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Generalized Solution of the Photon Transport Problem

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Abstract. The purpose of this paper is to show the existence of a generalized solution of the photon transport problem. By means of the theory of equicontinuous C_0-semigroup on a sequentially complete locally convex topological vector space we show that the perturbed abstract Cauchy problem has a unique solution when the perturbation operator and the forcing term function satisfy certain conditions. A consequence of the abstract result is that it can be directly applied to obtain a generalized solution of the photon transport problem.

1 Introduction

The motivation of this study is due to the problem of photon transport in a cloud. Meri Lisi and Silvia Totaro [5] consider the photon transport problem in a cloud that occupies a convex region of space with a localized source inside (for example, a star). They assume that the photon transport phenomenon is one-dimensional; that is, the photon number density U depends on the space variable x, on the angle variable μ, and time t. They also assume that the nebula is bounded by the two surfaces $x = a(t)$ and $x = b(t)$. In order to avoid a moving reference system, it is convenient to assume that the surface at the left end is fixed, *i.e.,* $x = a(t) = 0$. Hence, the boundary plane $x = b(t)$ moves with speed $\dot{b}(t)$, where $b(t)$ is a continuously differentiable real function of $t \in [0, +\infty)$ such that

$$|\dot{b}(t)| \leq \sup_{t \geq 0} |\dot{b}(t)| < \infty.$$

The following figure gives a sketch of the situation:

Vacuum (Part I)	Nebula (Part II)	Vacuum (Part III)

Each region is characterized by some different total and scattering cross sections. However, in each region, the relative cross sections can be considered constants; in particular, in the vacuum the total cross section and the scattering cross section are very small, because the particle density is low. Hence, in Parts I and III, if we denote the total cross section and the scattering cross section by $\widehat{\sigma}$, $\widehat{\sigma}_s$, respectively, then we may assume $\widehat{\sigma} > \widehat{\sigma}_s > 0$. On the other hand, in Part II, one has that $\sigma > \sigma_s > 0$, where σ, σ_s are the total cross section and the scattering cross section, respectively.

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Moreover, since the particle density in the nebula is higher than that in the vacuum, we may assume that $\sigma > \widehat{\sigma} > 0$ and $\sigma_s > \widehat{\sigma}_s > 0$.

The photon transport equation in interstellar space can be described as follows:

$$\begin{aligned}(1.1)\quad \frac{\partial}{\partial t}U(x,\mu,t) &= -c\mu\frac{\partial}{\partial x}U(x,\mu,t) - c[\widehat{\sigma} + (\sigma - \widehat{\sigma})\chi(x,t)]U(x,\mu,t)\\ &\quad + c\big[\widehat{\sigma}_s + (\sigma_s - \widehat{\sigma}_s)\chi(x,t)\big]\\ &\qquad \times \int_{-1}^{1} k(\mu,\mu')U(x,\mu',t)d\mu' + q_0\delta(x - x_0)\chi(x,t),\\ &\forall x \in (-\infty,+\infty),\quad \mu \in (-1,1),\ \text{and } t \in (0,+\infty),\end{aligned}$$

where $\chi = \chi(x,t)$ is the characteristic function of the interval $[0, b(\cdot)]$, q_0 is a constant, $x \in [0, b(t)]$, and $\delta(x - x_0)$ is the Dirac delta function. The scattering kernel $k(\mu,\mu')$ is a positive C^∞-function with compact support with respect to each variable in $(-1,1)$ such that

$$(1.2)\qquad k(\mu,\mu') = k(\mu',\mu), \int_{-1}^{1} k(\mu,\mu')d\mu = 1$$

and

$$(1.3)\qquad \Big|\frac{\partial^r}{\partial\mu^r}\, k(\mu,\mu')t\Big| \le \frac{\overline{k}}{2}\forall r \in N_0\big(\ N_0 = \{0,1,2,\dots\}\big),$$

where $\overline{k}$ is a suitable positive constant. They chose σ_s small enough such that

$$(1.4)\qquad \overline{k}\sigma_s < \sigma.$$

However, since the derivative of χ is not defined at $x = 0$ and $x = b(t)$, in order to avoid this difficulty, we consider a mollified version $\widehat{\chi}(x,t)$ of $\chi(x,t)$, which is defined as follows:

