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Hypergeometric polynomials and integer programming

MUTSUMI SAITO[1], BERND STURMFELS[2] and NOBUKI TAKAYAMA[3]
[1]*Department of Mathematics, Hokkaido University, Sapporo, Japan*
[2]*Department of Mathematics, University of California, Berkeley, CA 94720, U.S.A. and Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan*
[3]*Department of Mathematics, Kobe University, Kobe, Japan*

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Abstract. We examine connections between A-hypergeometric differential equations and the theory of integer programming. In the first part, we develop a 'hypergeometric sensitivity analysis' for small variations of constraint constants with creation operators and b-functions. In the second part, we study the indicial polynomial (b-function) along the hyperplane $x_i = 0$ via a correspondence between the optimal value of an integer programming problem and the roots of the indicial polynomial. Gröbner bases are used to prove theorems and give counter examples.

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1. Introduction

In this paper we examine connections between hypergeometric differential equations and the theory of integer programming. Let $A = (a_{ij})$ be a non-negative integer $d \times n$-matrix which has no zero column. Let a_i be the ith column vector of A. We obtain a linear map

$$T\colon \mathbf{N}^n \to \mathbf{N}^d, \quad u \mapsto A \cdot u, \tag{1.1}$$

where $\mathbf{N} = \{0, 1, 2, \ldots\}$. The fiber $T^{-1}(\alpha)$ over a point $\alpha \in \mathbf{N}^d$ is called the set of feasible points. *Integer programming* is concerned with the problem of minimizing a linear functional ω over $T^{-1}(\alpha)$. On the other hand, the matrix A and a parameter vector α define the A-*hypergeometric system* of partial differential equations due to Gel'fand, Kapranov and Zelevinsky [9]

$$\begin{aligned} &\left(\sum_{j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j} - \alpha_i \right) \psi = 0 \quad \text{for } i = 1, \ldots, d, \\ &\left(\left(\frac{\partial}{\partial x}\right)^u - \left(\frac{\partial}{\partial x}\right)^v \right) \psi = 0 \quad \text{for all } u, v \in \mathbf{N}^n \quad \text{with } Tu = Tv. \end{aligned} \tag{1.2}$$

This can be regarded as a holonomic $\mathcal{D}$-module (see [1], [3]) on affine n-space $\mathbf{C}^n$. The A-hypergeometric system (1.2) is an excellent test case for studying general problems in algebraic analysis, and there are many important and beautiful connections to combinatorics, algebraic geometry (see [8]) and theoretical physics (see e.g. [10]).

Our point of departure in this work is Proposition 2.1 which states that (1.2) has at most one linearly independent polynomial solution, namely, the *hypergeometric polynomial*

$$\Phi(\alpha;x) := \sum_{u\in T^{-1}(\alpha)} \frac{x^u}{u\,!} := \sum_{u\in T^{-1}(\alpha)} \frac{x_1^{u_1}x_2^{u_2}\cdots x_n^{u_n}}{u_1!\,u_2!\cdots u_n!}. \tag{1.3}$$

This polynomial encodes the fiber $T^{-1}(\alpha)$. In Sections 2 and 3 we develop a 'hypergeometric sensitivity analysis' for small variations of the right hand side α of our integer program. The key player is a certain differential operator C_i, called the *creation operator*, which transforms $\Phi(\alpha - a_i; x)$ into $\Phi(\alpha; x)$. The existence of C_i is proved in Theorem 3.1. In Algorithms 3.1 and 3.1 we show how to compute C_i using Gröbner bases.

Sections 4 and 5 are concerned with the *indicial polynomial* of the A-hypergeometric system along the hypersurface $x_i = 0$. The notion of indicial polynomial appears classically in the Frobenius method for solving ordinary linear differential equations. The roots of the indicial polynomial, called exponents, indicate the lowest order terms in a possible power series solution. For the modern approach in terms of $\mathcal{D}$-modules, see [11], [12], or [13].

Our main results are Theorem 4.2 and Theorem 5.1 which give formulas for the indicial polynomial, the first for arbitrary A and the second for normal A. These formulas involve polyhedral combinatorics and the value function of an integer program (see [2]).

One of our objectives is to supply users of Gröbner bases software with some new algorithmic tools. The Gröbner-minded reader will notice a surprising interplay between

– Gröbner bases for commutative polynomials (the classical version; see e.g. [4]),
– Gröbner bases for integer programming (as in [6], [22], [26]),
– Gröbner bases in the ring of differential operators (as in [16], [24]).

2. A generating function for feasible points

We fix a linear map $T\colon \mathbf{N}^n \to \mathbf{N}^d$ as in (1.1). For each $\alpha \in \mathbf{N}^d$ the *fiber* $T^{-1}(\alpha) = \{\, u \in \mathbf{N}^n\colon Au = \alpha\}$ is a finite set. The integer programming problem is to minimize a linear functional ω over this set. We encode the fiber $T^{-1}(\alpha)$ by the polynomial $\Phi(\alpha, x)$ in (1.3).

Throughout this paper we assume that $\mathrm{rank}(A) = d$, and that the vector $(1, 1, \ldots, 1)$ lies in the row space of A, or equivalently, that the column vectors $a_1, \ldots, a_n$ span affinely a hyperplane not passing through the origin in $\mathbf{R}^d$; see e.g. [22, Lem. 4.14]. This allows to define $\text{degree}(\alpha) := u_1 + \cdots + u_n$ for any $u = (u_1, \ldots, u_n) \in T^{-1}(\alpha)$, and we may compute the polynomials $\Phi(\alpha; x)$ by means of the generating function

$$\sum_{\alpha \in \mathbf{N}^d} \text{degree}(\alpha)\,! \cdot \Phi(\alpha; x) \cdot t_1^{\alpha_1} \cdots t_d^{\alpha_d} = \frac{1}{1 - \sum_{i=1}^n x_i \cdot t_1^{a_{1i}} \cdots t_d^{a_{di}}}. \tag{2.1}$$

We next recall the definition of the A-hypergeometric system due to Gel'fand, Kapranov and Zelevinsky [9]. Consider the *Weyl algebra* over the field of rational numbers

$$A_n := \mathbf{Q}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle.$$

The $2n$ variables satisfy the commutation relations

$$x_i x_j = x_j x_i, \quad \partial_i \partial_j = \partial_j \partial_i, \quad \partial_i x_j = x_j \partial_i$$
$$\text{if } i \neq j, \quad \text{and} \quad \partial_i x_i = x_i \partial_i + 1.$$

In the commutative polynomial subring $\mathbf{Q}[\partial_1, \ldots, \partial_n]$ of A_n we consider the *toric ideal*

$$I_A := (\partial^u - \partial^v \;:\; Au = Av).$$

Recall from [6], [22] and [26] that the integer programming problem can be solved by normal form reduction modulo the Gröbner basis of I_A with respect to ω.

For any $\alpha \in \mathbf{Q}^d$ we introduce the linear differential operators

$$Z_i(\alpha_i) := \sum_{j=1}^n a_{ij} x_j \partial_j \;-\; \alpha_i \quad \text{for } i = 1, \ldots, d.$$

The A-hypergeometric system with parameter vector α is the left A_n-module generated by the toric ideal I_A and the operators $Z_1(\alpha_1), \ldots, Z_d(\alpha_d)$. A function ψ on an open subset of $\mathbf{C}^n$ which is annihilated by this left module is called *A-hypergeometric with parameters* α. This definition agrees with the slightly more informal description in (1.2).

