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COMPLETE CONTINUITY PROPERTIES OF BANACH SPACES ASSOCIATED WITH SUBSETS OF A DISCRETE ABELIAN GROUP

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Abstract. We introduce and study the type I-, II-, and III-Λ-complete continuity property of Banach spaces, where Λ is a subset of the dual group of a compact metrizable abelian group G.

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1. Preliminaries. Throughout this paper G will denote a compact metrizable abelian group. We denote by $\mathcal{B}(G)$ the σ-field of the Borel subsets of G, and by λ a normalized Haar measure on G. The dual group of G will be denoted by $\widehat{G}$.

If X is a complex Banach space, then $B(X)$ will stand for the unit ball of the Banach space X, and $L^1(G, X)$ (resp. $L^\infty(G, X)$) denotes the Banach space of (all classes of) λ-Bochner integrable functions (resp. (all classes of) X-valued λ-measurable functions that are essentially bounded) on G with values in X. The space of all continuous X-valued functions on G will be denoted by $C(G, X)$. If $X = \mathbb{C}$, then $L^1(G, X)$, $L^\infty(G, X)$ and $C(G, X)$ will be denoted by $L^1(G)$, $L^\infty(G)$ and $C(G)$ respectively.

The symbol $\mathcal{M}^1(G, X)$ will be used to denote the space of countably additive X-valued measures that are of *bounded variation*, so $\mu \in \mathcal{M}^1(G, X)$ if the quantity

$$\|\mu\|_1 = \sup ||\sum_{A\in\pi} \frac{\mu(A)}{\lambda(A)}\chi_A||_1$$

is finite, where the supremum is taken over all finite partitions π consisting of Borel subsets of G. Here for each Borel subset A of G, χ_A denotes the characteristic function of A. An X-valued measure μ on G such that for every Borel subset A of G, $||\mu(A)||_X \leq c\lambda(A)$, for some positive contant c, is said to be of *bounded average range*. The infinimum of such constant c defines a norm on the space of vector measures and is denoted by $||\mu||_\infty$. The Banach space of all X-valued countably additive measures on G with $\|\mu\|_\infty < \infty$ is denoted by $\mathcal{M}^\infty(G, X)$.

If X and Y are Banach spaces, then $\mathcal{L}(X, Y)$ will denote the Banach space of all bounded linear operators from X to Y.

A bounded linear operator $T : X \longrightarrow Y$ is said to be *completely continuous* (also called Dunford-Pettis) if it maps weakly convergent sequences in the Banach space X into norm convergent sequences in the Banach space Y. Recall that a Banach space X has the *complete continuity property* (CCP) if every bounded linear operator $T : L^1(G) \to X$ is completely continuous.

2. The Λ-complete continuity property types. Let Λ be a subset of the dual group of G, and $\Lambda' = \{\gamma \in \widehat{G}, \overline{\gamma} \notin \Lambda\}$, where $\overline{\gamma}$ is the conjugate character of γ. For $\gamma \in \widehat{G}$, $f \in L^1(G, X)$, the Fourier coefficient of f at γ is defined by

$$\hat{f}(\gamma) = \int_G f(t)\bar{\gamma}(t)d\lambda(t).$$

More generally, if $\mu \in \mathcal{M}^1(G, X)$, the Fourier coefficient of μ at γ is defined by

$$\widehat{\mu}(\gamma) = \int_G \overline{\gamma}(t)d\mu(t).$$

In what follows we shall use the following:

$$\begin{aligned}
L^1_\Lambda(G, X) &= \{f \in L^1(G, X) : \widehat{f}(\gamma) = 0 \text{ for } \gamma \notin \Lambda\} \\
\mathcal{M}^1_\Lambda(G, X) &= \{\mu \in \mathcal{M}^1(G, X) : \widehat{\mu}(\gamma) = 0 \text{ for } \gamma \notin \Lambda\} \\
\mathcal{M}^1_{\Lambda ac}(G, X) &= \{\mu \in \mathcal{M}^1(G, X) : \mu \text{ is } \lambda\text{−continuous and } \widehat{\mu}(\gamma) = 0 \text{ for } \gamma \notin \Lambda\} \\
\mathcal{C}_\Lambda(G, X) &= \{f \in C(G, X) : \widehat{f}(\gamma) = 0 \text{ for } \gamma \notin \Lambda\}.
\end{aligned}$$

Each element of $L^1_\Lambda(G, X)$ (resp. $\mathcal{M}^1_\Lambda(G, X)$) will be termed as *$\Lambda$-function* (resp. Λ-measure). For the particular case where the Banach space $X = \mathbb{C}$, $L^1_\Lambda(G, \mathbb{C})$, $\mathcal{M}^1_\Lambda(G, \mathbb{C})$, and $\mathcal{C}_\Lambda(G, \mathbb{C})$ will be simply denoted by $L^1_\Lambda(G)$, $\mathcal{M}^1_\Lambda(G)$, and $\mathcal{C}_\Lambda(G)$ respectively.

In what follows we shall introduce types of complete continuity property associated to a subset Λ of the dual group $\widehat{G}$. These properties can be seen as the complete continuity counterpart of the types of Radon-Nikodým properties introduced by G. A. Edgar in [**6**], and P. Dowling in [**4**]. We recall that a Banach space X is said to have type *I-Λ-Radon-Nikodým property* (I-Λ-RNP), (resp. *II-Λ-Radon-Nikodým property* (II-Λ-RNP)) if every X-valued Λ-measure of bounded average range (resp.; of bounded variation) is differentiable (i.e. $\mathcal{M}^\infty_\Lambda(G, X) = L^\infty_\Lambda(G, X)$ (resp.; $\mathcal{M}^1_{\Lambda ac}(G, X) = L^1_\Lambda(G, X)$)) [**4**]. An element μ of $\mathcal{M}^1(G, X)$ is said to have a *relatively compact range* if the set $\{\mu(A) : A \in \mathcal{B}(G)\}$ is relatively compact in X.

