification of § 5 as for the general method, so too the rules for determining which pair of vertices is in the right hand plane remain unaltered. Thus if A_n is in the first quadrant for some n, then the sign of a_{n+1} completely determines the quadrant of A_{n+1}, and the sign of a_n determines the quadrant of A_{n-1}. Now A_0 is in the first quadrant, and so we may evaluate $M^{+}(f;P)$ by the following straight forward extension of (5.3).

It is clear that

$$
(7.2)\qquad M^{+}(f;P)=\inf_n M_n^{+}(f;P),
$$

where

$$
(7.3)\qquad
\begin{aligned}
M_0^{+}(f;P) &= \min\{M_0^{(3)}, M_0^{(4)}\},\ \text{and for } n>0,\\
M_n^{+}(f;P) &= \begin{cases}\min\{M_n^{(1)}, M_n^{(2)}\}, & \text{if } (-1)^n a_1a_2\cdots a_n<0,\\ \min\{M_n^{(3)}, M_n^{(4)}\}, & \text{if } (-1)^n a_1a_2\cdots a_n>0,\end{cases}\\
M_{-n}^{+}(f;P) &= \begin{cases}\min\{M_{-n}^{(1)}, M_{-n}^{(2)}\}, & \text{if } (-1)^n a_0a_{-1}\cdots a_{1-n}<0,\\ \min\{M_{-n}^{(3)}, M_{-n}^{(4)}\}, & \text{if } (-1)^n a_0a_{-1}\cdots a_{1-n}>0.\end{cases}
\end{aligned}
$$

Again we shall refer to the upper and lower alternatives at the nth step as X_n and Y_n, respectively. By consistently reversing the rules (7.3), taking X_0 as a reference point (i.e. $M_0^{-}(f;P)=\min\{M_0^{(1)}, M_0^{(2)}\}$), we may calculate $M^{-}(f;P)$. The chain for a grid defined by g and Q may be determined from the following lemma, suggested by Theorem 2.4.

LEMMA 7.1. *If the doubly infinite chain pair* $\{a_{n+1}, \varepsilon_n\}$ *is reversed about some point* (*e.g.* $n=0$), *the chain obtained corresponds to the grid derived from the form* g, *and the point* Q *of* (1.8) *and* (1.9).

PROOF. For a step n in the original chain, there corresponds a step n' in the resultant chain such that

$$
\begin{aligned}
\theta_n &= \varphi_{n'}, &\quad \varphi_n &= \theta_{n'},\\
\lambda_n &= \mu_{n'}, &\quad \mu_n &= \lambda_{n'}.
\end{aligned}
$$

Consequently, the groupings as a whole of the four products $M_n^{(r)}$ are preserved in the new chain (for some other value of n) but their order is not. In particular $M_n^{(2)}$ and $M_n^{(3)}$ are interchanged. Now as we have noted in § 2 of this paper, $M_n^{(1)}$, $M_n^{(2)}$, $M_n^{(3)}$, $M_n^{(4)}$ are derived from C_n, D_n,

B_n, A_n, respectively. Thus at the alternative $X_{n'}$ in the reversed chain, we are evaluating a minimum involving the coordinates at C_n and B_n, vertices of the nth divided cell in a grid associated with the original chain. Similarly, $Y_{n'}$ corresponds to an evaluation at A_n and D_n. The lemma may then be checked to hold in all cases, the result following from (1.8), (1.9), (2.2), and (7.3).

This lemma enables the calculation of $M^{\pm}(g; Q)$, and hence of $M^*(f; P)$.

In the previous section of this paper, certain results were stated which involved the lim inf of one-sided chain pairs. We will now deduce from them, various results on this restricted infimum of two-sided chains. We will use the following obvious extension for doubly infinite chains of a notation already used.

$$[_\infty(a_1, a_2, \cdots, a_n), \cdots].$$

Consider the following chain pairs, where $\{\varepsilon_n\}$ is chosen in accordance with Lemmas 6.2, 6.3, and the blocks A, B, C are given in (4.7).

$$\begin{aligned} &C_{-2}\colon && [_\infty A_\infty] \\ &C_{-1}\colon && [_\infty(BC)_\infty] \\ &C_0\colon && [_\infty B_\infty] \\ &C_r\colon && [_\infty(A(BC)_r)_\infty], \text{ for } r \geqq 1. \end{aligned} \tag{7.4}$$

Theorem 7.1. *If the chain pair C_r $(r \geqq -2)$ corresponds to a form f_r, and a point $P_r = P_r(x_r, y_r)$, then*

$$\max\{M^{\pm}(f_r; P_r)\} = \frac{\Delta}{\gamma_r}; \tag{7.5}$$

furthermore, if

$$g_r = f_r\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \text{ and } Q_r = Q_r(y_r, x_r), \tag{7.6}$$

then

$$\max\{M^{\pm}(g_r; Q_r)\} \leqq \frac{\Delta}{\gamma_r}, \tag{7.7}$$

where equality holds if and only if the chain C_r is symmetrical (identical with its inverse). If equality does not hold, then we may replace the constant in (7.7) by Δ/γ.

