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ON THE OSCILLATION OF SOLUTIONS OF CERTAIN LINEAR DIFFERENTIAL EQUATIONS IN THE COMPLEX DOMAIN

by STEVEN B. BANK and J. K. LANGLEY

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1. Introduction

Our starting point is the differential equation

$$y''+A(z)y=0 \tag{1.1}$$

where $A(z)$ is a transcendental entire function of finite order, and we are concerned specifically with the frequency of zeros of a non-trivial solution $f(z)$ of (1.1). Of course it is well known that such a solution $f(z)$ is an entire function of infinite order, and using standard notation from [7],

$$N\left(r,\frac{1}{f-b}\right)\sim T(r,f)$$

for all $b\in C\backslash\{0\}$, at least outside a set of r of finite measure. The same conclusions hold if y'' is replaced by a higher derivative in (1.1). Denoting by $\sigma(g)$ the order of an entire function g, and by $\lambda(g)$ the exponent of convergence of its zeros we have the following, proved in [**1**, **3**]:

Theorem A. *Let $A(z)$ be a transcendental entire function, and let f_1, f_2 be linearly independent solutions of* (1.1).

(a) *Suppose that $\sigma(A)$ is finite but is not a positive integer. Then* $\max\{\lambda(f_1),\lambda(f_2)\}$ *is not less than $\sigma(A)$, and is infinite if $\sigma(A)<\frac{1}{2}$.*

(b) *Suppose that $\lambda(A)<\sigma(A)<\infty$.*

Then for any $k\geqq 2$ and any non-trivial solution $f(z)$ of

$$y^{(k)}+A(z)y=0$$

we have $\lambda(f)\geqq\sigma(A)$.

We remark that it is conjectured that under the hypotheses of Theorem A, part (a), we always have

$$\max\{\lambda(f_1),\lambda(f_2)\}=\infty.$$

The following result was proved in [4], and gives conditions under which this stronger conclusion holds:

Theorem B. *Let $A(z)$ be a transcendental entire function of finite order ρ with the following property: there exists a set $H \subseteq \mathbb{R}$ of measure zero such that for each real θ not in H either*

(i) $$r^{-N}|A(re^{i\theta})| \to \infty \quad \text{as} \quad r \to +\infty \text{ for each } N>0, \quad \text{or}$$

(ii) $$\int_0^\infty r|A(re^{i\theta})|\,dr < +\infty, \quad \text{or}$$

(iii) *there exist positive real numbers K and b, and a non-negative real number n (all possibly depending on θ), such that $(n+2)<2\rho$ and*

$$|A(re^{i\theta})| \leqq Kr^n \quad \text{for all } r \geqq b.$$

Then if f_1 and f_2 are two linearly independent solutions of

$$y'' + A(z)y = 0$$

we have

$$\max\{\lambda(f_1), \lambda(f_2)\} = \infty.$$

This result is sharp in that (see [4]) there exist pairs of polynomials $P(z)$, $Q(z)$, whose degrees d_P, d_Q respectively satisfy $d_Q + 2 = 2d_P$, such that the equation

$$y'' + (e^P + Q)y = 0$$

has two linearly independent non-vanishing solutions. We mention two corollaries of Theorem B.

Corollary A. *Suppose that $A(z)$ is an entire function of finite order with zero as a Borel exceptional value. Then given any two linearly independent solutions f_1, f_2 of*

$$y'' + A(z)y = 0$$

we have

$$\max\{\lambda(f_1), \lambda(f_2)\} = \infty.$$

Corollary B. *Suppose that $P(z)$ is a non-constant, even polynomial with real coefficients and with positive leading coefficient. Then all non-trivial solutions f of*

$$y'' + e^P y = 0$$

satisfy

$$\lambda(f)=\infty.$$

Corollary B is obtained in [4] by coupling Theorem B with the Sturm theory for oscillation of real solutions of linear differential equations on the real line. The following problem is posed in [4], with reference to Corollary B: if $P(z)$ is any non-constant polynomial, must every non-trivial solution $f(z)$ of

$$y''+e^{P}y=0$$

satisfy $\lambda(f)=\infty$? In the present paper we settle this problem and rather more, and our methods extend to higher order equations and have a bearing on Corollary A. We shall prove:

Theorem. *Suppose that* $k\geqq 2$ *and that* $A(z)=\Pi(z)e^{P(z)}\not\equiv 0$ *where the entire function* $\Pi(z)$ *and the polynomial* $P(z)=a_nz^n+\cdots+a_0$ *satisfy:*

(*i*) $\sigma(\Pi)<n$;

(ii) *there exists* $\theta_0\in\mathbb{R}$ *with* $\delta(P,\theta_0)=Re(a_ne^{in\theta_0})=0$ *and a positive* ε *such that* $\Pi(z)$ *has only finitely many zeros in*

$$|\arg z-\theta_0|<\varepsilon.$$

Then if $n\geqq 2$ *and* Q *is a polynomial whose degree* d_Q *satisfies*

$$d_Q+k<kn,$$

all non-trivial solutions f *of*

$$y^{(k)}+(A(z)+Q(z))y=0 \tag{1.2}$$

satisfy $\lambda(f)=\infty$. *The same conclusion holds if* $n=1$ *and* Q *is identically zero.*

We remark that the theorem is sharp at least in the case $k=2$ in view of the examples mentioned after Theorem B. We do not know if condition (ii) is sharp; its presence serves to facilitate certain asymptotic representations for the solutions of (1.2). However we do have the following corollary to our theorem:

Corollary. *If* $A(z)$ *is a transcendental entire function of finite order having finitely many zeros, all non-trivial solutions of*

$$y^{(k)}+A(z)y=0$$

satisfy

$$\lambda(y)=\infty, \quad \textit{for any } k \geqq 2.$$

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2. Preliminaries

We need the following definition.

Definition. An R-set is a countable union of discs whose radii have finite sum.

We remark that the union of two R-sets is an R-set and that (see Hayman [**6**], also [**4**]) the set of θ for which the ray $re^{i\theta}$ meets infinitely many discs of a given R-set has measure zero.

Also, for a polynomial

$$P(z)=(\alpha+i\beta)z^n+\cdots+a_0$$

with α, β real, we define, for each real θ,

$$\delta(P,\theta)=\alpha\cos n\theta-\beta\sin n\theta$$

and denote the degree of P by d_P.

3. Lemmas needed for the theorem

Lemma 1. *Assume the hypotheses of the theorem. Then there is a constant $c>0$ such that the asymptotic relation*

$$\left|\frac{A'(re^{i\theta})}{A(re^{i\theta})}\right| \sim cr^{n-1} \qquad \textit{as } r\to\infty \tag{3.1}$$

holds uniformly for all real θ satisfying $|\theta-\theta_0|\leqq 2\varepsilon/3$. In addition, for any real θ satisfying $0<|\theta-\theta_0|\leqq \varepsilon/2$ the following are true:

(a) *if $\delta(P,\theta)>0$, the function $\log|A(re^{i\theta})|$ is increasing on an interval $[\alpha(\theta), +\infty)$ and we have*

$$|A(re^{i\theta})| \geqq \exp(\tfrac{1}{2}\delta(P,\theta)r^n) \tag{3.2}$$

there;

(b) *if $\delta(P,\theta)<0$, the function $\log|A(re^{i\theta})|$ is decreasing on an interval $[\alpha(\theta), +\infty)$ and on that interval*

$$|A(re^{i\theta})| \leqq \exp(\tfrac{1}{2}\delta(P,\theta)r^n). \tag{3.3}$$

Proof. We may write

$$A(z)=\Pi_1(z)\exp(a_n z^n)$$

where $\Pi_1(z)$ is an entire function of order less than n, and setting $\rho=(\sigma(\Pi_1)+n)/2$ standard estimates yield the inequality

$$\left|\log|\Pi_1(\zeta)|\right|\leqq R^\rho \tag{3.4}$$

for all ζ on $|\zeta|=R$ and for all R outside a set of finite linear measure L. Now, if $z=re^{i\theta}$, with $|\theta-\theta_0|\leqq 2\varepsilon/3$ and r sufficiently large, it follows from hypothesis (ii) of the theorem that all the zeros of Π_1 are at a distance at least $\delta|z|$ from z for some fixed $\delta>0$. For such a point z with $r>1$, since (3.4) must hold on some circle $|\zeta|=R$, with R belonging to $[(L+2)r,(2L+3)r]$, a routine application of the differentiated form of the Poisson–Jensen formula [7, p. 22] shows that

$$|\Pi_1'(z)/\Pi_1(z)|\leqq c_2|z|^{\rho-1}, \tag{3.5}$$

for a positive constant c_2 independent of z.