Definition 1. Let Λ be a subset of the dual group of a compact metrizable abelian group G. A Banach space X is said to have type I-*Λ-complete continuity property* (I-Λ-CCP) if every X-valued Λ-measure of bounded average range has a relatively compact range.

Definition 2. A Banach space is said to have type II-*Λ-complete continuity property* (II-Λ-CCP) if every X-valued λ-continuous Λ-measure of bounded variation has relatively compact range.

It is immediate that the type I-Λ-RNP (resp; II-Λ-RNP) implies the type I-Λ-CCP (resp; II-Λ-CCP). Moreover, since every element of $\mathcal{M}^\infty_\Lambda(G, X)$ is in particular an element of $\mathcal{M}^1_{\Lambda ac}(G, X)$, one easily notices that type II-Λ-CCP implies type I-Λ-CCP.

Every member $\mu \in \mathcal{M}_\Lambda^\infty(G, X)$ naturally defines a bounded linear operator $T : L^1(G) \longrightarrow X$ by $T(f) = \int_G f d\mu$, for all $f \in L^1(G)$. A simple computation shows that $T(\overline{\gamma}) = \widehat{\mu}(\gamma) = 0$ for all $\gamma \notin \Lambda$. Bounded linear operators from $L^1(G)$ into a Banach space X with the property $T(\overline{\gamma}) = 0$ for $\gamma \notin \Lambda$ will be called Λ-*operators*. Conversely, to a Λ-operator T from $L^1(G)$ into a Banach space X one can associate an element μ of $M_\Lambda^\infty(G, X)$ by $\mu(A) = T(\chi_A)$ for every $A \in \mathcal{B}(G)$. This leads us to the following:

THEOREM 2.1. *Let Λ be a subset of the dual group of a compact metrizable abelian group G. A Banach space X has type* I-Λ-CCP *if and only if every Λ-operator $T : L^1(G) \longrightarrow X$ is a completely continuous operator.*

One notices that for $\Lambda = \widehat{G}$, the I-Λ-CCP type and the II-Λ-CCP coincide with the complete continuity property. Also if $\Lambda_1 \subset \Lambda_2$ then type I-Λ_2-CCP (resp; II-Λ_2-CCP) implies type I-Λ_1-CCP (resp; II-Λ_2-CCP). In particular:

REMARK 2.2. If a Banach space X has the complete continuity property then it has the type I-Λ-CCP and II-Λ-CCP for any $\Lambda \subset \widehat{G}$.

It is known that the Banach space $L^1(G)$ fails the complete continuity property; however we will see that $L^1(G)$ has I-Λ-CCP for some $\Lambda \subset \widehat{G}$. The first example of a Banach space failing the I-Λ-CCP is provided by:

PROPOSITION 2.3. *Let Λ be an infinite subset of the dual group of a compact metrizable abelian group G. The sequence space c_0 fails* I-Λ-CCP.

Proof. To see this, let $(\gamma_n)_{n\in\mathbb{N}}$ be an enumeration of the elements of Λ. Define an operator $T : L^1(G) \longrightarrow c_0$ by

$$Tf = \left(\int_G f(t)\gamma_n(t) d\lambda(t) \right)_{n\in\mathbb{N}}$$

for all $f \in L^1(G)$. Then T is a bounded linear operator with $T(\overline{\gamma}) = 0$ for $\gamma \notin (\gamma_n)_{n\in\mathbb{N}}$. Since for every function $f \in L^1(G)$, $(\widehat{f}(\gamma) = \int_G f(t)\overline{\gamma}(t) d\lambda(t))_{\gamma\in\widehat{G}} \in c_0(\widehat{G})$ (see for example [**13**]), it is clear that the sequence $(\overline{\gamma}_n)_{n\in\mathbb{N}}$ is weakly null; however $\|T(\overline{\gamma}_n)\|_{c_0} = 1$ for $n = 1, 2, \ldots$. Thus the operator T is a Λ-operator which is not completely continuous. □

It is apparent that if a Banach space X has I-Λ-CCP (resp. II-Λ-CCP) type then so does each one of its subspaces. On the other hand, since the group G is compact metrizable, $\mathcal{B}(G)$ is countably generated, one sees that the I-Λ-CCP (resp. II-Λ-CCP) type is separably determined, i.e.:

THEOREM 2.4. *Let Λ be a subset of the dual group of a compact metrizable abelian group G. A Banach space X has type* I-Λ-CCP *(resp.* II-Λ-CCP*) if and only if so has each one of its separable subspaces.*

Also recall that a subset Λ of $\widehat{G}$ is said to be a *Riesz set* if $\mathcal{M}_\Lambda^1(G) = L_\Lambda^1(G)$ (cf. [**9**]), and Λ is a *Sidon set* if $C_\Lambda(G) = \ell^1(\Lambda)$. It can be deduced from [**4**] and [**11**] that

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types I-Λ-RNP and II-Λ-RNP are the same for Banach lattices provided Λ is Riesz, and they are equivalent to the non containment of isomorphic copies of c_0. In view of Proposition 2.3, we also have the following results.

THEOREM 2.5. *Let Λ be a Riesz subset of $\widehat{G}$. Then the following properties are equivalent for a Banach lattice X:*

(a) *X has type* II-Λ-RNP;
(b) *X has type* I-Λ-RNP;
(c) *X has type* II-Λ-CCP;
(d) *X has type* I-Λ-CCP;
(e) *X contains no subspaces isomorphic to c_0.*

We also have the following result which it can be deduced from a result of [**5**].