$$\begin{aligned}&\widehat{\chi}(x,t) = 0, \quad \forall x \in (-\infty,0] \cup [b(t),+\infty),\\ &\widehat{\chi}(x,t) = -\frac{2}{\varepsilon^2}x^3 + \frac{3}{\varepsilon^3}x^2, \quad \forall x \in (0,\varepsilon);\\ &\widehat{\chi}(x,t) = 1, \quad \forall x \in [\varepsilon, b(t) - \varepsilon];\\ &\widehat{\chi}(x,t) = -\frac{2}{\varepsilon^2}[b(t) - x]^3 + \frac{3}{\varepsilon^3}[b(t) - x]^2, \quad \forall x \in (b(t) - \varepsilon, b(t)).\end{aligned}$$

In the preceding equations, ε is a positive constant such that $\varepsilon \ll b(0)$, with $b(0) \neq 0$ (see [6] for details). Equation (1.1) is supplemented by the initial condition

$$(1.5)\qquad U(x,\mu,0) = U_0(x,\mu) \text{ for } x \in (-\infty,+\infty) \text{ and for } \mu \in (-1,1),$$

where U_0 is a given positive function. In order to study (1.1)–(1.5), Lisi and Totaro first considered the Banach space $X = L^1(R \times (-1, 1))$ endowed with the norm

$$\|f\| = \int_{-\infty}^{\infty} dx \int_{-1}^{1} |f(x, \mu)| d\mu \quad \forall f \in X.$$

They defined the operators $S\colon D(S) \subset X \to R(S) \subset X$ and $J\colon D(J) \subset X \to R(J) \subset X$ by

$$Sf(x, \mu) = -c\mu \frac{\partial}{\partial x} f(x, \mu) \quad \forall f \in D(S) \subset X,$$

$$Jf(x, \mu) = \int_{-1}^{1} k(\mu, \mu') f(x, \mu') d\mu' \quad \forall f \in X,$$

where $D(S) = \{f \in X : Sf \in X\}$. Moreover, they set

$$\sigma(t) = \widehat{\sigma} + (\sigma - \widehat{\sigma})\widehat{\chi}(x, t),$$

$$\sigma_s(t) = \widehat{\sigma}_s + (\sigma_s - \widehat{\sigma}_s)\widehat{\chi}(x, t),$$

$$Q(t) = q_0 \delta(x - x_0)\widehat{\chi}(x, t),$$

where $\sigma(t)$ and $\sigma_s(t)$ are functions from $[0, +\infty)$ into $L^\infty(R)$, and $Q(t)$ is considered as a function from $[0, +\infty)$ into X.

Hence, the problem (1.1)–(1.5) can be transformed into the form

$$\text{(1.6)} \qquad \begin{cases} \frac{d}{dt} U(t) = [S - c\sigma(t)I + c\sigma_s(t)J]U(t) + Q(t), & \forall t > 0; \\ U(0) = U_0, \end{cases}$$

where $U(t) = U(\cdot, \cdot, t)$ is considered as a function from $[0, +\infty)$ into X. It is reasonable to assume that the number of photons inside the cloud changes slowly in time, *i.e.*, $\frac{d}{dt}U(t)$ is small. For the same reasons they assumed that $\sigma(t) \equiv \sigma$, $\sigma_s(t) \equiv \sigma_s$, $Q(t) \equiv Q$, and $b(t) = \overline{b}$, do not depend on time t. They transformed the initial value problem (1.6) into the equation

$$\text{(1.7)} \qquad \begin{cases} \frac{d}{dt} U(t) = [S - c\sigma I + c\sigma_s J]U(t) + Q, & \forall\, t > 0; \\ U(0) = U_0. \end{cases}$$

However, they found that the initial value problem (1.7) has no solution in the Banach space $X = L^1(R \times (-1, 1))$, since $\delta(x - x_0)$ does not belong to X. To solve (1.7) they had to consider a more general space. It is for this reason that we consider the perturbed Cauchy problem in a sequentially complete locally convex space rather than in a Banach space. They solved equation (1.7) for the special case $\frac{d}{dt}U(t) = 0$. The new system

$$\begin{cases} [S - c\sigma I + c\sigma_s J]U(t) + Q(t) = 0, & \forall t > 0 \\ U(0) = U_0 \end{cases}$$

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is the so-called quasi-static equation in the space $\widetilde{X} = D'(R \times (-1, 1))$, where $\widetilde{X}$ is the space of all linear continuous functionals on the space D consisting of all test functions. We will give further descriptions of these spaces in Section 3 and show that the initial value problem (1.7) has a unique generalized solution for the case $\frac{d}{dt}U(t) \neq 0$.