Since

$$A'/A=(\Pi_1'/\Pi_1)+na_n z^{n-1}, \tag{3.6}$$

we obtain (3.1) from (3.5) and (3.6).

Now assume that θ satisfies $0<|\theta-\theta_0|\leqq\varepsilon/2$. Then by definition of $\delta(P,\theta)$ we have

$$|A(re^{i\theta})|=|\Pi_1(re^{i\theta})|\exp(\delta(P,\theta)r^n). \tag{3.7}$$

For r sufficiently large the point $\zeta=re^{i\theta}$ is sufficiently distant from the zeros of Π_1 that (3.4) holds with $R=r$, and so (3.2) and (3.3) hold.

Now, from (3.5) we see that if $r_0<r$ are both large then, setting $\psi(s)=\log|\Pi_1(se^{i\theta})|$ we have

$$|\psi(r)-\psi(r_0)|\leqq|\log(\Pi_1(re^{i\theta})/\Pi_1(r_0e^{i\theta}))|\leqq c_2(r-r_0)r^{\rho-1}. \tag{3.8}$$

It then follows that $|\psi'(r)|=O(r^{\rho-1})$ as $r\to+\infty$. Since by (3.7) the function $\phi(r)=\log|A(re^{i\theta})|$ satisfies the equation

$$\phi'(r)=n\delta(P,\theta)r^{n-1}+\psi'(r) \tag{3.9}$$

the rest of the lemma follows easily.

Lemma 2. *Assume the hypotheses of the Theorem with ε sufficiently small that $\delta(P,\theta)\neq 0$ for $0<|\theta-\theta_0|<\varepsilon$, and assume that $R(z)$ is analytic and satisfies*

$$|R(z)|\leqq K|z|^M \tag{3.10}$$

on the sectorial set A_0 given by

$$|\arg z-\theta_0|\leqq\varepsilon/2,\quad |z|\geqq\rho_0$$

where ρ_0 is large, and K and M are non-negative. For a fixed $a\in A_0$ define $H(z)$ in A_0 by

$$H(z)=A(z)^{1/k}\int_a^z R(t)A(t)^{-1/k}\,dt, \tag{3.11}$$

for some fixed branch of $A(z)^{1/k}$ in A_0. Then for some $\rho_1>0$ we have the following representation for $H(z)$ in the sectorial set A_1 given by

$$|\arg z-\theta_0|\leqq\varepsilon/4,\quad |z|\geqq\rho_1:$$

there exists an analytic function $S(z)$ on A_1 satisfying

$$\log^+|S(z)|=O(\log|z|) \tag{3.12}$$

in A_1, and such that for any θ with $0<|\theta-\theta_0|\leqq\varepsilon/4$, we have, as $r\to\infty$,

$$H(re^{i\theta})=S(re^{i\theta})+c(\theta)A(re^{i\theta})^{1/k}+O(r^{-2}) \tag{3.13}$$

if $\delta(P,\theta)>0$, where $c(\theta)$ is a constant, while if $\delta(P,\theta)<0$ we have

$$H(re^{i\theta})=S(re^{i\theta})+O(r^{-1}). \tag{3.14}$$

Proof. In view of (3.1) we may assume that ρ_0 is so large that

$$\left|\frac{A'(z)}{A(z)}\right|\geqq(c/2)|z|^{n-1} \tag{3.15}$$

on A_0. We choose $\rho_1>\rho_0$ so that there is a fixed constant ε_1 such that $0<\varepsilon_1<1$ and, for each z in A_1, the closed disc of radius $\varepsilon_1|z|$ and centre z is contained in A_0.