THEOREM 2.6. *Let Λ be a Sidon set of $\hat{G}$. The following properties of an arbitrarily Banach space X are equivalent:*

(a) *X has type* II-Λ-RNP;
(b) *X has type* I-Λ-RNP;
(c) *X has type* II-Λ-CCP;
(d) *X has type* I-Λ-CCP;
(e) *X contains no subspace isomorphic to c_0.*

3. Characterizations of the Λ-CCP types. For a compact metrizable abelian group G, a sequence $(i_n)_{n\in\mathbb{N}}$ of measurable functions $i_n : G \longrightarrow \mathbb{R}$ is called a *good approximate identity* on G if

(1) $i_n \geq 0$ *for all* $n \in \mathbb{N}$,
(2) $\int_G i_n(t)d\lambda(t) = 1$ *for all* $n \in \mathbb{N}$,
(3) $\sum_{\gamma\in\widehat{G}} \widehat{i_n}(\gamma) < \infty$ *for all* $n \in \mathbb{N}$, *and*
(4) $\lim_{n\to\infty} \int_U i_n(t)d\lambda(t) = 1$ *for every neighbourhood U of the identity element of G.*

We recall that for any compact metrizable abelian group G, a good approximate identity always exists on G (see for example [**6**], [**8**] or [**13**]).

For a Banach space X, and for an element f of $L^1(G, X)$ the *Pettis-norm* of f is given by

$$|||f||| = \sup\left\{\int_G |x^*f|d\lambda : x^* \in X^*, \|x^*\| \leq 1\right\}.$$

We say that a sequence (f_n) of elements of $L^1(G, X)$ is *Pettis-Cauchy* if it is a Cauchy sequence for the Pettis-norm.

In what follows we shall give characterizations of the I-Λ-CCP and II-Λ-CCP properties. Our results should be compared to the following theorems of [**4**] and [**6**] which characterize the different types of Λ-RNP spaces:

THEOREM 3.1. (Edgar). *Let G be a compact metrizable abelian group, let $\Lambda \subset \widehat{G}$ and let $(i_n)_{n\in\mathbb{N}}$ be a good approximate identity on G. Then the following properties are equivalent for a Banach space X:*

(a) *X has* I-Λ-RNP;

(b) *if* $(a_\gamma)_{\gamma\in\Lambda} \subset X$ *and* $(f_n = \sum_{\gamma\in\Lambda} \widehat{i_n}(\gamma)a_\gamma\gamma)_{n\in\mathbb{N}}$ *is bounded in* $L^\infty_\Lambda(G, X)$*, then the sequence* $(f_n)_{n\in\mathbb{N}}$ *converges in* $L^1(G, X)$*-norm.*

Theorem 3.2. (Dowling). *Let G be a compact metrizable abelian group, let* Λ *be a Riesz subset of* $\widehat{G}$ *and let* $(i_n)_{n\in\mathbb{N}}$ *be a good approximate identity on G. Then the following are equivalent for a Banach space X:*

(a) *X has* II-Λ-RNP;

(b) *if* $(a_\gamma)_{\gamma\in\Lambda} \subset X$ *and* $(f_n = \sum_{\gamma\in\Lambda} \widehat{i_n}(\gamma)a_\gamma\gamma)_{n\in\mathbb{N}}$ *is bounded in* $L^1_\Lambda(G, X)$*, then the sequence* (f_n) *converges in* $L^1(G, X)$*-norm.*

Theorem 3.3. *Let G be a compact metrizable abelian group, let* $\Lambda \subset \widehat{G}$ *and let* $(i_n)_{n\in\mathbb{N}}$ *be a good approximate identity on G. Then the following properties are equivalent for a Banach space X:*

(a) *X has* I-Λ-CCP;

(b) *if* $(a_\gamma)_{\gamma\in\Lambda} \subset X$ *and* $(f_n = \sum_{\gamma\in\Lambda} \widehat{i_n}(\gamma)a_\gamma\gamma)_{n\in\mathbb{N}}$ *is bounded in* $L^\infty(G, X)$*, then the sequence* $(f_n)_{n\in\mathbb{N}}$ *is Pettis-Cauchy.*

Proof. (a) $\Rightarrow$ (b). Let $(a_\gamma)_{\gamma\in\Lambda} \subset X$ and suppose the sequence $(f_n = \sum_{\gamma\in\Lambda} \widehat{i_n}(\gamma)a_\gamma\gamma)_{n\in\mathbb{N}}$ is bounded in $L^\infty(G, X)$. We want to show that

$$\lim_{n,m} |||f_n - f_m||| = \lim_{n,m} \sup\{\int_G |x^*f_n - x^*f_m|d\lambda, x^* \in X^*, \|x^*\| \le 1\} = 0.$$

To this end, we define, for each $n \in \mathbb{N}$, the operator $T_n : L^1(G) \longrightarrow X$ by $T_n(f) = \int_G ff_n d\lambda$, for all $f \in L^1(G)$. Then $\|T_n\| = \|f_n\|_{L^\infty(G,X)}$, for all $n \in \mathbb{N}$. Thus $\sup_n \|T_n\| < \infty$. Let (T_{n_α}) be a subnet of (T_n) that converges to an operator $T : L^1(G) \longrightarrow X^{**}$ in the weak* operator topology. In particular, for each $\gamma \in \widehat{G}$ and each $x^* \in B(X^*)$,

$$< T(\overline{\gamma}), x^* >= \lim_{n_\alpha} \int_G \overline{\gamma}(s)x^*f_{n_\alpha}(s)d\lambda(s) = \lim_{n_\alpha} x^*\widehat{f_{n_\alpha}}(\gamma).$$

Thus $T(\overline{\gamma}) = a_\gamma$ if $\gamma \in \Lambda$ and $T(\overline{\gamma}) = 0$ if $\gamma \notin \Lambda$. Since the characters form a total subset of $L^1(G)$, it follows that T is a bounded linear Λ-operator from $L^1(G)$ into X. Hence by our assumption, it is a completely continuous operator. Since the unit ball of $L^\infty(G)$ is relatively weakly compact in $L^1(G)$, the operator $S = T|_{L^\infty(G)}$ is compact.