Proof. It is clear from Theorems 17.1 in [4], and 6.2, that if the infimum in the definitions of $M^{\pm}(f_r; P_r)$ were replaced by lim inf, then (7.5) would hold. But since the chains C_r are totally periodic, there are only a finite number of different values for the $M_n^{(k)}(f_r; P_r)$. Consequently, the infimum will equal the lim inf, and so (7.5) follows.

By Lemma 7.1, the chain pair associated with g_r and Q_r will be the reverse of the chain C_r. In the case C_{-2} and C_0 (the only symmetrical C_r), equality will clearly hold in (7.7). But for $r = -1$, and $r \geqq 1$, C_r is not symmetrical, and its reverse provides a new periodic chain, and hence any right hand chain obtained from this chain by truncation can never be one of the semi-regular critical chains. Thus equality in (7.7) would contradict Theorem 6.2, as would equality with any constant exceeding Δ/γ.

COROLLARY. $M^*(f_r; P_r) = (\Delta/\gamma_r)$.
This is immediate upon (1.10)

THEOREM 7.2. $M^*(f_r) = (\Delta/\gamma_r)$.

PROOF. We may suppose that $f_r(x, y)$ is given by (1.3) where

$$\varphi = \beta_r, \quad \theta = 1/\bar{\varphi},$$

$\bar{\varphi}$ being the algebraic conjugate of φ. Then the chain for f_r, and some point S, such that the corresponding grid has no point on an axis, must contain a semi-regular expansion of φ, as a right hand chain (together with an associated ε-chain). Now the theory of § 6 clearly demonstrates that the A-expansion of φ must be taken if the infimum of the chain is to exceed Δ/γ. Hence the lim inf of the chain does not exceed Δ/γ_r, implying

$$M^{\pm}(f_r; S) \leqq \frac{\Delta}{\gamma_r},$$

for all such S. Similarly for g_r and points S' with the required property, we obtain

$$M^{\pm}(g_r; S') \leqq \frac{\Delta}{\gamma_r}.$$

The result follows by (1.11) and the previous corollary.

The following lemma will enable us to construct from a two-sided chain with a certain infimum, a one-sided chain with an arbitrarily close lim inf.

LEMMA 7.2. *If H is a finite set of integer pairs, and $\{a_{n+1}, \varepsilon_n\}$ any infinite sequence whose elements are taken from H, then for every integer $j > 0$, there exists a block containing j integer pairs of H, and this block occurs infinitely often in the sequence $\{a_{n+1}, \varepsilon_n\}$.*

PROOF. The lemma is clearly true for $j = 1$. Assuming the truth of the assertion for $j = k$, we have that there exists a block of k members of H which occur consecutively in the given sequence, infinitely often. We may suppose these blocks are well separated, by taking an infinite subset, if necessary. These blocks can be followed only by an element from a finite set, and so one such element must occur infinitely often. Thus the result holds for $j = k+1$.

THEOREM 7.3. *For all forms that do not represent zero, we have*

$$M^*(f) \leqq \frac{27\Delta}{28\sqrt{7}}.$$

Suppose that f is not equivalent to the form f_{-2} of Theorem 7.1, then

$$M^*(f) \leqq \frac{\Delta}{\gamma_{-1}} = \frac{359\Delta}{45\sqrt{510}},$$

where equality holds for forms equivalent to f_{-1}.

PROOF. Suppose C is an arbitrary doubly infinite chain, not identical to C_{-2}, and let f and P correspond to C. Then C falls into at least one of the following three types.

(a) C does not contain the subchain $[\cdots A_\infty]$.

(b) $C = [WA_\infty]$, where W is a one-sided chain which does not contain the subchain $[{}_\infty A \cdots]$.

(c) $C = [{}_\infty AVA_\infty]$, where V is a finite chain segment not equal to segment A_n, for any positive integer n.

Assume $M^+(f; P) = \rho\Delta$. Then for all n, we have

$$M_n^+(f; P) = M_n^+ \geqq \rho\Delta.$$

Assume, for an appropriate ε, that

$$0 < \varepsilon < \rho - \frac{1}{\gamma_{-1}}. \tag{7.8}$$

After Lemma 6.1, we have that $|\theta_n|$ and $|\varphi_n|$ are both bounded in the interval $(1{\cdot}5, 101)$. Thus we may apply Theorem 17.1 in [4], and the constant implied by the order notation is independent of the particular chain segment under consideration. There therefore exists an integer m, with the property that the respective products belonging to the centre of a common chain segment of length $2m$ from two chain pairs, differ by no more than ε.