Throughout this paper we will use the following notations. We let X be a sequentially complete locally convex space (*sclcs*) under a family of seminorms Γ. We denote by $\mathcal{L}(X)$ the space of all linear continuous operators on X and by E a collection of bounded subsets of X such that $(\bigcup_{M\in B} M) = X$. For each $B \in E$ and $q \in \Gamma$, a seminorm $p_{B,q}$ on $\mathcal{L}(X)$ is defined by

$$p_{B,q}(L) = \sup\{q(Lx) : x \in B\} \text{ for every } L \in \mathcal{L}(X).$$

Then the family $\{p_{B,q} : B \in E, q \in \Gamma\}$ induces a locally convex topology for $\mathcal{L}(X)$ (*e.g.*, see [4, p. 131]).

A family $\Im$ of linear operators on X is equicontinuous if for each $p \in \Gamma$, there is a continuous seminorm $q = q(p) \in \Gamma$ such that $p(Lx) \leq q(x)$ for all $L \in \Im$ all $x \in X$. For each $p \in \Gamma$ and a linear operator L on X, we define a corresponding seminorm for the linear operator L as

$$\widetilde{p}(L) = \sup\{p(Lx) : p(x) \leq 1\}.$$

A linear operator L on X is said to be p-continuous if

$$\widetilde{p}(L) = \sup\{p(Lx) : x \in X \text{ with } p(x) \leq 1\} < \infty.$$

A linear operator $L \in \mathcal{L}(X)$ is said to be Γ-continuous if it is p-continuous for every $p \in \Gamma$. Let $\mathcal{L}_\Gamma(X)$ denote the space of all Γ-continuous linear operators on X and let $\mathcal{B}_\Gamma(X)$ be the subspace of $\mathcal{L}_\Gamma(X)$ whose elements L satisfies

$$\|L\|_\Gamma = \sup\{p(Lx) : p \in \Gamma, x \in X \text{ with } p(x) \leq 1\} < \infty.$$

$\mathcal{B}_\Gamma(X)$ with the norm $\|\cdot\|_\Gamma$ is a Banach algebra. With these notations, we have the relation $\mathcal{B}_\Gamma(X) \subset \mathcal{L}_\Gamma(X) \subset \mathcal{L}(X)$. For any $K \in \mathcal{B}_\Gamma(X)$ we define the operator e^{tK} by

$$e^{tK} = \sum_{i=0}^{\infty} \frac{t^i}{i!} K^i, \text{ for each } t > 0 \text{ and } e^{0K} = I \text{ for } t = 0.$$

Definition 1.1 Let X be an *sclcs*. The family of continuous linear operators $\{T(t)\}_{t\geq 0}$ on X is called a strongly continuous C_0-semigroup if the following three conditions hold:

(i) $T(0) = I$,
(ii) $T(t)T(s) = T(t+s)$ for all $s, t \geq 0$ and
(iii) $T(t)x \longrightarrow x$ as $t \downarrow 0$, for every $x \in X$.

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We call a family of linear operators $\{T(t)\}_{t\geq 0}$ equicontinuous if for each continuous seminorm p on X, there exists a continuous seminorm q on X such that $p(T(t)x) \leq q(x)$ for all $t \geq 0$ and $x \in X$. Such a family $\{T(t)\}_{t\geq 0}$ is called an equicontinuous C_0-semigroup. Moreover, if there exists a number $\beta \geq 0$ such that $\{e^{-\beta t}T(t)\}_{t\geq 0}$ is equicontinuous, then it is called a quasi-equicontinuous C_0-semigroup. A semigroup $\{T(t)\}_{t\geq 0}$ is said to be locally equicontinuous if for any fixed $0 < T < \infty$, the subfamily $\{T(t) : 0 \leq t \leq T\}$ is equicontinuous.