For every function $g \in L^\infty(G)$, and for each $x^* \in X^*$, it is clear that

$$\begin{aligned} < S^*x^*, g > &=< x^*, Tg > \\ &= \lim_{n_\alpha} x^* \int_G f_{n_\alpha} g d\lambda \\ &= \lim_{n_\alpha} \int_G x^*f_{n_\alpha} g d\lambda. \end{aligned} \tag{3.1}$$

Equations 3.1 show that $S^*x^* = \text{weak-}\lim x^*f_{n_\alpha}$, and hence it shows that S^* takes its values in $L^1(G)$.

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Now let $R_n : L^1(G) \longrightarrow L^1(G)$ denote the convolution operator defined by $R_n f = i_n * f$ for all $f \in L^1(G)$, for each $n \in \mathbb{N}$. Since for each $f \in L^1(G)$, the sequence $(R_n(f))$ converges to $f \in L^1(G)$ (see for example [**13**]), the sequence of operators (R_n) converges uniformly on compact subsets of $L^1(G)$. For $x^* \in X^*$, $\|x^*\| \leq 1$, one has

$$\begin{aligned} R_n S^* x^* &= \sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma) \widehat{S^* x^*}(\gamma)\gamma \\ &= \sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma) x^* S(\overline{\gamma})\gamma \\ &= \sum_{\gamma \in \Lambda} \widehat{i_n}(\gamma) x^* a_\gamma \gamma = x^* f_n. \end{aligned}$$

Therefore,

$$\lim_{n,m} |||f_n - f_m||| = \lim_{n,m} \sup\{\|(R_n - R_m)S^* x^*\| : x^* \in X^*, \|x^*\| \leq 1\}.$$

The compactness of S now implies that this limit is 0 as desired.

(b) $\Rightarrow$ (a) Let $T : L^1(G) \longrightarrow X$ be a Λ-operator. Consider the sequence of functions $(f_n = \sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma) T(\overline{\gamma})\gamma)_{n \geq 1}$. One has, for each $t \in G$, and for $n \in \mathbb{N}$

$$f_n(t) = \sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma) T(\overline{\gamma})\gamma(t) = T\left(\sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma)\gamma(t - .) \right) = T(i_n(t - .)).$$

Then $\|f_n\|_{L^\infty(G,X)} \leq \|T\|$, for all $n \in \mathbb{N}$. Hence (f_n) is Pettis-Cauchy by our assumption.

Conversely, let $T_n \in \mathfrak{L}(L^1(G), X)$ be the bounded linear operator defined by $T_n f = \int_G f f_n d\lambda$, for every $f \in L^1(G)$, and denote by j_∞ the natural injection of $L^\infty(G)$ into $L^1(G)$. Consider the composition operator $S_n = T_n j_\infty$, for each $n \in \mathbb{N}$. Since T_n is completely continuous and the unit ball of $L^\infty(G)$ is relatively weakly compact in $L^1(G)$, the operator S_n is compact. For $x^* \in X^*$, and for every $f \in L^\infty(G)$,

$$S_n^* x^*(f) = x^* S_n(f) = x^* T_n j_\infty(f) = x^* \int_G f f_n d\lambda = \int_G f x^* f_n d\lambda.$$

Thus $S_n^* x^* = x^* f_n$, for each $n \in \mathbb{N}$. Hence, for $n, m \in \mathbb{N}$,

$$\begin{aligned} \|S_n - S_m\| &= \|S_n^* - S_m^*\| \\ &= \sup\{\|(S_n^* - S_m^*)(x^*)\|_1; x^* \in X^*, \|x^*\| \leq 1\} \\ &= \sup\{\|x^* f_n - x^* f_m\|_1; x^* \in X^*, \|x^*\| \leq 1\} \\ &= |||f_n - f_m|||. \end{aligned}$$

Thus the sequence $(S_n)_{n \geq 1}$ is Cauchy in $\mathfrak{L}(L^\infty(G), X)$, and hence it converges to an operator $S : L^\infty(G) \longrightarrow X$. Since each operator S_n is compact for each $n = 1, 2, \ldots$, so is the operator S.

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On the other hand, for $f \in L^\infty(G)$, one has

$$
\begin{aligned}
S_n f = T_n j_\infty f &= \int_G f \sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma) T(\overline{\gamma}) \gamma d\lambda \\
&= \sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma) \widehat{f}(\overline{\gamma}) T \overline{\gamma} \\
&= T\Big(\sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma) \widehat{f}(\overline{\gamma}) \overline{\gamma} \Big) \\
&= T(i'_n * f)
\end{aligned}
$$

where $i'_n(t) = i_n(-t)$, for $t \in G$ and for all $n \in \mathbb{N}$. Thus

$$\|(T - T_n)(f)\| = \|T(f - i'_n * f)\| \leq \|T\| \|f - i'_n * f\|_{L^1(G)},$$

for any positive integer n. It follows that the sequence of operators $(T_n)_{n>1}$ converges to T on $L^\infty(G)$, in the strong operator topology. Consequently, we have $T \equiv S$ on $L^\infty(G)$. Therefore, we can conclude that the restriction of the operator T on $L^\infty(G)$ is compact. This shows that the operator T is indeed completely continuous. □

The next theorem gives a characterization of the type II-Λ-CCP. This result can naturally be compared to the characterization theorem of the type II-Λ-RNP as given in [**4**] (see Theorem 3.2 above).