Now in the cases (a) and (b), C must contain some chain segment different from A, which occurs infinitely often. Consequently, by the method of Lemma 7.2, there exists a block, D say, of length $2m$ and containing this segment, which occurs in C infinitely often. In the case (c), let $D = A_{2m}$.

Consider the one-sided chain C^*, given by

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$$C^*: \ [(DZ)_\infty], \tag{7.9}$$

where in the cases (a) and (b), Z is a chain segment which separates two blocks D, and in the case (c) put $Z = V$. (Note that in the former cases we may always pick Z so that it separates two blocks D which commence at the same alternative, either X_n or Y_n.)

Since every step in the chain C^* (far enough along) is the centre of a chain segment of length $2m$ which also appears in C, then it follows from (5.3), (7.3), (7.8) and the choice of m, that, for some φ and α corresponding to the chain C^*,

$$k^+(\varphi, \alpha) \geqq \rho - \varepsilon > \frac{1}{\gamma_{-1}}.$$

But, by construction, C^* is not one of the critical chains of Theorem 6.2, and so we have a contradiction. The theorem now follows.

8. Further results for $M^*(f)$

It would be of interest to know whether $M^*(f)$ takes only the discrete values Δ/γ_r, $r \geqq -2$, greater than Δ/γ. In the previous section we have seen that for certain equivalence classes of forms $\{f_r\}$, we have $M^*(f_r) = \Delta/\gamma_r$. But it is possible that there are other doubly infinite chains for which $M^*(f) > \Delta/\gamma$.

To enable such results to be obtained, it seems certain that we would need lemmas of the type Lemma 28 ([7], p. 349), whereby the products at certain 'privileged' points of the following chain segments are compared.

$$\cdots A'(B'C')_r A'(B'C')_k A' \cdots, \qquad 0 \leqq k \leqq k',$$
$$\cdots A'(B'C')_{r'} A'(B'C')_{k'} A' \cdots, \qquad 1 \leqq r < r'.$$

It is probable that similar results would follow through for the semi-regular algorithm, by the same type of argument used in [7] (p. 326—355). If this were so, then the following chain

$$[_\infty(A(BC)_k)(A(BC)_r)_\infty],$$

for corresponding f and P, would have

$$M^+(f; P) \leqq \frac{\Delta}{\gamma} \text{ for } k > r, \quad \text{and} \quad \liminf_{|n|\to\infty} M_n^+(f; P) = \frac{\Delta}{\gamma_r}, \text{ for } k < r.$$

The question as to whether Δ/γ_r is approached from above or below could be settled by detailed, but straight-forward analysis of the type needed for Lemma 28 of [7]. We will not undertake such an investigation in this paper, but it seems reasonable to conjecture that the Δ/γ_r are, in fact,

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the only values taken by $M^*(f)$ which exceed Δ/γ. For example, if $\{r_s\}$ is a finite, strictly increasing sequence of positive integers, with k members, then it would be consistent with [7] to conjecture that the chain

$$[_\infty(A(BC)_{r_1})A(BC)_{r_2}\cdots A(BC)_{r_{k-1}}(A(BC)_{r_k})_\infty]$$

has infimum equal to Δ/γ_{r_k}.

References

[1] E. S. Barnes, 'The inhomogeneous minima of binary quadratic forms IV', *Acta Math.* 92 (1954), 235–264.
[2] E. S. Barnes, 'On linear inhomogeneous diophantine approximation', *J. London Math. Soc.* 31 (1956), 73–79.
[3] E. S. Barnes, and H. P. F. Swinnerton-Dyer, 'The inhomogeneous minima of binary quadratic forms III', *Acta Math.* 92 (1954), 199–234.
[4] P. E. Blanksby, 'On the product of two linear forms, one homogeneous and one inhomogeneous', *J. Aust. Math. Soc.* 8 (1968), 457—511.
[5] P. E. Blanksby, *Ph. D. Thesis* (submitted to University of Adelaide, Australia, 1967).
[6] J. W. S. Cassels, 'Über $\underline{\lim}_{x\to+\infty}\, x|\theta x+\alpha-y|$', *Math. Ann.* 127 (1954) 288–304.
[7] R. Descombes, 'Sur la répartition des sommets d'une ligne polygonale régulière non fermée', *Ann. Sci. Ec. Norm. Sup.* 75 (1956), 284–355.
[8] R. Descombes, 'Problèmes d'approximation diophantienne', *L'Enseignement Math.* 6 (1960), 18–26.
[9] L. E. Dickson, *Introduction to the Theory of Numbers* (Dover, 1957).
[10] E. J. Pitman *Ph. D. Thesis* (University of Sydney, Australia, 1957).
[11] E. J. Pitman, 'The inhomogeneous minima of a sequence of symmetric Markov forms', *Acta Arith.*, 5 (1958), 81–116.
[12] E. J. Pitman, 'Davenport's Constant for indefinite binary quadratic forms', *Acta Arith.*, 6 (1960), 37–46.

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