PROPOSITION 2.1. *The A-hypergeometric system has a nonzero polynomial solution if and only if the parameter vector α is integral and lies in the image of T. In this case $\Phi(\alpha; x)$ is the unique (up to scaling) A-hypergeometric polynomial with parameters α.*

Proof. Let $\psi = \Sigma c_u x^u$ be an A-hypergeometric polynomial for some α. The relations $Z_i(\alpha_i)\psi = 0$ imply $T(u) = \alpha$ for all u appearing in ψ. In particular, we find that α is integral and lies in the image of T. For any two terms $c_u x^u$ and $c_v x^v$ in ψ we have $\partial^u - \partial^v \in I_A$. The relation $\partial^u \psi = \partial^v \psi$ implies $u\,! \cdot c_u = v\,! \cdot c_v$. Therefore the space of A-hypergeometric polynomials with parameters α is one-dimensional. It is easily checked that $\Phi(\alpha; x)$ is annihilated by all operators in I_A, and hence it spans this space.

EXAMPLE 2.1. (The twisted cubic curve). Let $n = 4, d = 2$ and

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

Here I_A is the defining ideal of the twisted cubic curve in projective 3-space. It is generated by the 2×2-minors of $\begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \\ \partial_2 & \partial_3 & \partial_4 \end{pmatrix}$. A function $\psi = \psi(x_1, x_2, x_3, x_4)$ is hypergeometric with parameters (α_1, α_2) if and only if

$$\begin{aligned} &\partial_1\partial_3\psi = \partial_2^2\psi, \quad \partial_1\partial_4\psi = \partial_2\partial_3\psi, \quad \partial_2\partial_4\psi = \partial_3^2\psi. \\ &3x_1\partial_1\psi + 2x_2\partial_2\psi + x_3\partial_3\psi = \alpha_1 \cdot \psi, \\ &x_2\partial_2\psi + 2x_3\partial_3\psi + 3x_4\partial_4\psi = \alpha_2 \cdot \psi. \end{aligned}$$

Here is a small example of an A-hypergeometric polynomial

$$\Phi(6, 6; x) = \tfrac{1}{4}x_1^2x_4^2 + x_1x_2x_3x_4 + \tfrac{1}{6}x_1x_3^3 + \tfrac{1}{6}x_2^3x_4 + \tfrac{1}{4}x_2^2x_3^2.$$

EXAMPLE 2.2. (The transportation problem, [22, Exam. 5.1]). Fix positive integers r and s and let $\mathbf{N}^{r\times s}$ be the monoid of non-negative integer $r \times s$-matrices. We consider the linear operator T: $\mathbf{N}^{r\times s} \to \mathbf{N}^{r+s}$ which maps a matrix to its vector of row sums and column sums. Here $n = r \cdot s$ and $d = r + s$, and A is a unimodular $\{0, 1\}$-matrix of rank $d - 1$. The columns of A are the vertices of the product of regular simplices $\Delta_{r-1} \times \Delta_{s-1}$. The integer programming problem associated with A is called the *transportation problem*. The toric ideal I_A is generated by the 2×2-minors of an $r \times s$-matrix of indeterminates. The variety defined by I_A is the Segre embedding of the product of projective spaces $P^{r-1} \times P^{s-1}$. The corresponding system of hypergeometric differential equations is called *the hypergeometric system of type* $(r, r + s)$. It equals

$$\begin{aligned} &\frac{\partial^2\psi}{\partial x_{ij}\partial x_{kl}} = \frac{\partial^2\psi}{\partial x_{il}\partial x_{kj}} \quad \text{for } 1 \leqslant i < j \leqslant r,\ 1 \leqslant k < l \leqslant s, \\ &\sum_{i=1}^{r} x_{ij}\frac{\partial\psi}{\partial x_{ij}} = \gamma_j \cdot \psi \quad \text{for } j = 1, \ldots, s, \\ &\sum_{j=1}^{s} x_{ij}\frac{\partial\psi}{\partial x_{ij}} = \rho_i \cdot \psi \quad \text{for } i = 1, \ldots, r. \end{aligned} \tag{2.2}$$

The system (2.2) was associated with the Grassmannian of r-planes in $\mathbf{C}^{r+s}$ in [7]. It is holonomic of rank $\binom{r+s-2}{r-1} = \mathrm{vol}(\Delta_{r-1} \times \Delta_{s-1})$. See [9] for details.

Let $\gamma = (\gamma_1, \ldots, \gamma_s)$ and $\rho = (\rho_1, \ldots, \rho_r)$ be non-negative integer vectors such that $\gamma_1 + \cdots + \gamma_s = \rho_1 + \cdots + \rho_r$. The fiber $T^{-1}(\gamma, \rho)$ is the set of all non-negative integer $r \times s$-matrices with row sums ρ and column sums γ. We encode this set by the polynomial

$$\Phi(\gamma, \rho; x) = \sum_{u \in T^{-1}(\gamma,\rho)} \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{x_{ij}^{u_{ij}}}{u_{ij}\,!}. \tag{2.3}$$

This is the unique (up to scaling) polynomial solution to (2.2).

The hypergeometric polynomials in (2.3) satisfy the following relations for all i, j, γ, ρ

$$\frac{\partial}{\partial x_{ij}} \Phi(\gamma, \rho; x) = \Phi(\gamma - e_i, \rho - e_j; x). \tag{2.4}$$

These are the simplest *contiguity relations*. They are straightforward to check.

We shall be interested in inverting the effect of the differential operator $\partial/\partial x_{ij}$ in (2.4). This is accomplished by the following non-trivial contiguity relations due to Sasaki [20].

THEOREM 2.1. [20]. The operator $C_{ij} := x_{ij} + \Sigma_{p=1}^{r} \Sigma_{q=1}^{s} x_{pj} x_{iq} (\partial/\partial x_{pq})$ satisfies

$$C_{ij}\, \Phi(\gamma, \rho;\, x\,) = (\gamma_i + 1) \cdot (\rho_j + 1) \cdot \Phi(\gamma + e_i, \rho + e_j;\, x\,). \tag{2.5}$$

The contiguity relations (2.5) can be used iteratively to compute any of the hypergeometric polynomials $\Phi(\gamma, \rho; x)$ and hence to enumerate any set of non-negative integer matrices with fixed row and column sums. The idea is to start with the trivial hypergeometric polynomial $\Phi(0, 0; x) = 1$ and then to apply an appropriately scaled sequence of the *creation operators* $C_{11}, C_{12}, \ldots, C_{rs}$. Here is a little example for $r = s = 3$

$$\begin{aligned}
&\Phi((2,2,1),(1,3,1);x) \\
&\quad = \tfrac{1}{24} \cdot C_{11}(C_{12}(C_{22}(C_{23}(C_{32}(1))))) \\
&\quad = x_{11}x_{12}x_{22}x_{23}x_{32} + x_{12}x_{13}x_{21}x_{22}x_{32} \\
&\qquad + \tfrac{1}{2} \cdot (x_{12}x_{22}^2x_{13}x_{31} + x_{12}^2x_{22}x_{23}x_{31} + x_{11}x_{13}x_{22}^2x_{32} \\
&\qquad\qquad + x_{12}^2x_{21}x_{23}x_{32} + x_{12}^2x_{21}x_{22}x_{33} + x_{11}x_{12}x_{22}^2x_{33})
\end{aligned} \tag{2.6}$$

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3. Computing creation operators

An important issue in integer programming is to understand how the fiber $T^{-1}(\alpha)$ behaves under a small change in the right-hand side α. Equivalently, how does the hypergeometric polynomial $\Phi(\alpha; x)$ change under a small variation of α? It is easy to see that subtracting a column vector a_i from the right-hand side α corresponds to taking a partial derivative

$$\partial_i \Phi(\alpha; x) = \Phi(\alpha - a_i; x). \tag{3.1}$$

This is the simplest *contiguity relation*. We call ∂_i the ith annihilation operator.