Let $\Im$ be an equicontinuous family of linear operators on X and let Γ be a calibration for X. We define, for each $p \in \Gamma$, a continuous seminorm p' on X by

$$p'(x) = \sup\{p(Lx) : L \in \Im \text{ or } L = I\} \text{ for every } x \in X.$$

This implies that $p' \geq p$ for each $p \in \Gamma$. Choe [1] showed that the new calibration $\Gamma' = \{p' : p \in \Gamma\}$ induces the same topology on X.

If $\{T(t)\}_{t\geq 0}$ is an equicontinuous C_0-semigroup on X, then Choe's result allows us to define a new calibration Γ' on X such that $\|T(t)\|_{\Gamma'} \leq 1$ for all $t \geq 0$. In this case, $\{T(t)\}_{t\geq 0}$ is called a Γ'-contraction C_0-semigroup. In fact, we have the following proposition.

Proposition 1.2 *If $\{T(t)\}_{t\geq 0}$ is an equicontinuous C_0-semigroup on X, then there is a new calibration Γ' on X such that $\{T(t)\}_{t\geq 0}$ is a Γ'-contraction C_0-semigroup.*

Proof Let Γ' be a new calibration on X that is defined by

$$p'(x) = \sup_{t\geq 0}\{p(T(t)x) : p(x) \leq 1\} \text{ for each } p \in \Gamma.$$

Then for every $p \in \Gamma$

$$\begin{aligned} p'(T(t)x) &= \sup_{s\geq 0}\{p(T(s)T(t)x) : p(x) \leq 1\} = \sup_{s\geq 0}\{p(T(t+s)x) : p(x) \leq 1\} \\ &= \sup_{k\geq t}\{p(T(k)x) : p(x) \leq 1\} \leq \sup_{k\geq 0}\{p(T(k)x) : p(x) \leq 1\} = p'(x). \end{aligned}$$

This shows that $\|T(t)\|_{\Gamma'} \leq 1$ for all $t \geq 0$, and hence $\{T(t)\}_{t\geq 0}$ is a Γ'-contraction C_0-semigroup. ■

For convenience, if no confusion arises, we will still denote this new calibration by Γ instead of Γ'.

Definition 1.3 Let X be a locally convex linear space. Then any convex, balanced, and absorbing closed set is called a barrel. X is called a barrel space if each of its barrels is a neighborhood of zero.

Let X_Γ be the subspace of X such that $X_\Gamma = \{x \in X : \sup_{p\in\Gamma} p(x) < \infty\}$. We defined $\|\cdot\|_\Gamma$ on X_Γ by $\|X\|_\Gamma = \sup_{p\in\Gamma} p(x)$ for every $x \in X_\Gamma$. Then $\|\cdot\|_\Gamma$ is a norm on X_Γ for which $(X_\Gamma, \|\cdot\|_\Gamma)$ is a Banach space. (For details, see [7, Proposition 2.5]). Since X_Γ is a Banach space we can consider the Bochner integrable function on X_Γ.

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Let A_i be measurable set on $[0, T]$ with the measure $\mu(A_i) < \infty$ for all $i = 1, 2, \ldots, m$ and $\cup_{i=1}^{m} A_i = [0, T]$. We say that a function $f\colon [0, T] \to X_\Gamma$ is a simple measurable function if $f = \sum_{i=1}^{m} x_i \chi_{A_i}$, where $x_i \in X_\Gamma$.

Naturally we may define $\int_0^T f d\mu = \sum_{i=1}^{m} x_i \mu(A_i)$ for every simple measure function f. A measurable function f is called Bochner integrable (or just integrable for simplicity) if there exists a sequence of simple measurable functions $\{f_n\}$ that converge almost everywhere to f so that $\int_0^T \|f_n - f_m\|_{\Gamma'} d\mu \to 0$ and the integral $\int_0^T f d\mu$ is then defined as $\lim_{n\to\infty} \int_0^T f_n d\mu$.

Choe showed that if A generates an equicontinuous C_0-semigroup on X and $B \in \mathcal{B}_\Gamma(X)$, then $(A + B)$ generates an equicontinuous C_0-semigroup on X ([1, Corollary 5.4]). In fact, by means of some estimates of resolvent operators, Choe proved a more general result for both A and B that depend on t and satisfy certain conditions (see [1, Theorem 5.3]). However, for discourse on the photon transport problem we need only to consider the linear operators A and B that are independent of t. By choosing a suitable calibration on the *sclcs* X, we can prove Choe's Corollary 5.4 in a different approach in Section 2 (see Theorem 2.1).