We now define two sequences (R_m) and (S_m) of functions analytic on A_0 by the equations

$$R_1=RA/A',\quad S_1=R_1', \tag{3.16}$$

and for $m\geqq1$,

$$R_{m+1}=S_mA/A',\quad S_{m+1}=R'_{m+1}. \tag{3.17}$$

Now let $z_0=re^{i\theta}$, with $0<|\theta-\theta_0|\leqq\varepsilon/4$, belong to A_1. In view of (3.10) and (3.15), a simple induction using Cauchy's formula for derivatives shows that for each $m=1,2,\ldots$, we have, on $|z-z_0|\leqq\varepsilon_1|z_0|2^{-m}$, the estimate

$$|S_m(z)|\leqq K_m|z_0|^{M-nm}$$

where K_m is a positive constant independent of z_0. Integration by parts yields, for each $m=1,2,\dots,$

$$\int_a^z R(t)A(t)^{-1/k}\,dt=\Big[_a^z -kR_1A^{-1/k}+k\int_a^z S_1A^{-1/k}\,dt$$

$$\vdots$$

$$=\Big[_a^z S_m^*A^{-1/k}+k^m\int_a^z S_mA^{-1/k}\,dt \tag{3.19}$$

where

$$S_m^*=-kR_1-k^2R_2-\cdots-k^mR_m.$$

We now choose m so that $M-nm\leqq -3$. Now if $S_m\equiv 0$ we need only set $S=S_m^*$, while if S_m does not vanish identically it remains only to estimate the last integral in (3.19). In the latter case, if $z=re^{i\theta}$ lies in A_1 (3.18) and the choice of m imply that

$$|S_m(\zeta)|\leqq K_m|\zeta|^{-3} \tag{3.20}$$

for all ζ in A_1, and it follows from (3.2) that if $\delta(P,\theta)>0$, the integral

$$\int_a^\infty S_mA^{-1/k}\,dt=c(\theta)$$

converges, where the path of integration is eventually along the ray $\arg z=\theta$. Thus

$$\int_a^z S_mA^{-1/k}\,dt=c(\theta)-\int_z^\infty S_mA^{-1/k}\,dt \tag{3.21}$$

and in view of (3.20) and the fact that, by Lemma 1, the function $|A(se^{i\theta})|$ is eventually increasing as $s\to+\infty$, we see that the integral on the right-hand side of (3.21) is bounded by

$$K_m|A(z)|^{-1/k}\int_r^\infty s^{-3}\,ds.$$

The representation (3.13) now holds with $S=S_m^*$.

Now suppose that $z=re^{i\theta}$ lies in A_1, with $\delta(P,\theta)<0$, If r is sufficiently large the point $z^*=\sqrt{r}e^{i\theta}$ also lies in A_1 and we may write

$$\int_a^z S_mA^{-1/k}\,dt=\int_a^{z^*} S_mA^{-1/k}\,dt+\int_{z^*}^z S_mA^{-1/k}\,dt \tag{3.22}$$

where as before we integrate eventually along the ray $\arg z = \theta$. By Lemma 1, $|A(se^{i\theta})|$ is eventually decreasing, so that in view of (3.20) and (3.3) we have, provided that r is large enough,

$$\left|\int_a^{z^*} S_m A^{-1/k}\,dt\right| \leqq B_1 |A(\sqrt{r}e^{i\theta})|^{-1/k}, \tag{3.23}$$

where B_1 is a positive constant while

$$\left|\int_{z^*}^{z} S_m A^{-1/k}\,dt\right| \leqq K_m |A(re^{i\theta})|^{-1/k} \int_{\sqrt{r}}^{r} t^{-3}\,dt. \tag{3.24}$$

Since

$$\log|A(se^{i\theta})| \sim \delta(P,\theta)s^n \tag{3.25}$$

as $s \to +\infty$, using (3.7) and observing that (3.4) will hold with $R = s$ and $\zeta = se^{i\theta}$, we deduce from (3.23) and (3.24) that (3.14) holds, again with $S = S_m^*$.

Lemma 3. *Suppose that $A(z)$ is analytic in a sector S containing the ray $z = re^{i\theta}$ and that, for some non-negative K and n, and positive b we have*

$$|A(re^{i\theta})| \leqq Kr^n$$

for all $r \geqq b$. Then if $k \geqq 2$, any non-trivial solution $w(z)$ of

$$w^{(k)} + A(z)w = 0$$

satisfies

$$\log^+|w(re^{i\theta})| \leqq M(1 + r^{(n+k)/k})$$

for some $M > 0$ and for all $r \geqq b$.