In this section we address the problem of inverting the annihilation operator ∂_i. The goal is to compute a differential operator whose action on hypergeometric polynomials corresponds to adding a column vector a_i to the right-hand side α. For the transportation problem (Example 2.2) such operators C_{ij} were given in Theorem 2.1. In what follows we explain how to preprocess an arbitrary matrix A for subsequent derivations like (2.6).

We call the matrix A *normal* if the monoid spanned by its columns is normal, i.e.,

$$\sum_{i=1}^{n} \mathbf{N} a_i = \sum_{i=1}^{n} \mathbf{Z}\, a_i \cap \sum_{i=1}^{n} \mathbf{R}_+ a_i. \tag{3.2}$$

Let $s_1, \ldots, s_d$ be indeterminates and form the Weyl algebra over these parameters

$$A_n[s_1, \ldots, s_d] = \mathbf{Q}[s_1, \ldots, s_d]\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle.$$

Let $\mathcal{B}_{A,i}(s)$ be the left ideal in $A_n[s_1, \ldots, s_d]$ generated by the toric ideal I_A, the annihilation operator ∂_i and the parametric linear operators $Z_1(s_1), \ldots, Z_d(s_d)$. We are interested in its intersection with the commutative polynomial subring $\mathbf{Q}[s_1, \ldots, s_d]$.

THEOREM 3.1.

(a) *The elimination ideal* $\mathcal{B}_{A,i}(s) \cap \mathbf{Q}[s_1, \ldots, s_d]$ *is nonzero.*
(b) *If A is normal then* $\mathcal{B}_{A,i}(s) \cap \mathbf{Q}[s_1, \ldots, s_d]$ *is a principal ideal.*
(c) *If α is not a zero of the above elimination ideal, then the annihilation operator ∂_i possesses an inverse r_i modulo the A-hypergeometric module with parameters α.*

Proof. Part (a) is proved in [19, page 560]. Part (b) is proved in [18]. For part (c) choose an element $b = b(s)$ in the elimination ideal such that $b(\alpha) \neq 0$. There exists a relation

$$b(s) = r_i(x, \partial, s) \cdot \partial_i \quad \text{modulo } A_n[s] \cdot I_A + \sum_{j=1}^{d} A_n[s] \cdot Z_j(s_j). \tag{3.3}$$

Using the relations $Z_j(s_j)$ we may eliminate the occurrence of $s = (s_1, \ldots, s_d)$ in the operator $r_i(x, \partial, s)$ and write $r_i(x, \partial)$ instead. The resulting operator $(1/b(\alpha)) \cdot r_i(x, \partial)$ is an inverse to ∂_i modulo the A-hypergeometric system.

We call $b(s)$ a *b-polynomial* for ∂_i. It is essentially unique if A is normal. The operator

$$C_i(x, \partial, \alpha) := \frac{1}{b(\alpha)} \cdot r_i(x, \partial) \tag{3.4}$$

is called an ith *creation operator*. From (3.1) we conclude the desired relation

$$C_i(x, \partial, \alpha)\, \Phi(\alpha - a_i; x) = \Phi(\alpha; x). \tag{3.5}$$

for all $\alpha \in \mathbf{N}^d$ with $b(\alpha) \neq 0$.

We shall present two algorithms for computing creation operators. The first algorithm is a straightforward application of Gröbner bases in the Weyl algebra. See e.g. [4] for Gröbner basics. Algorithm 3.1 can be run in the computer algebra system `kan/sm1` [24]. Its correctness follows immediately from the basic facts in [16], [23] or [24].

ALGORITHM 3.1. (Computing an ith creation operator from scratch).

(1) Compute a set $\mathcal{F}$ of generating binomials $\partial^u - \partial^v$ for the toric ideal I_A (e.g. using one of the two algorithms presented in [22, Sect. 12.A]).
(2) Let $\prec$ be any term order on the Weyl algebra $A_n[s_1, \ldots, s_d]$ which refines the weights

$$\begin{array}{ccccccccc} s_1 & \ldots & s_d & x_1 & \ldots & x_n & \partial_1 & \ldots & \partial_n \\ 0 & \ldots & 0 & 1 & \ldots & 1 & 1 & \ldots & 1 \end{array}$$

(3) Compute the reduced Gröbner basis $\mathcal{G}$ in the Weyl algebra $A_n[s_1, \ldots, s_d]$ for the input set $\mathcal{F} \cup \{\partial_i\} \cup \{Z_1(s_1), \ldots, Z_d(s_d)\}$ with respect to the term order $\prec$.
(4) Choose an element $b(s)$ of minimal degree in $\mathcal{G} \cap \mathbf{Q}[s_1, \ldots, s_d]$.
(5) Derive an identity (3.3) by tracing back the Gröbner basis computation in step (3).
(6) Output the resulting creation operator (3.4).

EXAMPLE 3.1. (continuation of Example 2.1). If A is the matrix of the twisted cubic curve then the ideals of b-polynomials are principal

$$\begin{aligned} \mathcal{B}_{A,1}(s) \cap \mathbf{Q}[s_1, s_2] &= (s_1(s_1 - 1)(s_1 - 2)) \\ \mathcal{B}_{A,2}(s) \cap \mathbf{Q}[s_1, s_2] &= (s_1(s_1 - 1)s_2) \\ \mathcal{B}_{A,3}(s) \cap \mathbf{Q}[s_1, s_2] &= (s_1 s_2(s_2 - 1)) \\ \mathcal{B}_{A,4}(s) \cap \mathbf{Q}[s_1, s_2] &= (\, s_2(s_2 - 1)(s_2 - 2)). \end{aligned} \tag{3.6}$$

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We computed the following four explicit creation operators for the twisted cubic.

The operator $r_1 = 27x_1^3\partial_1^2 + 54x_1^2x_2\partial_1\partial_2 + 27x_1^2x_3\partial_1\partial_3 + 36x_1x_2^2\partial_1\partial_3 + 36x_1x_2x_3\partial_1\partial_4 + 9x_1x_3^2\partial_2\partial_4 + 8x_2^3\partial_1\partial_4 + 12x_2^2x_3\partial_2\partial_4 + 6x_2x_3^2\partial_3\partial_4 + x_3^3\partial_4^2 + 54x_1^2\partial_1 + 54x_1x_2\partial_2 + 18x_1x_3\partial_3 + 12x_2^2\partial_3 + 6x_2x_3\partial_4 + 6x_1$

satisfies $r_1\Phi(\alpha_1,\alpha_2;x_1,\,x_2,x_3,x_4) = (\alpha_1+3)(\alpha_1+2)(\alpha_1+1)\Phi(\alpha_1+3,\alpha_2;\,x_1,x_2,x_3,x_4)$,