Instead of solving the photon transport problem (1.6) directly, we consider the inhomogeneous term $Q(t)$ is not a constant function and $\frac{d}{dt}U(t) \neq 0$. We consider the abstract initial value problem

$$\begin{cases} \frac{d}{dt}(u(t)) = Au(t) + f(t), t > 0; \\ u(0) = x, x \in D(A), \end{cases} \tag{1.8}$$

where A is the generator of a equicontinuous C_0-semigroup and $f\colon [0, T] \to X_\Gamma$ is a Bochner integrable function. We will show that the abstract initial value problem (1.8) has a unique mild solution if A is a generator of the quasi-equicontinuous C_0-semigroup and f is a Bochner integrable function on X_Γ (see Theorem 2.3).

2 Main Results

Theorem 2.1 *Suppose $\{T(t)\}_{0\le t\le T}$ is a locally equicontinuous C_0-semigroup on a barrelled space X generated by a closed linear operator A. If B is a closed linear operator on X with $\|B\|_\Gamma = M < \infty$, then there exists a locally equicontinuous C_0-semigroup $\{S(t)\}_{0\le t\le T}$ generated by $(A + B)$.*

Before proving Theorem 2.1 we state the following lemma, which was proved in [2, Corollary 4.11].

Lemma 2.2 *Consider the abstract Cauchy problem*

$$\begin{cases} \frac{d}{dt}(u(t)) = Au(t), t > 0; \\ u(0) = x, x \in X. \end{cases} \tag{2.1}$$

The following are equivalent.

(i) *The operator A generates a locally equicontinuous semigroup.*

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(ii) *There exists a unique mild solution of* (2.1) *for all* $x \in X$.

Although this lemma was proved in the Fréchet space, it can be easily extended to the general locally convex space. We leave it to the interested reader.

Proof of Theorem 2.1 By Proposition 1.2, we may assume that $\{T(t)\}_{0 \le t \le T}$ is a local Γ-contraction C_0-semigroup. Let

$$S_0(t)x \equiv T(t)x \quad \text{for } 0 \le t \le T, x \in X \tag{2.2}$$

and define $S_n(t)$ inductively by

$$S_{n+1}(t)x \equiv \int_0^t T(t-s)BS_n(s)xds \text{ for } 0 \le t \le T, x \in X \text{ and } n = 0, 1, 2, \dots. \tag{2.3}$$

From this definition it is obvious that for each $x \in X$ and $n \ge 0$, $t \to S_n(t)x$ is continuous mapping from $[0, T]$ into X. From (2.2) and (2.3) we see that

$$\begin{aligned} \|S_1(t)x\|_\Gamma &= \| \int_0^t T(t-s)BT(s)xds\|_\Gamma \\ &\le \int_0^t \|T(t-s)BT(s)x\|_\Gamma ds \\ &\le \|Bx\|_\Gamma \int_0^t ds = t\|Bx\|_\Gamma \end{aligned}$$

for any $0 \le t \le T$ and for any $x \in X$. This implies that $\|S_1(t)\|_\Gamma \le t\|B\|_\Gamma$. By induction, one can show that

$$\| S_k(t)\|_\Gamma \le \frac{t^k}{k!}\|B\|_\Gamma^k \quad \text{for every } k \in N \text{ and } 0 \le t \le T.$$

Let

$$S(t)x = \sum_{n=0}^\infty S_n(t)x \quad \text{for every } x \in X \text{ and } 0 \le t \le T, \tag{2.4}$$

then

$$\|S(t)\|_\Gamma \le \| \sum_{k=0}^\infty S_k(t)\|_\Gamma \le \sum_{k=0}^\infty \|S_k(t)\|_\Gamma \le \sum_{k=0}^\infty \frac{t^k}{k!}\|B\|_\Gamma^k = e^{t\|B\|_\Gamma}.$$

This implies that the series (2.4) converges uniformly in $\mathcal{B}_\Gamma(X)$ under the uniform operator topology on $0 \le t \le T$. Therefore for each $x \in X$, $t \to S(t)x$ is continuous mapping from $[0, T]$ into X. According to (2.2) and (2.3) it follows that for any $x \in X$ and any $t \in [0, T]$, $S(t)x$ satisfies the equation