Proof. Take $L > 0$ and set

$$v(r) = \exp\left(\int_b^r (Lt^n)^{1/k}\,dt\right).$$

Then (see e.g. [7, p. 73]) we clearly have

$$v^{(k)}(r)/v(r) = Lr^n + O(r^{n-1}) \geqq (L/2)r^n$$

for all $r \geqq b$, provided L is large enough. Now set $h(r) = w(re^{i\theta})$ so that h solves the equation

$$h^{(k)}(r) - B(r)h = 0,$$

where $B(r) = -e^{ik\theta}A(re^{i\theta})$. Choose a positive constant c such that

$$|h(b)| \leqq cv(b),$$

$$|h'(b)| \leqq cv'(b),$$

$$\vdots$$

$$|h^{(k-1)}(b)| \leqq cv^{(k-1)}(b).$$

Then, since

$$|B(r)| \leqq (L/2)r^n$$

for $r \geqq b$, provided L is large enough, Herold's comparison theorem [8] applies and we deduce that $|h(r)| \leqq cv(r)$ for all $r \geqq b$ and the lemma is proved.

Lemma 4. *Suppose that $a(z)$ is analytic in a sector S containing the ray $re^{i\theta}$ and suppose that $k \geqq 2$ and*

$$\int^{\infty} r^{k-1}|a(re^{i\theta})|\,dr < \infty.$$

Then any solution $w(z)$ of

$$w^{(k)} + aw = 0$$

satisfies

$$w(re^{i\theta}) = 0(r^{k-1})$$

as $r \to +\infty$.

Proof. We may write

$$w(z) = c_1 + c_2 z + \cdots + c_k z^{k-1} - \frac{1}{(k-1)!}\int_{e^{i\theta}}^{z} (z-s)^{k-1}a(s)w(s)\,ds.$$

This gives, for $z = re^{i\theta}$, with $r > 1$, and with $h(z) = w(z)r^{1-k}$,

$$|h(z)| \leqq O(1) + \frac{1}{(k-1)!}\int_1^r t^{k-1}|a(te^{i\theta})||h(te^{i\theta})|\,dt$$

and we now apply Gronwall's lemma [5, p. 35].

4. Proof of the theorem

The outline of the proof is as follows. Assuming that (1.2) has a non-trivial solution f with $\lambda(f)<\infty$, we obtain, using the first-order differential equation satisfied by e^P, a representation $f=We^G$, where W is analytic and of finite order of growth in a sector. Using the asymptotic relations of Lemmas 1 and 2, and the growth estimates of Lemmas 3 and 4, we then obtain a contradiction.

Suppose then that $f=\Pi_1 e^h$ is a non-trivial solution of (1.2), where k, A and Q are as in the statement (and $Q\equiv 0$ if $\sigma(A)=1$), and suppose that Π_1 has finite order. Now (1.2) gives

$$(h')^k+P_{k-1}(h')+A+Q=0, \tag{4.1}$$

where $P_{k-1}(h')$ is a differential polynomial of total degree at most $(k-1)$ in $h', h'', \ldots$ and with coefficients which are polynomials in $\Pi_1'/\Pi_1, \ldots, \Pi_1^{(k)}/\Pi_1$, having constant coefficients. Clunie's lemma (see [2]) shows that $\sigma(h')$ is finite. Differentiating (4.1) we obtain

$$k(h')^{k-1}h''+Q_{k-1}(h')+A'+Q'=0, \tag{4.2}$$

where $Q_{k-1}(h')$ is again a differential polynomial of degree at most $(k-1)$ whose coefficients are polynomials in $\Pi_1'/\Pi_1, \ldots, \Pi_1^{(k+1)}/\Pi_1$. Multiplying (4.1) by A'/A and subtracting from (4.2) we obtain

$$kR(h')^{k-1}-(A'/A)P_{k-1}(h')+Q_{k-1}(h')-(A'/A)Q+Q'=0 \tag{4.3}$$

where

$$R=h''-(A'/kA)h'. \tag{4.4}$$

From hypothesis (ii) and (3.1) R is analytic and of finite order of growth on a sectorial set given by $|\arg z-\theta_0|\leqq 2\varepsilon/3$, $|z|$ large. Since Π_1, A and h' are all of finite order, standard estimates yield an $N>0$ such that for $j=1,\ldots,k+1$,