the operator $r_2 = 9x_1^2x_2\partial_1^2+18x_1^2x_3\partial_1\partial_2+27x_1^2x_4\partial_2^2+12x_1x_2^2\partial_1\partial_2+30x_1x_2x_3\partial_2^2+36x_1x_2x_4\partial_2\partial_3 + 12x_1x_3^2\partial_2\partial_3 + 18x_1x_3x_4\partial_2\partial_4 + 4x_2^3\partial_2^2 + 12x_2^2x_3\partial_2\partial_3+12x_2^2x_4\partial_2\partial_4 + 9x_2x_3^2\partial_2\partial_4 + 12x_2x_3x_4\partial_3\partial_4 + 2x_3^3\partial_3\partial_4 + 3x_3^2x_4\partial_4^2 + 18x_1x_2\partial_1+24x_1x_3\partial_2 + 18x_1x_4\partial_3 + 10x_2^2\partial_2 + 16x_2x_3\partial_3 + 6x_2x_4\partial_4 + 4x_3^2\partial_4 + 2x_2$

satisfies $r_2\Phi(\alpha_1,\alpha_2;x_1,x_2,x_3,x_4) = (\alpha_1+2)(\alpha_1+1)(\alpha_2+1)\Phi(\alpha_1+2,\alpha_2+1;\,x_1,x_2,x_3,x_4)$,

the operator $r_3 = 3x_1x_2^2\partial_1^2 + 12x_1x_2x_3\partial_1\partial_2 + 18x_1x_2x_4\partial_1\partial_3 + 12x_1x_3^2\partial_1\partial_3 + 36x_1x_3x_4\partial_2\partial_3+27x_1x_4^2\partial_3^2+2x_2^3\partial_1\partial_2+9x_2^2x_3\partial_1\partial_3+12x_2^2x_4\partial_2\partial_3+12x_2x_3^2\partial_2\partial_3+30x_2x_3x_4\partial_3^2 + 18x_2x_4^2\partial_3\partial_4 + 4x_3^3\partial_3^2 + 12x_3^2x_4\partial_3\partial_4 + 9x_3x_4^2\partial_4^2 + 6x_1x_3\partial_1+18x_1x_4\partial_2 + 4x_2^2\partial_1 + 16x_2x_3\partial_2 + 24x_2x_4\partial_3 + 10x_3^2\partial_3 + 18x_3x_4\partial_4 + 2x_3$

satisfies $r_3\Phi(\alpha_1,\alpha_2;x_1,x_2,x_3,x_4) = (\alpha_1+1)(\alpha_2+2)(\alpha_2+1)\Phi(\alpha_1+1,\alpha_2+2;\,x_1,x_2,x_3,x_4)$, and

the operator $r_4 = x_2^3\partial_1^2+6x_2^2x_3\partial_1\partial_2+9x_2^2x_4\partial_1\partial_3+12x_2x_3^2\partial_1\partial_3+36x_2x_3x_4\partial_1\partial_4+27x_2x_4^2\partial_2\partial_4 + 8x_3^3\partial_1\partial_4 + 36x_3^2x_4\partial_2\partial_4 + 54x_3x_4^2\partial_3\partial_4 + 27x_4^3\partial_4^2 + 6x_2x_3\partial_1 + 18x_2x_4\partial_2 + 12x_3^2\partial_2 + 54x_3x_4\partial_3 + 54x_4^2\partial_4 + 6x_4$

satisfies $r_4\Phi(\alpha_1,\alpha_2;x_1,\,x_2,x_3,x_4) = (\alpha_2+3)(\alpha_2+2)(\alpha_2+1)\Phi(\alpha_1,\alpha_2+3;\,x_1,x_2,x_3,x_4)$.

We next present a polyhedral formula for the b-polynomial which generalizes the specific expressions for the twisted cubic in (3.6). This additional information will then be used to give an alternative algorithm for computing creation operators. Choose any element h in the monoid $\Sigma_{i=1}^n \mathbf{N}a_i$ which satisfies the property

$$h + \left(\sum_{i=1}^n \mathbf{Z}a_i \cap \sum_{i=1}^n \mathbf{R}_+a_i\right) \subseteq \sum_{i=1}^n \mathbf{N}a_i. \tag{3.7}$$

The existence of such elements h is proved in [19, Appendix, Lem. 1]. We can choose $h=0$ if and only if A is normal. In general, h is a 'common denominator' for all Hilbert basis elements of the normalization, and it can be found using Algorithm 13.2 in [22].

We identify the matrix A with the set $\{a_1,\dots,a_n\}$. Its convex hull $\operatorname{conv}(A)$ is a $(d-1)$-dimensional polytope. Every facet Γ of $\operatorname{conv}(A)$ has a unique primitive integral support function

$$F_\Gamma\colon \mathbf{Z}^d \to \mathbf{Z}.$$

This function is linear, vanishes on the facet Γ, takes positive values on $A\backslash\Gamma$ and is surjective. The extension of F_Γ to $\mathbf{C}^d$ is also denoted by F_Γ. Our b-polynomial for ∂_i is expressed in terms of the primitive support functions of those facets Γ which are visible from $h + a_i$.

THEOREM 3.2. [19, p. 560], [18, Thm. 6.4]. *For any element h as in* (3.7)*, the polynomial*

$$b_h(s_1, \ldots, s_d) := \prod_{\Gamma : F_\Gamma(h+a_i)>0} \; \prod_{m=0}^{F_\Gamma(h+a_i)-1} (F_\Gamma(s_1, \ldots, s_d) - m) \tag{3.8}$$

lies in the ideal $\mathcal{B}_{A,i}(s) \cap \mathbf{Q}[s_1, \ldots, s_d]$. Moreover, b_h generates this ideal if A is normal.

Note the following obvious congruence in the Weyl algebra

$$b_h(\alpha_1, \ldots, \alpha_d) - b_h\left(\sum_{j=1}^n a_{1j}x_j\partial_j, \ldots, \sum_{j=1}^n a_{dj}x_j\partial_j\right) \in \sum_{l=1}^d A_n Z_l(\alpha_l).$$

It gives rise to the following algorithm for computing an ith creation operator. One advantage of Algorithm 3.2 over Algorithm 3.1 is that it can be run in any computer algebra system (e.g. maple) since it does not require non-commutative Gröbner bases.

ALGORITHM 3.2. (Computing an ith creation operator from a given b-polynomial).

(1) Compute a Gröbner basis $\mathcal{G}$ for the toric ideal I_A in $\mathbf{Q}[\partial_1, \ldots, \partial_n]$ with respect to any *reverse* lexicographic term order which has lowest variable ∂_i.
(2) Expand the following expression in the Weyl algebra A_n

$$b_h\left(\sum_{j=1}^n a_{1j}x_j\partial_j, \ldots, \sum_{j=1}^n a_{dj}x_j\partial_j\right),$$

into a $\mathbf{Q}$-linear combination of monomials $x_1^{i_1} \cdots x_n^{i_n}\partial_1^{j_1} \cdots \partial_n^{j_n}$.
(3) Reduce that $\mathbf{Q}$-linear combination modulo the Gröbner basis $\mathcal{G}$; either in A_n or in the commutative polynomial ring

$$\mathbf{Q}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n].$$

(4) The normal form computed in Step 3 has ∂_i as a right factor. Divide by ∂_i and output the result. It is a creation operator for ∂_i.