THEOREM 3.4. *Let G be a compact metrizable abelian group, let Λ be a Riesz subset of $\widehat{G}$ and let $(i_n)_{n \in \mathbb{N}}$ be a good approximate identity on G. Then the following are equivalent for a Banach space X:*

(a) *X has* II-Λ-CCP;

(b) *if $(a_\gamma)_{\gamma \in \Lambda} \subset X$ and $(f_n = \sum_{\gamma \in \Lambda} \widehat{i_n}(\gamma) a_\gamma \gamma)_{n \in \mathbb{N}}$ is bounded in $L^1(G, X)$, then the sequence $(f_n)_{n \in \mathbb{N}}$ is Pettis-Cauchy.*

Proof. (a) ⇒ (b) Let $(a_\gamma)_{\gamma \in \Lambda} \subset X$ and assume that $(f_n = \sum_{\gamma \in \Lambda} \widehat{i_n}(\gamma) a_\gamma \gamma)_{n \in \mathbb{N}}$ is bounded in $L^1(G, X)$. For each $n \geq 1$, let $\mu_n \in \mathcal{M}^1(G, X)$ be defined by

$$\mu_n(A) = \int_G \chi_A(t) f_n(t) d\lambda(t),$$

for each $A \in \mathcal{B}(A)$. Then $||\mu_n||_1 = ||f_n||_1$, for each $n \geq 1$.

Consider the space $\mathcal{M}^1(G, X^{**})$. It is well known [**2**], that $\mathcal{M}^1(G, X^{**})$ is isometrically isomorphic to the dual space $\mathcal{C}(G, X^*)^*$. Since by our assumption the sequence (μ_n) is bounded in $\mathcal{M}^1(G, X)$, it is also bounded in $\mathcal{M}^1(G, X^{**})$. Let (μ_{n_α}) be a subnet of (μ_n) that converges to an element ν in $\mathcal{M}^1(G, X^{**})$ in the weak* topology. Then in particular for each character $\gamma \in \widehat{G}$, and for each element $x^* \in X^*$, we have

$$\widehat{\nu}(\gamma) x^* = \lim_{n_\alpha} \int_G \overline{\gamma} x^* f_{n_\alpha} d\lambda = x^*(\lim_{n_\alpha} \widehat{f}_{n_\alpha}(\gamma)).$$

Thus

$$\widehat{\nu}(\gamma) = \begin{cases} a_\gamma, & \text{if} \quad \gamma \in \Lambda, \quad \text{and} \\ 0, & \text{if} \quad \gamma \notin \Lambda. \end{cases}$$

Since the characters form a total subset of $\mathcal{C}(G)$, it follows that the mapping $x^* \longrightarrow \nu(\cdot)x^*$ of X^* into $\mathcal{C}(G)^*$ is weak* to weak* continuous. Therefore, we can define a bounded linear operator $T : \mathcal{C}(G) \longrightarrow X$ by $x^*T(f) = \int_G f d(x^*\nu)$, for each $f \in \mathcal{C}(G)$ and for each $x^* \in X^*$ [**2**, Theorem 1]. Since by our assumption X has II-Λ-CCP, X contains no isomorphic copy of c_0. Thus the operator T is weakly compact and consequently the measure ν takes its values in X [**2**, p. 238]. Since $\hat{\nu}(\gamma) = 0$ if $\gamma \notin \Lambda$, and Λ is a Riesz set, then ν is absolutely continuous with respect to Haar measure on G. Thus, by our assumption, the measure ν has relatively compact range and hence the operator T is compact.

On the other hand, it is easily seen that $\lim_n x^*f_n$ exists in $L^1(G)$ and that

$$< \lim_n x^*f_n, f >=< x^*, Tf >=< T^*x^*, f >,$$

for each $x^* \in X^*$ and for each $f \in \mathcal{C}(G)$. That is, the adjoint operator of the operator T is given by $T^*x^* = \lim_n x^*f_n$, for each $x^* \in X^*$, and thus $T^*x^* \in L^1(G)$. From here we just repeat the last part of the proof of the implication (a) $\Rightarrow$ (b) of the Theorem 3.3. This establishes (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a) Let $\mu \in \mathcal{M}^1_{\Lambda ac}(G, X)$. Set $\widehat{\mu}(\gamma) = a_\gamma$, $\gamma \in \widehat{G}$ and let $f_n = \sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma) a_\gamma \gamma$. Thus for $n \in \mathbb{N}$, and for $t \in G$,

$$\begin{aligned} i_n * \mu(t) &= \int_G i_n(t-s) d\mu(s) \\ &= \int_G \sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma)\gamma(t)\overline{\gamma}(s) d\mu(s) \\ &= \sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma)\widehat{\mu}(\gamma)\gamma(t) \\ &= \sum_{\gamma \in \widehat{G}} \widehat{i_n}(\gamma) a_\gamma \gamma(t) = f_n(t). \end{aligned}$$

Therefore $\|f_n\|_{L^1(G,X)} = \|i_n * \mu\|_{L^1(G,X)} \leq \|\mu\|_1$, for all $n \in \mathbb{N}$. Thus the sequence (f_n) is Pettis-Cauchy.

For each $n \in \mathbb{N}$, let $\mu_n = f_n \cdot \lambda$. For $n, m \in \mathbb{N}$, and $E \in \mathcal{B}(G)$,

$$\|\mu_n(E) - \mu_m(E)\| \leq \|\|f_n - f_m\|\|.$$

Thus there exists a set function $\nu : \mathcal{B}(G) \longrightarrow X$ such that $\nu(E) = \lim_n \mu_n(E)$ uniformly on $\mathcal{B}(G)$. An appeal to Vitali-Hahn-Saks' Theorem (cf. [**2**]), shows that ν is λ-continuous.