$$|A'/A|+|\Pi_1^{(j)}/\Pi_1|+|h^{(j+1)}/h'|=O(|z|^N) \tag{4.5}$$

at least outside an R-set, and thus (4.5) holds as $z=re^{i\theta}\to\infty$ along the ray $\arg z=\theta$ for all θ outside a set E_1 of measure zero (see Section 2). By writing $R=h'(h''/h'-(A'/kA))$ we see that if $\theta\notin E_1$ and r is large enough, the inequality $|h'(re^{i\theta})|\leqq 1$ implies that $|R(re^{i\theta})|\leqq r^{N+1}$. On the other hand, solving (4.5) for R, we see that if $\theta\notin E_1$, if r is large, and $|h'(re^{i\theta})|>1$, then $|R(re^{i\theta})|\leqq r^U$ for some constant U. It now follows from the Phragmén–Lindelöf principle that for some $V>0$, the inequality

$$|R(re^{i\theta})|\leqq r^V \quad \text{as } r\to+\infty \tag{4.6}$$

holds uniformly in θ for $|\theta-\theta_0|\leqq\varepsilon/2$.

In view of (4.4) we have, for a suitable point a, and a constant K,

$$h'(z)=A(z)^{1/k}\int_a^z R(t)A(t)^{-1/k}\,dt+KA(z)^{1/k}. \tag{4.7}$$

Now (4.6) implies that the hypothesis of Lemma 2 is fulfilled, and we obtain an analytic function $S(z)$ on the sectorial region A_1 given by

$$|\arg z-\theta_0|\leqq\varepsilon/4,\quad |z|\geqq\rho_1,$$

which satisfies (3.12), (3.13) and (3.14), where

$$H(z)=A(z)^{1/k}\int_a^z R(t)A(t)^{-1/k}\,dt. \tag{4.8}$$

We now define $W(z)$ on A_1 by the equation

$$f(z)=W(z)A(z)^{\alpha}\exp\left(h(z)-\int_a^z S(t)\,dt\right), \tag{4.9}$$

where $\alpha=(1-k)/2k$. It follows easily from (3.12) and the representation $f=\Pi_1 e^h$, and hypothesis (ii) of the Theorem, that $W(z)$ is analytic and of finite order of growth in A_1.

Now, in view of (3.12), (4.6) and (3.1), it is easy to see using Cauchy's integral formula that in a sectorial set A_2 given by $|\arg z-\theta_0|\leqq\varepsilon/8$, $|z|\geqq\rho_2$, we have, for each $m=0,1,\dots,k$, and for some $q>0$,

$$|S^{(m)}(z)|+|R^{(m)}(z)|+|(A'(z)/A(z))^{(m)}|\leqq|z|^q. \tag{4.10}$$

Now (4.5) and the remark in Section 2 imply that for θ outside a set E_2 of measure zero, and for $j=1,\dots,k$,

$$|\Pi_1^{(j)}(re^{i\theta})/\Pi_1(re^{i\theta})|\leqq r^N\quad\text{for } r\geqq r(\theta). \tag{4.11}$$

It now follows from (4.9), (4.10), (4.11) and the representation $f=\Pi_1 e^h$, that if $\theta\notin E_2$ and $|\theta-\theta_0|\leqq\varepsilon/8$, then for $j=1,\dots,k$, we have

$$|W^{(j)}(re^{i\theta})/W(re^{i\theta})|\leqq r^M\quad\text{for } r\geqq r(\theta) \tag{4.12}$$

where M is a fixed constant.

We choose θ_1, θ_2 such that $0<|\theta_j-\theta_0|<\varepsilon/8$ and

$$\delta(P,\theta_1)<0,\quad \delta(P,\theta_2)>0,\quad\text{and}\quad \theta_2\notin E_2. \tag{4.13}$$

We now assert that

$$W(re^{i\theta_1})\to 0\quad\text{as } r\to\infty. \tag{4.14}$$

We set $d=d_Q$ if $Q\not\equiv 0$, and $d=0$ otherwise, and note that by (3.3) and (4.13), it follows that if $\sigma(A)>1$ then Lemma 3 or Lemma 4 applies and we obtain

$$\log^+|f(re^{i\theta_1})|=O(r^s) \quad \text{as } r\to+\infty, \tag{4.15}$$

where $s=(d+k)/k<\sigma(A)$. On the other hand, if $\sigma(A)=1$ (and hence $Q\equiv 0$), Lemma 4 applies and we obtain

$$\log^+|f(re^{i\theta_1})|=O(\log r) \quad \text{as } r\to+\infty. \tag{4.16}$$

However, by (3.14), (4.7) and (4.8) we have as $z\to\infty$ on $\arg z=\theta_1$,