Proof of Correctness. The toric ideal I_A is homogeneous by our assumption that $a_1, \ldots, a_n$ lie on an affine hyperplane. The expression computed in step (2) looks like

$$p = \sum C_{i_1 \cdots i_n j_1 \cdots j_n} \cdot x_1^{i_1} \cdots x_n^{i_n} \cdot \partial_1^{j_1} \cdots \partial_n^{j_n}.$$

Note that p is the unique polynomial in the Weyl algebra A_n which is congruent to $b_h(s)$ modulo the left $A_n[s]$-ideal generated by the $Z_j(s_j)$. By Theorem 3.2, $b_h(s)$ lies in $\mathcal{B}_{A,i}(s)$. This implies that p lies in the left A_n-ideal generated by I_A and ∂_i. Therefore there exists another polynomial

$$q = \sum D_{k_1 \cdots k_n l_1 \cdots l_n} \cdot x_1^{k_1} \cdots x_n^{k_n} \cdot \partial_1^{l_1} \cdots \partial_n^{l_n},$$

which has ∂_i as a right factor and is congruent to p modulo the left A_n-ideal generated by I_A.

Now apply the reduction modulo $\mathcal{G}$ in (3) to p. In each reduction step we replace a right monomial factor $\partial_1^{u_1} \cdots \partial_n^{u_n}$ of a term of p by another such monomial. The result is the same, regardless of whether it was done over the Weyl algebra A_n or over $\mathbf{Q}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$. Moreover, we get the same normal form if we reduce q modulo $\mathcal{G}$, since p and q are congruent modulo I_A. Since q has ∂_i as a (right) factor, the normal form has ∂_i as a (right) factor, by the property of the reverse lexicographic order.

EXAMPLE 3.2. (A non-normal matrix). The ideal of b-polynomials is generally not principal. Let $n = 4, d = 2$ and

$$A = \begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

The Gröbner basis of I_A for the reverse lexicographic order $\partial_1 > \partial_2 > \partial_3 > \partial_4$ equals

$$\mathcal{G} = \{\underline{\partial_3^3} - \partial_2\partial_4^2,\ \underline{\partial_2\partial_3} - \partial_1\partial_4,\ \underline{\partial_2^3} - \partial_1^2\partial_3,\ \underline{\partial_1\partial_3^2} - \partial_2^2\partial_4\}.$$

The underlined monomials are the initial terms. Note that neither I_A nor its initial ideal are Cohen–Macaulay. Suppose we wish to compute a creation operator for $i = 4$. In step (3) of Algorithm 3.1 we would find that the elimination ideal is not principal

$$\begin{aligned} &\mathcal{B}_{A,4}(s) \cap \mathbf{Q}[s_1, s_2] \\ &\quad = (s_2) \cap (s_2 - 1) \cap (s_2 - 2) \cap (s_2 - 3) \cap (s_1 - 2, s_2 - 6). \end{aligned}$$

Applying steps (2)–(4) of Algorithm 3.2 to any element of this ideal will result in a creation operator. For example, we can substitute $s_1 = 4x_1\partial_1 + 3x_2\partial_2 + x_3\partial_3$ and $s_2 = x_2\partial_2 + 3x_3\partial_3 + 4x_4\partial_4$ into $s_2(s_2 - 1)(s_2 - 2)(s_2 - 3)(s_1 - 2)$ and reduce its expansion modulo $\mathcal{G}$. Removing a factor ∂_4 from each term in the output gives a creation operator.

This example also shows that the polyhedral b-polynomials predicted by Theorem 3.2 are generally not best possible with respect to minimizing degree. For instance, we have the following

$$h=(4,0), b_h=s_2(s_2-1)(s_2-2)(s_2-3)(s_2-4)(s_2-5)(s_2-6)(s_2-7),$$
$$h=(3,1), b_h=s_2(s_2-1)(s_2-2)(s_2-3)(s_2-4)(s_2-5)(s_2-6)s_1,$$
$$h=(1,3), b_h=s_1(s_2-1)(s_2-2)(s_2-3)(s_2-4)s_1(s_1-1)(s_1-2),$$
$$h=(0,4), b_h=s_2(s_2-1)(s_2-2)(s_2-3)s_1(s_1-1)(s_1-2)(s_1-3).$$

4. Optimal value and indicial polynomial

Every integer programming problem can be transformed into a standard form in which the linear objective function is simply the last coordinate

$$\text{Minimize } u_n \text{ subject to } \quad A\cdot u=\alpha \quad \text{and} \quad u\in \mathbf{N}^n. \tag{4.1}$$

For instance, if we are given the integer program

$$\text{Minimize } w\cdot u \text{ subject to } \quad A'\cdot u=\alpha' \quad \text{and} \quad u\in \mathbf{N}^{n-1},$$

then we transform it into (4.1) by adding a row and a column to get

$$A=\begin{pmatrix} -1 & w \\ 0 & A' \end{pmatrix}.$$

In this section we study the optimal value of the integer program (4.1) as a function of $\alpha=(\alpha_1,\dots,\alpha_d)$. We shall express the optimal value as root of the indicial polynomial along the singularity $\{x_n=0\}$ of the A-hypergeometric system.

The notion of an *indicial polynomial* appears classically in the Frobenius method for solving ordinary linear differential equation. The roots of the indicial polynomial are called *exponents*; they indicate the lowest order terms in a possible power series solution.

The following modern approach to indicial polynomials is used for holonomic systems and $\mathcal{D}$-modules ([11, Thm. 2.7], [12, Thm. 1], and [13, Thm. 4.1.1], see also [15] and [17]). Let P be an element of the Weyl algebra $A_n=\mathbf{Q}\langle x_1,\dots,x_n, \partial_1,\dots,\partial_n\rangle$. We abbreviate $\theta_n := x_n\partial_n$ and $x'=(x_1,\dots,x_{n-1})$ and $\partial'=(\partial_1,\dots,\partial_{n-1})$. We define a filtration $\{F_m\}_{m\in\mathbf{Z}}$ of A_n as follows: for each integer m, put

$$F_m=\left\{\sum_{s-q\leqslant m} a_{p,q,r,s}x'^p x_n^q \partial'^r {\partial_n}^s \in A_n, a_{p,q,r,s}\in\mathbf{Q}\right\}.$$

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It can be easily shown that $F_pF_q \subseteq F_{p+q}$. For nonzero $P \in A_n$, the minimum integer m satisfying $P \in F_m$ is called the F-order of P. When the F-order of an operator $P \in A_n$ is m, we define $\hat{\sigma}(P)$ as the top degree component of P

$$\hat{\sigma}(P) = \sum_{s-q=m} a_{p,q,r,s} x'^p x_n^q \partial'^r \partial_n{}^s.$$

Noting that

$$x_n^k \partial_n^k = \theta_n(\theta_n - 1) \cdots (\theta_n - k + 1)$$

and

$$\partial_n^k x_n^k = (\theta_n + 1)(\theta_n + 2) \cdots (\theta_n + k),$$

we can see that

$$x_n^m \hat{\sigma}(P) = \sum a_{p,q,r,s} x'^p \partial'^r \theta_n(\theta_n - 1) \cdots (\theta_n - s + 1) =: p(x', \partial', \theta_n)$$

for $m \geqslant 0$, and

$$\begin{aligned} \partial_n^{-m} \hat{\sigma}(P) &= \sum a_{p,q,r,s} x'^p \partial'^r (\theta_n + 1)(\theta_n + 2) \cdots (\theta_n - m) \\ &\quad \times \theta_n(\theta_n - 1) \cdots (\theta_n - s + 1) \\ &=: p(x', \partial', \theta_n) \quad \text{for } m < 0. \end{aligned}$$

In either case we replace the operator θ_n by a new scalar variable t and we define

$$\psi(P) := p(x', \partial', t) \in A_{n-1}[t].$$

When P is an ordinary differential operator then $\psi(P)$ is the classical indicial polynomial.