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)xds.$$

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This shows that $S(t)x$ is a mild solution of (1.3). To prove the uniqueness of the solution we let $\{V(t) : 0 \le t \le T\}$ be a family of operators for which $t \to V(t)x$ is continuous for every $x \in X$, and it satisfies that

$$V(t)x = T(t)x + \int_0^t T(t-s)BV(s)xds \text{ for every } x \in X \text{ and for every } 0 \le t \le T.$$

Estimating the difference of $S(t)$ and $V(t)$ yields

$$\|(S(t) - V(t))x\|_\Gamma \le \int_0^t \|B\|_\Gamma \|(S(s) - V(s))x\|_\Gamma ds.$$

Gronwall's inequality implies that $S(t) = V(t)$ for every $t \in [0, T]$. According to Lemma 2.2, $\{S(t)\}_{0\le t\le T}$ is a locally equicontinuous C_0-semigroup generated by $(A+B)$. ■

To solve the photon transport problem (1.6), where the inhomogeneous term $Q(t)$ is not a constant function, we should consider the initial value problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t), t > 0; \\ u(0) = u_0 \in D(A); \end{cases} \tag{2.5}$$

where A is a generator of an equicontinuous C_0- semigroup and f is a Bochner integrable function on X_Γ. Instead of proving that (2.5) has a mild solution directly, we prove the more general case that (2.5) has a mild solution as long as A is the generator of a quasi-equicontinuous C_0- semigroup. In fact we have following theorem.

Theorem 2.3 *If A is the generator of a quasi-equicontinuous C_0- semigroup $\{T(t)\}_{t\ge 0}$ and f is a Bochner integrable on X_Γ, then (2.5) has a unique mild solution given by*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds.$$

Moreover, if f is continuous, then $u(t)$ is a solution of (2.5).

Proof If u is a solution of (2.5), then the X_Γ valued function $g(s) = T(t-s)u(s)$ is differentiable for $0 < s < t$ and

$$\begin{aligned} \frac{d}{ds}g(s) &= -AT(t-s)u(s) + T(t-s)u'(s) \\ &= -AT(t-s)u(s) + T(t-s)Au(s) + T(t-s)f(s) \\ &= T(t-s)f(s). \end{aligned}$$

Moreover, if f is Bochner integrable on X_Γ, then $T(t-s)f(s)$ is also Bochner integrable. Integrating it from 0 to t yields

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds.$$

To see that $u(t)$ is a solution of (2.5) when f is continuous, we need only to show that $u(t)$ satisfies (2.5). Since

$$\begin{aligned}
u'(t) &= \lim_{h\to 0}\frac{1}{h}\big(u(t+h)-u(t)\big)\\
&= \lim_{h\to 0}\frac{1}{h}\Big(T(t+h)u_0+\int_0^{t+h}T(t+h-s)f(s)ds - T(t)u_0\\
&\qquad -\int_0^t T(t-s)f(s)ds\Big)\\
&= \lim_{h\to 0}\frac{1}{h}(T(t+h)u_0-T(t)u_0)\\
&\quad +\lim_{h\to 0}\frac{1}{h}\Big(\int_0^{t+h}T(t+h-s)f(s)ds-\int_0^t T(t-s)f(s)ds\Big)\\
&= AT(t)u_0+\lim_{h\to 0}\frac{1}{h}(T(h)-I)\int_0^t T(t-s)f(s)ds\\
&\quad +\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}T(t+h-s)f(s)ds\\
&= AT(t)u_0 + A\int_0^t T(t-s)f(s)ds+f(t)\\
&= Au(t)+f(t),
\end{aligned}$$

we see that $u(t)$ is differentiable on $(0,\infty)$, and it satisfies (2.5). ■

3 Generalized Solution of the Photon Transport Problem

To find a generalized solution of the photon transport problem, we let X be an *sclcs* and $\{T(t)\}_{t\ge 0}\subset \mathcal{L}(X,X)$ be an equicontinuous C_0-semigroup. Also, let X_s' be the dual space of X endowed with the seminorm $p_{B,q} = \sup\{q(Lx') : x'\in B\}$ for every $L\subset \mathcal{L}(X_s',X_s')$, and let $T^*(t)$ denotes the dual operator of $T(t)$, where B is an arbitrary bounded subsets of X'. Then the family $\{T^*(t)\}_{t\ge 0}$ of linear operators are in $\mathcal{L}(X_s',X_s')$ and satisfy the semigroup property

$$T^*(t)T^*(s)=T^*(t+s),\quad T^*(0)=I^*,$$

where I^* is the identity operator on X_s'. Notice that $\{T^*(t)\}_{t\ge 0}$ is not a C_0-semigroup in general. However, T. Komura [3] showed the following theorem.