$$h'(z)=S(z)+KA(z)^{1/k}+O(|z|^{-1}) \tag{4.17}$$

so that using (3.3) we have, for r sufficiently large,

$$h(re^{i\theta_1})-\int_a^{re^{i\theta_1}} S(t)\,dt=O(\log r). \tag{4.18}$$

Thus from (4.9) we have, for some $b>0$,

$$W(re^{i\theta_1})=f(re^{i\theta_1})A(re^{i\theta_1})^{(k-1)/2k}O(r^b), \tag{4.19}$$

as $r\to+\infty$. But then, using (3.3), (4.15) or (4.16) and the fact that $s<\sigma(A)$, we obtain (4.14) from (4.19).

We now assert that there is a finite, non-zero constant J such that

$$W(re^{i\theta_2})\to J \quad \text{as } r\to\infty. \tag{4.20}$$

We remark that once this claim is established, (4.20) and (4.14) together provide a contradiction as follows. Since θ_1 and θ_2 satisfying (4.13) can both be chosen arbitrarily close to θ_0, the finite order of W in A implies, using the Phragmén–Lindelöf principle, that W is bounded in the sector between the two rays $re^{i\theta_1}$, $re^{i\theta_2}$ which in turn implies that $J=0$ (see [**9**, p. 179]), which is impossible.

To establish (4.20), we write $f=We^G$, where, using (4.9),

$$G'=h'-S+\alpha(A'/A). \tag{4.21}$$

Substituting in (1.2) we obtain an expression of the form

$$W^{(k)}/W+\sum_{j=0}^{k-1} F_j(W^{(j)}/W)+A+Q=0 \tag{4.22}$$

where each F_j is a polynomial in $G',\dots,G^{(k)}$, with constant coefficients, satisfying the following conditions:

$$\text{for } j \geqq 2,\ F_j \text{ has total degree at most } (k-2); \tag{4.23}$$

$$F_1 = k(G')^{k-1} + B_1; \tag{4.24}$$

$$F_0 = (G')^k + (k(k-1)/2)(G')^{k-2}G'' + B_2 \tag{4.25}$$

and where the total degrees of B_1 and B_2 are at most $k-2$. We need estimates for the derivatives of G' on the ray $\arg z = \theta_2$ which we obtain as follows.

From (4.7), (4.8) and (3.13) it follows that for each θ sufficiently close to θ_2 there exists a constant $c_1(\theta)$ such that as $r \to +\infty$,

$$G'(re^{i\theta}) = c_1(\theta)A(re^{i\theta})^{1/k} + \alpha(A'(re^{i\theta})/A(re^{i\theta})) + O(r^{-2}). \tag{4.26}$$

We take a positive δ so small that the interval $|\theta - \theta_2| < \delta$ lies in $|\theta - \theta_0| < \varepsilon/8$ and such that $\delta(P, \theta) \geqq \delta_0 > 0$, say, on this smaller interval. Now $G'A^{-1/k}$ has finite order of growth as $z \to \infty$ in the sectorial set A_3 given by

$$|\arg z - \theta_2| < \delta, \quad |z| \geqq \rho_3,$$

and from (3.2), (3.1) and (4.26) we see that for each θ in $|\theta - \theta_2| < \delta$,

$$G'(re^{i\theta})A(re^{i\theta})^{-1/k} \to c_1(\theta)$$

as $r \to +\infty$, so that by the Phragmén–Lindelöf principle $c_1(\theta) = c_1$ is independent of θ in this interval. Now $c_1 \neq 0$ for otherwise we should have, for each $j = 1, \dots, k$, and for some $b_1 > 0$,

$$|G^{(j)}(z)| = O(|z|^{b_1})$$

for z lying in $|\arg z - \theta_2| \leqq \delta/2$, $|z| \geqq \rho_4$, using (4.26), (4.10), the Phragmén–Lindelöf principle and Cauchy's estimate. Substituting in (4.22) and using (4.12) we would obtain

$$\log^+ |A(re^{i\theta_2})| = O(\log r)$$

as $r \to +\infty$, contradicting (3.2). We deduce from (4.26) that in the sectorial set A_2 given by $|\arg z - \theta_2| \leqq \delta/2$, $|z| \geqq \rho_5$ we have

$$G'(z) = c_1 A(re^{i\theta})^{1/k}(1 + \phi(z)) \tag{4.27}$$

where $|\phi(z)| \leqq |z|^{-2}$, and, for some $M_2 > 0$, using (4.10),

$$|G^{(j)}(re^{i\theta_2})| \leqq |z|^{M_2} |G'(re^{i\theta_2})| \tag{4.28}$$

for each $j = 2, \dots, k$, and for all $r \geqq \rho_6$, say.