Consider any left ideal I of A_n. Let $\psi(I)$ be the left ideal in $A_{n-1}[t]$ generated by the operators $\psi(P)$ for all $P \in I$. We are interested in the elimination ideal $\psi(I) \cap \mathbf{Q}[t]$. This ideal is principal. If it is nonzero then its unique (up to scaling) generator of $\psi(I) \cap \mathbf{Q}[t]$ is called the (*global*) *indicial polynomial* along $x_n = 0$ of the left A_n-module A_n/I.

It was shown by Oaku (see [15], [17]) that the *indicial ideal* $\psi(I) \cap \mathbf{Q}[t]$ is gotten from any generating set of I by Gröbner basis computation with respect to the variable weights

$$\begin{array}{cccccccc} \partial_1 & \cdots & \partial_{n-1} & \partial_n & x_1 & \cdots & x_{n-1} & x_n \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1. \end{array}$$

In this computation special care must be taken because of the negative weight -1. The ordinary Buchberger algorithm may not terminate when negative weights are present. This difficulty can be dealt with by an extra homogenizing variable, or by adapting the standard basis algorithm in local rings [5, Sect. 4.4]. The following version of Oaku's algorithm has been implemented in kan/sm1 and risa/asir [14] to compute examples for this paper.

Define a **Q**-linear map h from A_n to $A_n[s, s^{-1}]$ by the **Q**-linear extension of the map

$$x^\alpha \partial^\beta \mapsto s^{\beta_n - \alpha_n} x^\alpha \partial^\beta$$

defined on monomials in A_n. For $\ell \in A_n$, we call $h(\ell)$ *the F-homogenization* of ℓ.

ALGORITHM 4.1. (Computing the global indicial polynomial).

(1) Given generators $\ell_1, \ldots, \ell_m$ of the left ideal I, compute their F-homogenizations $h(\ell_k)$ and find a monomial s^p so that $s^p h(\ell_k)$ is a polynomial in s for all k.

(2) Let $\succ$ be any term order on the Weyl algebra $A_n[s]$ which refines the weights

$$v = \begin{array}{cccccccc} s & x_1 & \cdots & x_n & \partial_1 & \cdots & \partial_n \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0. \end{array}$$

(3) Compute a Gröbner basis $\mathcal{G}$ in the Weyl algebra $A_n[s]$ for the input set

$$\{\, s^p h(\ell_1),\ s^p h(\ell_2)\,,\ \ldots\ ,\ s^p h(\ell_m)\,\}.$$

(4) Eliminate $x_1, \ldots, x_{n-1}, \partial_1, \ldots, \partial_{n-1}$ from the leading terms

$$\{in_v(g_i) \mid g_i \in \mathcal{G}\}$$

with respect to the variable s and choose an element $c(s, x_n, \partial_n)$ of minimal degree in ∂_n.

(5) The polynomial $\psi(c(1, x_n, \partial_n))$ is the global indicial polynomial of I along $x_n = 0$.

We now fix an integer vector $\alpha \in \mathbf{N}^d$ and consider the A-hypergeometric ideal

$$I_{A,\alpha} := A_n \cdot I_A + \sum_{j=1}^{d} A_n \cdot Z_j(\alpha_j).$$

THEOREM 4.1. *Let $(u_1, \ldots, u_n)$ be an optimal solution to the integer program* (4.1). *Then the optimal value $t = u_n$ is a zero of the indicial ideal $\psi(I_{A,\alpha}) \cap \mathbf{Q}[t]$.*

Proof. Let $c(t) \in \psi(I_{A,\alpha}) \cap \mathbf{Q}[t]$. There exists an operator r such that

$$c(x_n\partial_n) + x_n \cdot r(x', \partial', x_n, \theta_n) \in I_{A,\alpha}. \tag{4.2}$$

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The hypergeometric polynomial for the right-hand side α can be written as follows

$$\Phi(\alpha; x) = x_n^{u_n} \cdot P(x') + x_n^{u_n+1} \cdot Q(x', x_n). \tag{4.3}$$

Note that $P(x')$ contains the term $x_1^{u_1} \cdots x_{n-1}^{u_{n-1}}/(u_1! \cdots u_{n-1}! u_n!)$ and possibly others. When we apply the operator (4.2) to (4.3) then we get zero. In particular, the lowest term of (4.2) with respect to x_n must be annihilated by the lowest term of (4.3)

$$c(x_n\partial_n)\,(x_n^{u_n} \cdot P(x')) = c(u_n) \cdot x_n^{u_n} \cdot P(x') = 0.$$

This implies $c(u_n) = 0$, as desired.

We next present a general construction and lemma which will be used in the proof of Theorem 4.2 below. A subset $\mathcal{U}$ of $\mathbf{N}^d$ is called an *order ideal* if $u \in \mathcal{U}$ and $v \leqslant u$ (componentwise) implies $v \in \mathcal{U}$. The complement of an order ideal $\mathcal{U}$ can be identified with the monomial ideal $M_{\mathcal{U}} = (x^w\colon\ w \notin \mathcal{U})$ in $\mathbf{Q}[x] = \mathbf{Q}[x_1, \ldots, x_n]$. Let $I_{\mathcal{U}}$ be the radical ideal consisting of all polynomials in $\mathbf{Q}[x]$ which vanish at the points in $\mathcal{U}$.

LEMMA 4.1. *Let $\mathcal{U}$ be an order ideal in $\mathbf{N}^n$ and $I_{\mathcal{U}}$ its vanishing ideal. Then the reduced Gröbner basis of $I_{\mathcal{U}}$ with respect to any term order consists of the polynomials*

$$f_w := \prod_{i=1}^{n} \prod_{j=0}^{w_i-1} (x_i - j),$$

where $x^w = x_1^{w_1} \cdots x_n^{w_n}$ runs over all minimal generators of $M_{\mathcal{U}}$.

Proof. Fix an arbitrary term order $\prec$ on $\mathbf{Q}[x]$. We first assume that $\mathcal{U}$ is finite. Then both $\mathbf{Q}[x]/I_{\mathcal{U}}$ and $\mathbf{Q}[x]/M_{\mathcal{U}}$ are artinian rings of $\mathbf{Q}$-dimension $\#(\mathcal{U})$. Let F be the ideal generated by the polynomials f_w above. Since f_w vanishes on $\mathcal{U}$, we have $F \subseteq I_{\mathcal{U}}$. This inclusion lifts to initial monomial ideals and we get $in_\prec(F) \subseteq in_\prec(I_{\mathcal{U}})$. The observation $in_\prec(f_w) = x^w$ implies $M_{\mathcal{U}} \subseteq in_\prec(F)$. Consider now the following chain of inequalities

$$\begin{aligned}\#(\mathcal{U}) &= \dim \mathbf{Q}[x]/I_{\mathcal{U}} &&= \dim \mathbf{Q}[x]/in_\prec(I_{\mathcal{U}}) \leqslant\\ &\leqslant \dim \mathbf{Q}[x]/in_\prec(F) &&\leqslant \dim \mathbf{Q}[x]/M_{\mathcal{U}} = \#(\mathcal{U}).\end{aligned}$$

All inequalities are equalities, and hence $in_\prec(I_{\mathcal{U}}) = in_\prec(F) = M_{\mathcal{U}}$. This shows that the set $\{f_w\}$ is a Gröbner basis for $I_{\mathcal{U}}$ with respect to $\prec$.