Theorem 3.1 *Let X be an sclcs such that its strong dual space X_s' is also sequentially complete. Let $\{T(t)\}_{t\ge0}$ be a C_0-semigroup with the generator A. Let us denote by X^+ the closure of the domain $D(A^*)$ in the strong topology of X_s'. If $T^+(t)$ is the restriction of $T^*(t)$ of to X^+, then $\{T^+(t)\}_{t\ge0}\subset\mathcal{L}(X^+,X^+)$ and $\{T^+(t)\}_{t\ge0}$ is a C_0-semigroup with the generator A^+, which is the largest restriction of A^* with domain and range in X^+. In particular, if a C_0-semigroup $\{T(t)\}_{t\ge0}$ is locally equicontinuous (resp. equicontinuous), then $\{T^+(t)\}_{t\ge0}$ is also locally equicontinuous (resp. equicontinuous).*

Now we are able to show the existence of a solution for the photon transport problem (1.6). As we mentioned in Section 1, it is reasonable to assume that $\sigma(t) = \sigma$ and $\sigma_s(t) = \sigma_s$ are independent of t and (1.6) may be rewritten as

$$\begin{cases} \frac{d}{dt}U(t) = [S - c\sigma I + c\sigma_s J]U(t) + Q, & \forall t > 0, \\ U(0) = U_0. \end{cases} \tag{3.1}$$

We will show that problem (3.1) has a unique generalized (or weak) solution U^* in some space $\widetilde{X}(= D'(R \times (-1,1)))$. To describe the space $\widetilde{X}$, we use following notations: Let $\{K_m\}_{m=1}^{\infty}$ be a sequence of compact subsets of $R \times (-1,1)$ such that $K_1 \subset K_2 \subset \cdots$ and $R \times (-1,1) = \bigcup_{m=1}^{\infty} K_m$, and let D_{K_m} be the set

$$D_{K_m}(R \times (-1,1)) = \{\phi \in C^{\infty}(R \times (-1,1)) : \text{ supp } \phi \subset K_m\}(m \in N)$$

with the calibration of seminorms $\Gamma = \{p_{m,\alpha}; m \in N, \alpha \in N_0^2\}$ such that

$$p_{m,\alpha}(\phi) = \sup_{(x,\mu)\in K_m} |(\partial^{\alpha}\phi)(x,\mu)|, \phi \in D_{K_m}. \tag{3.2}$$

Here, N_0 is the set of all nonnegative integers. Let the space $D = D(R \times (-1,1))$ be defined by

$$D = D(R \times (-1,1)) = \bigcup_{m=1}^{\infty} D_{K_m}(R \times (-1,1)).$$

Then D is a Fréchet space with topology induced by the calibration of seminorms (3.2). Since every Fréchet space is a barrelled space, this implies that D is a barrelled space. Let $\widetilde{X} = D'$ be the dual space of D.

It can be shown that (see *e.g.*, [8]) $X = L^1(R \times (-1,1)) \subset \widetilde{X} = D'$ in the following sense. We say that $f \in D'$ can be identified with $f \in L^1$ if

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} \Big[\int_{-1}^{1} f(x,\mu)\phi(x,\mu)d\mu \Big] dx \quad \forall \phi \in D.$$

We also extend the operator $T = c\sigma I - S - c\sigma_s J$ to the operator

$$\widetilde{T} \colon \widetilde{X} = D' \to \widetilde{X} = D'$$

such that

$$\langle \widetilde{T}\widetilde{f}, \phi \rangle = \langle \widetilde{Tf}, \phi \rangle = \int_{-\infty}^{\infty} \Big[\int_{-1}^{1} Tf(x,\mu)\phi(x,\mu)d\mu \Big] dx \quad \forall \phi \in D.$$