Now since by construction the μ_n have relatively compact ranges, we claim that ν also has the same property. Indeed, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ large enough such that

$$\|\nu(E) - \mu_{n_\epsilon}(E)\| < \epsilon/3, \text{ for } E \in \mathcal{B}(G).$$

Thus it follows that

$$\{\nu(E); E \in \mathcal{B}(G)\} \subset \{\mu_{n_\epsilon}(E); E \in \mathcal{B}(G)\} + \epsilon B(X),$$

where $B(X)$ denotes the unit ball of X. As mentioned above, we have that the set $\{\mu_{n_\epsilon}(E); E \in \mathcal{B}(G)\}$ is relatively compact for each $\epsilon > 0$, and so is $\{\nu(E); E \in \mathcal{B}(G)\}$ by a standard argument. This proves our claim. Finally, for $\gamma \in \widehat{G}$, we have

$$\widehat{\nu}(\gamma) = \lim_n \int_G \overline{\gamma} f_n d\lambda = \lim_n \widehat{f_n}(\gamma) = a_\gamma = \widehat{\mu}(\gamma).$$

We conclude that $\mu = \nu$ and thus μ has relatively compact range. □

Remark 3.5. The hypothesis that Λ is a Riesz set was only needed in the implication (a)⇒(b).

Finally, let us introduce the following type of Λ-CCP which has very interesting properties as did its Radon-Nikodým counterpart [**4**].

Defintion 3. Let Λ be a subset of the dual group of a compact metrizable abelian group G. A Banach space X is said to have type III-Λ-*complete continuity property* (III-Λ-CCP), if every absolutely summing operator [**3**] $T : \mathcal{C}(G) \longrightarrow X$ with $T \equiv 0$ on $\mathcal{C}_{\Lambda'}(G)$ is compact.

The following two interesting results were shown in [**4**].

Proposition 3.6 (Dowling). *Let Λ be a Riesz subset of the dual group of a compact metrizable abelian group G. Then a Banach space X has type* II-Λ-RNP *if and only if it has* III-Λ-RNP.

Proposition 3.7 (Dowling). *Let Λ be a non Riesz subset of the dual group $\widehat{G}$ of a compact metrizable abelian group G. Then a Banach space X has type* III-Λ-RNP *if and only if it has the Radon-Nikodým property.*

As it was shown in the above results, the next two propositions show that the type III-Λ-CCP is not an isolated property. It coincides with either of type II-Λ-CCP or CCP depending on whether or not Λ is a Riesz set.

First, it is known and easy to see that if Λ is Riesz then $\mathcal{M}^1_\Lambda(G, X) = \mathcal{M}^1_{\Lambda ac}(G, X)$, for any Banach space X. Consequently, we obtain the following result.

Proposition 3.8. *Let Λ be a Riesz subset of the dual group of a compact metrizable abelian group G. Then a Banach space X has type* II-Λ-CCP *if and only if it has* III-Λ-CCP.

Proof. First note that type III-Λ-CCP implies type II-Λ-CCP for any subset $\Lambda \subset \hat{G}$. To see this, assume that the Banach space X has type III-Λ-CCP and let μ be in $\mathcal{M}^1_{\Lambda ac}(G, X)$. A simple computation shows that the integration operator

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$T : \mathcal{C}(G) \longrightarrow X$ defined by $T(f) = \int_G f d\mu$, for all $f \in \mathcal{C}(G)$ is absolutely summing and $T(\gamma) = \int_G \gamma d\mu = \widehat{\mu}(\overline{\gamma}) = 0$ for every $\gamma \in \Lambda'$. Therefore T is compact. Since for each Borel subset A of G

$$\mu(A) = T^{**}(\chi_A),$$

where χ_A denotes the characteristic function of A. It follows that the measure μ has relatively compact range. Therefore X has type II-Λ-CCP.

For the converse, suppose the Banach space X has type II-Λ-CCP and let $T : \mathcal{C}(G) \longrightarrow X$ be an absolutely summing operator such that $T \equiv 0$ on $\mathcal{C}_{\Lambda'}(G)$. Let $\mathfrak{F} : \mathcal{B}(G) \to X^{**}$ be the vector measure representing the operator T, i.e. for each Borel subset A of G,

$$\mathfrak{F}(A) = T^{**}(\chi_A).$$

Since T is absolutely summing, it is in particular weakly compact and hence its representing measure $\mathfrak{F}$ takes its values in X. On the other hand, $\mathfrak{F}(\gamma) = T(\overline{\gamma})$ for all γ in $\widehat{G}$. It follows that $\mathfrak{F} \in \mathcal{M}^1_\Lambda(G, X)$. Now since Λ is a Riesz set, the measure $\mathfrak{F}$ is λ -continuous. Therefore the representing measure $\mathfrak{F}$ of the operator T has relatively compact range since X has type II-Λ-CCP. This shows that the operator T is compact (see [**2**, p. 161]). Thus X has type III-Λ-CCP. The proof is complete. □

On the other hand, for a non Riesz subset of $\widehat{G}$, we shall proceed as in [**4**] to show that the situation is completely different.