Next consider the case where $\mathcal{U}$ is infinite. Suppose, by contradiction, that $\{f_w\}$ is not a Gröbner basis for $I_{\mathcal{U}}$ with respect to $\prec$. Then there exists a non-zero polynomial $f \in I_{\mathcal{U}}$ such that no term of f lies in $M_{\mathcal{U}}$. Let $\mathcal{U}'$ be the smallest order ideal in $\mathbf{N}^n$ which contains all the terms of f. Then $\mathcal{U}'$ is finite and $\mathcal{U}' \subset \mathcal{U}$. We have $f \in I_{\mathcal{U}'}$ and no term of f lies in $M_{\mathcal{U}'}$. This is a contradiction to Lemma 4.1 for finite order ideals.

Therefore $\{f_w\}$ is a Gröbner basis for $I_{\mathcal{U}}$ in both cases. Since x^w is the only term of f_w which lies in $M_{\mathcal{U}}$, we conclude that $\{f_w\}$ must be the reduced Gröbner basis for $I_{\mathcal{U}}$.

We are now prepared to prove the existence of a nonzero indicial polynomial. First consider the case of generic parameters. Let $s = (s_1, \dots, s_d)$ be indeterminates and consider

$$I_{A,s} := A_n[s] \cdot I_A + \sum_{j=1}^{d} A_n[s] \cdot Z_j(s_j).$$

This is a left ideal in $A_n[s] = A_n[s_1, \dots, s_d]$, and $\psi(I_{A,s})$ is a left ideal in $A_{n-1}[t, s_1, \dots, s_d]$.

THEOREM 4.2. *The ideal* $\psi(I_{A,s}) \cap \mathbf{Q}[t, s_1, \dots, s_d]$ *is the vanishing ideal of all points* (τ, α) *where* τ *is the optimal value of the integer program* (4.1) *with right-hand side* $\alpha = (\alpha_1, \dots, \alpha_d)$. *This radical ideal has height* 1 *and contains a polynomial monic in* t.

Proof. Let J denote the vanishing ideal of all points (τ, α) where τ is the optimal value of the integer program with right hand side α. It follows from Theorem 4.1 that $\psi(I_{A,s}) \cap \mathbf{Q}[t, s_1, \dots, s_d] \subseteq J$. We must prove the reverse inclusion. Let $f = f(t, s_1, \dots, s_d) \in J$. We replace s_i by $\Sigma_{j=1}^{n-1} a_{ij}\theta_j + a_{in}t$ for $i = 1, \dots, d$ to get a polynomial $\tilde{f} = \tilde{f}(\theta_1, \dots, \theta_{n-1}, t)$ which is congruent to f modulo $\psi(I_{A,s})$. We identify $t = \theta_n$.

By hypothesis, $\tilde{f} = \tilde{f}(\theta_1, \dots, \theta_n)$ vanishes at all non-negative integer points $v = (v_1, \dots, v_n)$ which are optimal in their fiber. These points form an order ideal in $\mathbf{N}^n$ ([26, Lem. 2.1.4]). Lemma 4.1 implies that $\tilde{f}$ can be written as a linear combination of the polynomials

$$\prod_{j=1}^{n} \prod_{k=0}^{u_j-1} (\theta_j - k), \quad \text{where } u = (u_1, \dots, u_n) \text{ is not optimal in its fiber.} \tag{4.4}$$

If u is not optimal then there exists another point v in the same fiber which satisfies $v_n < u_n$. We have $\partial^u - \partial^v \in I_A$. This implies that

$$\psi(x^u \cdot (\partial^u - \partial^v)) = \prod_{j=1}^{n} \prod_{k=0}^{u_j-1} (\theta_j - k) \quad \text{lies in } \psi(A_n \cdot I_A).$$

We conclude $\tilde{f} \in \psi(A_n \cdot I_A)$ and, hence, $f \in \psi(I_{A,s})$, as desired.

For the second assertion we recall the following familiar result from integer programming (cf. [2, Thm. 4.6]): There exist finitely many linear functionals $L_1, L_2, \dots, L_r$ on $\mathbf{Q}^d$ such that for every feasible $\alpha \in \mathbf{N}^d$ there exists $j \in$

$\{1,2,\ldots,r\}$ such that the optimal value of (4.1) for the right-hand side α is equal to $L_j(\alpha)$. (In particular, $L_j(\alpha)$ is an integer, for such α). This shows that the monic (in t) polynomial $\prod_{j=1}^{r}(t-L_j(s_1,\ldots,s_d))$ lies in the radical ideal J. Hence J is a proper ideal. It follows from [2, Thm. 4.6] that at least one index $j\in\{1,\ldots,r\}$ is attained on the intersection of a d-dimensional cone with an affine sublattice of finite index in $\mathbf{Z}^d$. Such a set is Zariski dense in $\mathbf{Q}^d$, and therefore J has height one.

Remark 4.1. One may be tempted to conjecture from the previous argument that $\psi(I_{A,s})\cap\mathbf{Q}[t,s_1,\ldots,s_d]$ equals the principal ideal generated by $\prod_{j=1}^{r}(t-L_j(s_1,\ldots,s_d))$. This is generally not true, as the following example shows. However, it is true under a suitable normality hypothesis. This will be shown in the next section.

EXAMPLE 4.1. (The ideal of optimal values need not be principal). Let $n=5$, $d=3$ and

$$A=\begin{pmatrix}1&1&1&1&1\\0&2&3&4&3\\0&1&1&0&2\end{pmatrix}.$$

Here

$$\begin{aligned}&\psi(I_{A,s})\cap\mathbf{Q}[t,s_1,s_2,s_3]\\&\quad=(t)\cap(ts_2-2s_3)\cap(t+s_1-s_3)\\&\quad\cap\,(t+4s_1-s_2-s_3)\cap(s_3-2,t-1).\end{aligned}$$

The last prime component shows that this ideal is not principal.

The monic polynomial in Theorem 4.2 guarantees the existence of a nonzero indicial polynomial for every right-hand side vector α.

COROLLARY 4.1. *For every $\alpha\in\mathbf{Q}^n$, the ideal $\psi(I_{A,\alpha})\cap\mathbf{Q}[t]$ is nonzero. The unique (up to scaling) minimal generator is called the indicial polynomial of the integer program* (4.1).

EXAMPLE 4.2. (continuation of Example 4.1). For generic right-hand sides α the indicial polynomial equals

$$t(t+\alpha_2-2\alpha_3)(t+\alpha_1-\alpha_3)(t+4\alpha_1-\alpha_2-\alpha_3). \tag{4.5}$$

For $\alpha=(0,0,2)$ the indicial polynomial equals $t(t-4)(t-2)^2(t-1)$. For $\alpha=(0,0,0)$ it equals t^3. Thus the degree may be higher or lower than in the degree in the generic case.

5. The indicial polynomial in the normal case

In this section a geometric construction of the indicial polynomial will be presented. We retain the notation from Section 4, and we make the following assumptions throughout:

(a) The vectors $a_1, \ldots, a_{n-1}, a_n$ lie on an affine hyperplane in $\mathbf{R}^d$.
(b) The vectors $a_1, \ldots, a_{n-1}$ span $\mathbf{Z}^d$.
(c) The matrix $A' := (a_1, \ldots, a_{n-1})$ is normal.