Now we define the operator $\widehat{T}$on D by

$$\widehat{T}\varphi(x,\mu)=c\sigma\varphi(x,\mu)-c\mu\frac{\partial\varphi(x,\mu)}{\partial x}-c\sigma_s\int_{-1}^{1}k(\mu,\mu')\varphi(x,\mu')d\mu'\quad\forall\varphi\in D.$$

Then we have the relation $\langle\widetilde{T}\widetilde{f},\phi\rangle=\langle\widetilde{f},\widehat{T}\phi\rangle\quad\forall\widetilde{f}\in D',\ \phi\in D$. In other words, $\widetilde{T}$ is the adjoint of $\widehat{T}$. Let L denote the operator $\frac{d}{dt}+\widehat{T}$ and let L^* denotes its formal adjoint of L. We say that a distribution U^* is a generalized solution of (3.1) if $\langle L^*U^*,\phi\rangle=\langle U^*,L\phi\rangle=\langle Q,\phi\rangle$ is satisfied for every $\phi\in D$.

Let the operator A, B, C be on the space D as follows

$$A\varphi(x,\mu)=-c\mu\frac{\partial\varphi(x,\mu)}{\partial x},$$

$$B\varphi(x,\mu)=c\sigma_s\int_{-1}^{1}k(\mu,\mu')\varphi(x,\mu')d\mu',\ \text{and}$$

$$C\varphi(x,\mu)=c\sigma\varphi(x,\mu).$$

Meri Lisi and Silvia Torato [5] showed that there exists a Γ-contraction C_0-semigroup $\{W(t)\}_{t\geq 0}$on D that is generated by $-A$, and B and C are in $\mathcal{B}_\Gamma(D)$ with the operator norm $\|B\|_\Gamma=c\sigma_s\bar{k}$ and $\|C\|_\Gamma=c\sigma$, respectively.

By Theorem 2.1, there exists a locally equicontinuous C_0-semigroup $\{V(t)\}_{t\geq 0}$ on D generated by $-\widehat{T}$. Let $\{V^*(t)\}_{t\geq 0}$ be the C_0-semigroup on D' generated by $-\widetilde{T}$. According to Theorem 3.1, $\{V^*(t)\}_{t\geq 0}$ is also locally equicontinuous on D' since $\widetilde{T}$ (the adjoint of $\widehat{T}$) is an automorphism on D', *i.e.,* the domain of $\widetilde{T}$ is $\widetilde{X}$ (see [5, Remark 3.2]). Clearly, Q belongs to D'_Γ. This implies that (3.1) can be considered the special case $f(t)=Q$ for all $t\in[0,T]$. Then by Theorem 2.3, we conclude that (3.1) has a unique generalized solution U^*.

References

[1] Y. H. Choe, *C_0-semigroups on a locally convex space.* J. Math. Anal. Appl. **106**(1985), no. 2, 293–320. doi:10.1016/0022-247X(85)90115-5

[2] R. deLaubenfels, *Existence families, functional calculi and evolution equations.* Lecture Notes in Mathematics, 1570, Springer-Verlag, Berlin, 1994.

[3] T. Komura, *Semigroups of operators in locally convex spaces.* J. Functional Analysis **2**(1968), 258–296. doi:10.1016/0022-1236(68)90008-6

[4] G. Köthe, *Topological vector spaces. II.* Grundlehren der Mathematischen Wissenschaften, 237, Springer-Verlag, New York-Berlin, 1979.

[5] M. Lisi and S. Totaro, *Photon transport with a localized source in locally convex spaces.* Math Methods Appl. Sci. **29**(2006), no. 9, 1019–1033. doi:10.1002/mma.713

[6] ———, *Inverse problems related to photon transport in an interstellar cloud.* Transport Theory Statist. Phys. **32**(2003), no. 3–4, 327–345. doi:10.1081/TT-120024767

[7] E. V. Teixeira, *Strong solutions for differential equations in abstract spaces.* J. Differential Equations **214**(2005), no. 1, 65–91. doi:10.1016/j.jde.2004.11.006

[8] F. Trèves, *Topological vector spaces, distributions and kernels.* Academic Press, New York-London, 1967.

[9] K. Yosida, "Functional analysis," AcademicPress, NewYork, 1968 Second Edition.

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