Proposition 3.9. *Let Λ be a non Riesz subset of the dual group $\widehat{G}$ of a compact metrizable abelian group G. Then a Banach space X has type* III-Λ-CCP *if and only if it has the complete continuity property.*

Proof. It is clear that a Banach space with CCP has type III-Λ-CCP. For the converse, suppose the Banach space X has III-Λ-CCP, where Λ is a non Riesz subset of $\widehat{G}$. Let $S : \mathcal{C}(G) \longrightarrow X$ be an absolutely summing operator. We want to show that S is compact. Let $q : \mathcal{C}(G) \longrightarrow \mathcal{C}(G)/\mathcal{C}_{\Lambda'}(G)$ be the natural quotient map. Since Λ is not a Riesz set, the dual space $(\mathcal{C}(G)/\mathcal{C}_{\Lambda'}(G))^* = \mathcal{M}^1_\Lambda(G)$ is not separable, and hence $q^*((\mathcal{C}(G)/\mathcal{C}_{\Lambda'}(G))^*)$ is not separable. Exactly as in the proof of [**4**, Theorem 11], by a result of H. P. Rosenthal [**12**], there exists a subspace Z of $\mathcal{C}(G)$ isometric to $\mathcal{C}(G)$ such that the restriction map $q|_Z : Z \longrightarrow q(Z)$ is an isomorphism. Thus we have the following diagram

$$\begin{array}{ccc}
\mathcal{C}(G) & \xrightarrow{S} & X \\
j \uparrow\downarrow j^{-1} & & \uparrow T \\
q(Z) & & \mathcal{C}(G) \\
i \searrow & & \swarrow q \\
& \mathcal{C}(G)/\mathcal{C}_{\Lambda'}(G) &
\end{array}$$

where j is an isomorphism, i is the inclusion map.

Let $\tilde{S} = Sj$. Then since S is absolutely summing, $\tilde{S}$ is Pietsch integral (see for example [**2**, p. 165]). Let $\tilde{T}$ be the Pietsch integral extension of $\tilde{S}$ to $\mathcal{C}(G)/\mathcal{C}_{\Lambda'}(G)$, and

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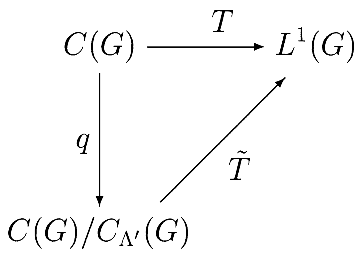
define $T = \tilde{T}q$. Then the operator T is Pietsch integral and thus it is absolutely summing. Also $T(f) = \tilde{T}(q(f)) = 0$ for every function $f \in \mathcal{C}_{\Lambda'}(G)$. Since the Banach space X has type III-Λ-CCP T is compact and so is $T_{|Z} = \tilde{T}q_{|Z} : Z \longrightarrow X$.

Now $\tilde{S} = \tilde{T}_{|q(Z)} = (\tilde{T}q_{|Z}) \circ (q_{|Z})^{-1} : q(Z) \longrightarrow X$. Thus the operator $\tilde{S}$, and consequently $S = \widetilde{\tilde{S}j}^{-1}$, is compact. The proof is complete. □

Let us finish this section with the following interesting result.

Theorem 3.10. *Let Λ be a subset of $\hat{G}$. The following properties are equivalent:*
(i) $\mathcal{M}^1_\Lambda(G)$ *has* CCP;
(ii) $\mathcal{M}^1_\Lambda(G)$ *has* RNP.

Proof. We need only show that (i)⇒(ii). Assume $\mathcal{M}^1_\Lambda(G)$ has CCP. We claim that this implies $L^1(G)$ has III-Λ-CCP. To see this, let $T : C(G) \to L^1(G)$ be a 1-summing operator such that $T_{|C_{\Lambda'}(G)} = 0$. Let $\tilde{T} : C(G)/C_{\Lambda'}(G) \to L^1(G)$ be such that the following diagram commutes.



It was pointed out in [**4**] that since T is Pietsch integral, then it follows from a result of Grothendick [**2**] that $\tilde{T}$ is also Pietsch integral. Hence $\tilde{T}^* : L^1(G)^* \to (C(G)/C_{\Lambda'}(G))^*$ is Pietsch integral. Since $(C(G)/C_{\Lambda'}(G)^*$ is isometric to $\mathcal{M}^1_\Lambda(G)$ and $\mathcal{M}^1_\Lambda(G)$ is assumed to have CCP, and since Pietsch integral operators factor through L^1 spaces, it follows that $\tilde{T}$ is compact, hence T is compact. This proves the claim. Moreover, if $L^1(G)$ has III-Λ-CCP, then it follows from Proposition 3.9 that Λ should be a Riesz set. This of course implies that $\mathcal{M}^1_\Lambda(G) = L^1_\Lambda(G)$ and thus $\mathcal{M}^1_\Lambda(G)$ has RNP since it is a separable dual Banach space [**2**]. □

4. G_δ-embedding and concluding remarks. In [**7**], N. Ghoussoub and H. P. Rosenthal proved the following:

Proposition 4.1. *Let T be a bounded linear operator from L^1 to a Banach space Y and let S be a G_δ-embedding of Y into a Banach space X. Then the operator T is completely continuous if and only if so is the operator ST.*

Recall that given two Banach spaces X and Y, an element $T \in \mathcal{L}(X, Y)$ is a G_δ-*embedding* if for any closed subset F of Y, $T(F)$ is a G_δ-subset of Y.

Proposition 4.1 establishes in particular that the CCP is stable under G_δ-embedding. In this section, we shall see that this result can also be used to prove the stability property of the types I-, II- and III-Λ-CCP under G_δ-embedding, where Λ is a subset of the dual group of a compact metrizable abelian group G.

The proof of the stability of type I-Λ-CCP under G_δ -embedding is immediate by Proposition 4.1.

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THEOREM 4.2. *Let Λ be a subset of the dual group of a compact metrizable abelian group G. Let X be a Banach space with type* I-Λ-CCP. *Then every Banach space that G_δ-embeds in X has type* I-Λ-CCP.

The fact that the II-Λ-CCP is also stable by G_δ-embedding is straight forward as shown in the following theorem.

THEOREM 4.3 *Let Λ be a subset of the dual group of a compact metrizable abelian group G. Let X be a Banach space with type* II-Λ-CCP. *Then every Banach space that G_δ-embeds in X has type* II-Λ-CCP.