Here the hypothesis (c) is the most restrictive one. As we shall see in Lemma 5.1, this hypothesis implies that the integer programming problem (4.1) can be solved by rounding up the objective function value of the associated *linear programming problem*

$$\text{Minimize} \quad u_n \quad \text{subject to} \quad u \in \mathbf{R}^n, \quad A \cdot u = \alpha \quad \text{and} \quad u \geqslant 0. \tag{5.1}$$

To solve (5.1) geometrically, we consider the convex hull

$$\text{conv}(A') = \text{conv}\{a_1, \ldots, a_{n-1}\}.$$

This is a $(d-1)$-polytope. The cone over $\text{conv}(A')$ is the d-dimensional cone $\text{pos}(A')$. For any facet Γ of $\text{conv}(A')$ let L_Γ denote its primitive integral support function. This is the unique epimorphism $\mathbf{Z}^d \to \mathbf{Z}$ which is non-negative on $\text{conv}(A')$ and vanishes on Γ. We say that a facet Γ is *visible from* a_n if $L_\Gamma(a_n) < 0$. Let $\mathcal{F}$ denote the set of all facets Γ of $\text{conv}(A')$ which are visible from a_n. Note that $a_n \in \text{conv}(A')$ if and only if $\mathcal{F} = \emptyset$. The linear program (5.1) is feasible if and only if the right-hand side α lies in $\text{pos}(A) = \text{pos}(A' \cup \{a_n\})$ if and only if $\alpha \in \text{pos}(A')$ or $\alpha \in \text{pos}(\Gamma \cup \{a_n\})$ for some $\Gamma \in \mathcal{F}$.

PROPOSITION 5.1. *Let u_n be the optimum value of a feasible linear program* (5.1). *Then*

$$u_n = \begin{cases} 0 & \text{if } \ \alpha \in \text{pos}(A'), \\ L_\Gamma(\alpha)/L_\Gamma(a_n) & \text{if } \ \alpha \in \text{pos}(\Gamma \cup \{a_n\}) \quad \text{for } \ \Gamma \in \mathcal{F}. \end{cases}$$

Proof. The first case $\alpha \in \text{pos}(A')$ is obvious. Suppose we are in the second case. The optimal value is the smallest real number u_n such that $\alpha - u_n a_n$ lies in $\text{pos}(A')$. Since L_Γ is non-negative on $\text{pos}(A')$, we find that $L_\Gamma(\alpha - u_n a_n) = L_\Gamma(\alpha) - u_n L_\Gamma(a_n) \geqslant 0$. The assumption $\alpha \in \text{pos}(\Gamma \cup \{a_n\})$ implies that the last inequality is attained.

THEOREM 5.1. *Under the hypotheses* (a)–(c) *above, the ideal $\psi(I_{A,s}) \cap \mathbf{Q}[t, s_1, \ldots, s_d]$ is principal. Its generator equals the following product of linear polynomials*

$$t \cdot \prod_{\Gamma \in \mathcal{F}} \prod_{k=0}^{-L_\Gamma(a_n)-1} \left(L_\Gamma(s_1, \ldots, s_d) - t \cdot L_\Gamma(a_n) - k \right). \tag{5.2}$$

Theorem 5.1 is our main result in this section. For the proof we need one lemma.

LEMMA 5.1. *If the integer progam* (4.1) *is feasible and* u_n *is the optimum value of the linear program* (5.1), *then the least integer* $\lceil u_n \rceil$ *that is greater than or equal to* u_n *is the optimum value of* (4.1).

Proof. First suppose $\alpha \in \mathrm{pos}(A')$. Then $u_n = 0$ by Proposition 5.1. By the normality hypothesis (c), the right-hand side α is a non-negative integer linear combination of $a_1, \ldots, a_{n-1}$. Hence $u_n = \lceil u_n \rceil = 0$ is also the optimal value of the integer program (4.1).

Next suppose $\alpha \in \mathrm{pos}(\Gamma \cup \{a_n\})$ for $\Gamma \in \mathcal{F}$. The optimal value of the integer program (4.1) is the smallest integer U_n such that $\alpha - U_n \cdot a_n \in \mathbf{N}A'$. The optimal value u_n for the linear program (5.1) satisfies $u_n \leqslant U_n$ and $\alpha - u_n \cdot a_n \in \mathrm{pos}(\Gamma) \subset \mathrm{pos}(A')$.

If $u_n = U_n$ we are done, hence assume $u_n < U_n$. The identity

$$\alpha - \lceil u_n \rceil a_n = \frac{\lceil u_n \rceil - u_n}{U_n - u_n} \cdot (\alpha - U_n a_n) + \frac{U_n - \lceil u_n \rceil}{U_n - u_n} \cdot (\alpha - u_n a_n)$$

shows that $\alpha - \lceil u_n \rceil a_n$ lies in $\mathrm{pos}(A')$. By normality we conclude $\alpha - \lceil u_n \rceil \cdot a_n \in \mathbf{N}A'$. This implies $U_n = \lceil u_n \rceil$, as desired.

Proof of Theorem 5.1. Lemma 5.1 and Proposition 5.1 imply that (5.2) lies in the ideal $\psi(I_{A,s}) \cap \mathbf{Q}[t, s_1, \ldots, s_d]$. Conversely, let $f = f(t, s_1, \ldots, s_d)$ be any element of that ideal. Consider the set of all feasible $\alpha \in \mathbf{Q}^d$ such that the optimal value of (4.1) equals $(L(\alpha) - k)/L(a_n)$, for some fixed k. This set equals the intersection of the d-dimensional cone $\mathrm{pos}(\Gamma \cup \{a_n\})$ with an affine sublattice of finite index in $\mathbf{Z}^d$. Hence this set is Zariski dense in $\mathbf{Q}^d$. We conclude that the polynomial f vanishes on the hyperplane in $\mathbf{Q}^{d+1}$ defined by any of the linear factors in (5.2). Therefore f is a multiple of (5.2).

COROLLARY 5.1. *For every* $\alpha \in \mathbf{Q}^d$, *the indicial polynomial is a factor of*

$$t \cdot \prod_{\Gamma \in \mathcal{F}} \prod_{k=0}^{-L_\Gamma(a_n)-1} \left(L_\Gamma(\alpha) - t \cdot L_\Gamma(a_n) - k \right). \tag{5.3}$$

For generic values of α, *this expression is square-free and it equals the indicial polynomial.*

EXAMPLE 5.1. (Transportation problem and hypergeometric system of type $(r, r+$

s).) We retain the notations of Example 2.2. The indicial polynomial along $x_{ij} = 0$ is equal to

$$t \cdot \left(t - \gamma_j + \sum_{k \neq i} \rho_k \right),$$

for generic values of parameters. Computer experiments indicate that the indicial polynomial is equal to this quadratic polynomial for all values of parameters.

EXAMPLE 5.2. Let $n = 8, d = 3$ and consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

This is the normalization of the matrix in Example 4.1. Here $a_n = (1, 3, 2)$, the polygon conv(A') is a quadrangle, the set $\mathcal{F}$ of visible facets has three elements, and we have $L_\Gamma(a_n) = -1$ for all $\Gamma \in \mathcal{F}$. The product (5.3) equals the expression (4.5), but, in contrast to Example 4.1, the generic indicial ideal is now principal

$$\begin{aligned} &\psi(I_{A,s}) \cap \mathbf{Q}[t, s_1, s_2, s_3] \\ &\quad = \left(t \cdot (t + s_2 - 2s_3) \cdot (t + s_1 - s_3) \cdot (t + 4s_1 - s_2 - s_3)\right). \end{aligned}$$

For special values of α the indicial polynomial may be a proper factor of (5.3). For $\alpha = (0, 0, 0)$ we get here the same answer as in Example 4.2: $\psi(I_{A,\alpha}) \cap \mathbf{Q}[t] = (t^3)$.

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