We proceed to obtain (4.20) using the estimates (4.27) and (4.28). Now, if $r \geqq \rho_6$ and $z = re^{i\theta_2}$, we have, using (4.12), (4.22)–(4.25) and (4.27) and (4.28),

$$k(G')^{k-1}(W'/W) + (G')^k + c_k(G')^{k-2}G'' + A = H_1 \tag{4.29}$$

where $c_k = k(k-1)/2$ and

$$|H_1(z)| \leqq |z|^{M_3}(|G'(z)|)^{k-2} \tag{4.30}$$

for some $M_3 > 0$. Now, from (4.26), we may write

$$(G')^k = c_1^k A + kc_1^{k-1}A^{(k-1)/k}(\alpha(A'/A) + O(r^{-2})) + H_2 \tag{4.31}$$

and

$$(G')^{k-2} = c_1^{k-2}A^{(k-2)/k} + H_3, \tag{4.32}$$

where for some $M_4 > 0$, and for all $z = re^{i\theta_2}$ with $r \geqq \rho_7$, say

$$|H_2(z)| \leqq |z|^{M_4}(|G'(z)|)^{k-2} \tag{4.33}$$

and

$$|H_3(z)| \leqq |z|^{M_4}(|G'(z)|)^{k-3} \tag{4.34}$$

unless $k = 2$, in which case $H_3 \equiv 0$.

We need a more precise estimate for G'' than (4.28). Now (4.26) and the Phragmén–Lindelöf principle imply that $G' - c_1 A^{1/k}$ is analytic and bounded by a power of $|z|$ in a sector about the ray $re^{i\theta_2}$, so that Cauchy's formula for derivatives yields

$$G'' = (c_1/k)A^{(1-k)/k}A' + H_4 \tag{4.35}$$

for $z = re^{i\theta_2}$ and $r \geqq \rho_8$, say, where

$$|H_4(z)| \leqq r^{M_5} \tag{4.36}$$

for some $M_5 > 0$. Substituting (4.31), (4.32) and (4.35) in (4.29) we obtain

$$\begin{aligned} &k(G')^{k-1}(W'/W) + (c_1^k + 1)A + kc_1^{k-1}A^{(k-1)/k}(\alpha(A'/A) + O(r^{-2})) \\ &\quad + c_k(c_1^{k-2}A^{(k-2)/k} + H_3)\left(\frac{c_1}{k}A^{(1-k)/k}A' + H_4\right) \\ &= H_1 - H_2. \end{aligned}$$

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Using (4.27), (4.30), (4.33), (4.34) and (4.36) we obtain, noting that $c_k = k(k-1)/2$,

$$k(G')^{k-1}(W'/W) + (c_1^k + 1)A + kc_1^{k-1}A^{(k-1)/k}(\alpha(A'/A) + O(r^{-2})) + \frac{k-1}{2}c_1^{k-1}A^{(k-1)/k}(A'/A) = H_5$$

where

$$|H_5(re^{i\theta_2})| \leqq r^{M_6}(|G'(re^{i\theta_2})|)^{k-2} \tag{4.37}$$

for some $M_6 > 0$ and all $r \geqq \rho_9$, say.

But $\alpha = (1-k)/2k$ and we therefore have

$$k(G')^{k-1}(W'/W) + (c_1^k + 1)A + A^{(k-1)/k}O(r^{-2}) = H_5. \tag{4.38}$$

Now, (3.2), (4.27), (4.37) and (4.12) imply that $c_1^k + 1 = 0$. The same estimates now imply that $(W'/W) = O(r^{-2})$ in (4.38) and (4.20) is proved.

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Department of Mathematics
University of Illinois
1409 West Green St.
Urbana 61801, USA

Department of Pure Mathematics
University of St. Andrews
North Haugh
St. Andrews KY16 9SS
Scotland