Proof. Suppose that the Banach space Y G_δ-embeds in X. Let $S : Y \longrightarrow X$ denote the G_δ-embedding. Let $\mu \in M^1_{\Lambda ac}(G, Y)$. Define $\nu : \mathcal{B}(G) \longrightarrow X$ by $\nu(A) = S(\mu(A))$, for $A \in \mathcal{B}(G)$. It is easy to see that ν is a is λ-continuous Λ-measure of bounded variation. Therefore by our hypothesis, the measure ν has relatively compact range. On the other hand, by the Hahn decomposition theorem, there exists a sequence (E_n) of disjoint members of $\mathcal{B}(G)$ such that $G = \bigcup_{n=1}^{\infty} E_n$ and with the property that for each Borel subset A of G

$$(n-1)\lambda(A \cap E_n) \leq |\mu|(A \cap E_n) \leq n\lambda(A \cap E_n).$$

For each positive integer n, consider the increasing sequence of measurable subsets of G defined by $\widetilde{E}_n = \bigcup_{\nu=1}^{\nu=n} E_\nu$. It is clear that $G = \bigcup_{n=1}^{\infty} \widetilde{E}_n$, and thus

$$\lim_n \lambda(G \backslash \widetilde{E}_n) = 0. \tag{4.1}$$

For each $n \in \mathbb{N}$, let μ_n be the measure defined by $\mu_n(A) = \mu(A \cap \widetilde{E}_n)$, for every $A \in \mathcal{B}(G)$. Then by construction the measures μ_n are of bounded average range and as such define bounded linear operators $T_n : L^1(G) \longrightarrow Y$ by $T_n(f) = \int_G f d\mu_n$, for $f \in L^1(G)$. It is clear that for each $n \in \mathbb{N}$, and for every $A \in \mathcal{B}(G)$,

$$\nu(A \cap E_n) = ST_n(A).$$

Since the measure ν has relatively compact range, we see that the operator ST_n is completely continuous. Proposition 4.1 ensures that, for each $n \in \mathbb{N}$, the operator T_n is also completely continuous and therefore the measure μ_n has relatively compact range, for each $n \in \mathbb{N}$.

Now for each $n \in \mathbb{N}$, and for every $A \in \mathcal{B}(G)$, we have

$$\begin{aligned} ||\mu(A) - \mu_n(A)|| &= ||\mu(A) - \mu(A \cap \widetilde{E}_n)|| \\ &= ||\mu(A \cap (G \backslash \widetilde{E}_n))|| \\ &\leq ||\mu(G \backslash \widetilde{E}_n)||. \end{aligned} \tag{4.2}$$

It follows from (4.1) and (4.2) that $\lim_n \mu_n = \mu$ uniformly on $\mathcal{B}(G)$. Hence for every $\epsilon > 0$, there exists n_ϵ large enough so that

$$\{\mu(A) : A \in \mathcal{B}(G)\} \subset \{\mu_{n_\epsilon}(A) : A \in \mathcal{B}(G)\} + \epsilon B(Y).$$

Since $\{\mu_{n_\epsilon}(A) : A \in \mathcal{B}(G)\}$ is relatively compact for any arbitrary $\epsilon > 0$, a standard argument shows that $\{\mu(A) : A \in \mathcal{B}(G)\}$ is also relatively compact. This finishes the proof. □

Finally for the case of type III-Λ-CCP, we saw that this property is equivalent to either: type II-Λ-CCP, for Λ Riesz (see Proposition 3.8), or CCP, for Λ non Riesz (see Proposition 3.9). Therefore, we immediately have the following.

THEOREM 4.4. *Let Λ be a subset of the dual group of a compact metrizable abelian group G. Let X be a Banach space with type* III-Λ-CCP. *Then every Banach space that G_δ-embeds in X has type* III-Λ -CCP.

The next theorem is a known result of J. Bourgain and H. P. Rosenthal [1].

THEOREM 4.5. *The sequence space c_0 G_δ-embeds in a Banach space X if and only if it embeds in X.*

Proof. One implication is obvious. For the other implication suppose c_0 fails to embed in X. Then X has type I-Λ-CCP for any Sidon set Λ by Theorem 2.6, hence c_0 cannot G_δ-embed in X. □

Finally, we can show the following result.

PROPOSITION 4.6. *Let Λ be an infinite subset of the dual group $\widehat{G}$ of a compact metrizable abelian group G. Then $L^1(G)/L^1_{\Lambda'}(G)$ fails* I-Λ-CCP.

Proof. Let $q : L^1(G) \to L^1(G)/L^1_{\Lambda'}(G)$ be the natural quotient mapping. It is clear that $q(\bar{\gamma}) = 0$ for any $\gamma \notin \Lambda$, thus q is a Λ-operator but q is not completely continuous for the sequence $(\bar{\gamma}_n)$ where $\gamma_n \in \Lambda$ is a weakly null sequence, yet the sequence $\|q(\bar{\gamma}_n)\| \geq 1$ for all $n \geq 1$. □

In [**10**], A. Pełczyński showed that if $L^1(\mathbb{T})/H^1(\mathbb{T})$ embeds in a Banach lattice X, then X must contain an isomorphic copy of c_0. The following result reveals that in fact the conclusion of the statement of the above proposition remains true for the Banach lattice X if we replace "embeds" by "G_δ-embeds" in the statement.

PROPOSITION 4.7. *Let Λ be a Riesz subset of the dual group $\widehat{G}$ of a compact metrizable abelian group G. Then if $L^1(G)/L^1_{\Lambda'}(G)$ G_δ-embeds in a Banach lattice X, then X must contain an isomorphic copy of c_0.*

Proof. If the Banach lattice X contains no copy of c_0, then X has type I-Λ-CCP by Theorem 2.5. If we combine the result of Proposition 4.6 with that of Theorem 4.2, we see that $L^1(G)/L^1_{\Lambda'}(G)$ cannot G_δ-embed in X. □

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