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REMARKS ON LIFTING OF COHEN-MACAULAY PROPERTY

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Let (R, m) be a local noetherian ring and I a proper ideal in R. Let $\mathscr{R}(I)$ be the Rees-ring $\oplus_{n\geq 0} I^n$ with respect to I. In this note we describe conditions for I and R in order that the Cohen-Macaulay property (C-M for short) of R/I can be lifted to R and $\mathscr{R}(I)$, see Propositions 1.2, 1.3 and 1.4.

§ 1. Preliminaries, examples and results

The statements in the following proposition are well known. We give here a short proof.

PROPOSITION 1.1. *For a prime ideal $p \subset R$ let R_p be regular and p/p^2 flat over R/p. If R/p is* C-M *then R is a* C-M *domain and $\mathscr{R}(p^\tau)$ is* C-M *for all $\tau \geqq 1$.*

Proof. By assumption p is generated by a regular sequence (see [HSV], Lemma 3.17, p. 75), in particular we have $\dim R = \dim R/p + \mathrm{ht}(p)$. Therefore by [D] and [HSV], p. 72 R is a domain. Then the C-M property of R/p can be used to get a regular sequence with $\dim R$ elements in R, so R is C-M. Hence by [V] we know that $\mathscr{R}(p^\tau)$ is C-M for all $\tau \geqq 1$.

The statement of Proposition 1.1 is false if the regularity of R_p is replaced by the C-M property. Here is an example of Hesselink (see [HSV], p. 76): Let S be a discrete valuation ring and t a generator of its maximal ideal. Take the ideal

$$J = (X^2, XY - tZ^2, XZ^2, Z^4)$$

in the polynomial ring $H = S[X, Y, Z]$. Then we consider the local ring

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$$R = H_M/JH_M \qquad \text{where } M = tH + (X, Y, Z)H\,,$$

and the prime ideal $p = (X, Y, Z)R$. Then [HSV] Lemma 1.53, p. 34 yields that R is normally flat along p (i.e. p^n/p^{n+1} is flat over R/p for all $n \geq 0$). Furthermore we get:

(i) R_p is C-M-ring with $\dim R_p = 1$,

(ii) $\dim R = 2$,

(iii) $\operatorname{depth} R = 1$, since t is R-regular and the ideal (J, t) has M as an embedded component.

So R is not C-M.

One reason for missing the C-M property of R in this example can be seen in the fact that pR_p has proper reductions. From this point of view we need the regularity of R_p in the sketched proof. On the other hand flatness of p/p^2 over R/p is a rather strong condition. (Note that in the case of Proposition 1.1 we have even normal flatness of R along p.) Therefore we have asked if normal pseudo-flatness of R along I is the "correct" condition to be put on I in this context. This last condition means $\operatorname{ht}(I) = \ell(I)$, where $\operatorname{ht}(I)$ is height and $\ell(I)$ the analytic spread of the ideal I. The nicest result one could perhaps expect is that for a Buchsbaum ring R and a prime ideal p such that

(i) R_p is regular,

(ii) $\operatorname{ht}(p) = \ell(p)$,

(iii) R/p is C-M,

we get the C-M property of R.

Unfortunately this is not true. We are indebted to S. Goto for the following *example*:

$R = k[[s^2, s^3, st, t]]$ is a Buchsbaum ring with multiplicity $e(R) = 2$. Let $p = (t, st)$. Note that R_p is regular. Furthermore we have $p^2 = tp$, i.e. $\ell(p) = 1$. Hence we know that $\operatorname{ht}(I) = \ell(I) = 1$. Finally $R/p \simeq k[[s^2, s^3]]$ is C-M. But $\operatorname{depth} R = 1$ (t is a regular element), so R is not C-M. (Note that p is not generated by a regular sequence.)

What we can really prove is stated in the following Propositions 1.2, 1.3 and 1.4.

PROPOSITION 1.2. *Let (R, m) be a local ring*[1] *such that R_p is* C-M *for all $p \neq m$. Let I be an ideal in R with the following properties*:

1) To avoid technical complications we always assume $|R/m| = \infty$.

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(i) *I is locally a complete intersection*[2],

(ii) $0 < \mathrm{ht}(I) \leqq \operatorname{depth} R - 1$,

(iii) $\mathrm{ht}(I) = \ell(I)$.

Then I can be generated by a regular sequence.

The following examples (see also [H-O-2]) show that Proposition 1.2 is false for a prime ideal $I = p$ which satisfies the conditions (ii) and (iii) but not (i).

EXAMPLE 1. Let

$$R = k[[X, Y, Z, W]]/(Z^2 - W^5,\ Y^2 - XZ)$$
$$= k[[x, y, z, w]]$$

and $p = (y, z, w)$.

We have $wp^3 = p^4$, hence $\ell(p) = \mathrm{ht}(p) = 1$. Furthermore $R/p \simeq k[[x]]$ is regular. Therefore by [H-O-1] we get equimultiplicity: $e(R) = e(R_p)$. Surely $e(R) > 1$, hence $e(R_p) \geq 2$, i.e. R_p is not regular. But R is C-M, hence $\operatorname{depth} R = 2 \geq \mathrm{ht}(p) + 1$. Now in this case p is not generated by a regular sequence. Furthermore $\mathscr{R}(p)$ is not C-M (otherwise p could be generated by one element; see Proposition 1.5).

EXAMPLE 2. Let

$$R = k[[X_1, X_2, X_3, Y_1, Y_2, Y_3]]/(X_1Y_1 + X_2Y_2 + X_3Y_3,\ (Y_1, Y_2, Y_3)^2)$$
$$= k[[x_1, x_2, x_3, y_1, y_2, y_3]]$$

and $p = (x_3, y_1, y_2, y_3)$.

Then R/p is regular, $\mathrm{ht}(p) = \ell(p) = 1$ since $p^2 = x_3 p$ and $\operatorname{depth}(R) = 2 \geq \mathrm{ht}(p) + 1$ (x_1, x_2 is a regular sequence in R). Note that R_p is not regular and indeed R and $\mathscr{R}(p)$ are not C-M.

As a corollary of Proposition 1.2 we have

PROPOSITION 1.3. *Under the same assumptions as in Proposition* 1.2 *we get the implication: If R/I is* C-M *then R and $\mathscr{R}(I^\tau)$ are* C-M *for $\tau \geq 1$.*

The next proposition gives a characterization of the C-M property of $\mathscr{R}(I)$ if R is normally flat along I and R/I is C-M (but generally not regular). It is based on a result of S. Ikeda (see Proposition 2.1).

PROPOSITION 1.4. *Let (R, m) be a Buchsbaum ring and I an ideal in*

2) IR_p is a complete intersection (i.e. $\mathrm{ht}(IR_p) =$ minimal number of generators of IR_p) for all $p \in \mathrm{Ass}(R/I)$.

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R with $\mathrm{ht}(I) > 0$ *such that* R/I *is* C-M *and* R *is normally flat along* I. *Then* $\mathscr{R}(I)$ *is* C-M *if and only if*

(i) $H^i_M(G)_n = \begin{cases} H^i_m(R) & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}$ *and* $i < d = \dim R$,

(ii) $H^d_M(G)_n = 0$ for $n \geq 0$,

where $G = \mathrm{gr}_I(R)$ *and* $M = m \oplus \sum_{n>0} I^n$.

The next result describes necessary conditions for $\mathscr{R}(I)$ to be C-M.

PROPOSITION 1.5. *Let* (R, m) *be a local ring and* I *an ideal in* R *with* $\mathrm{ht}(I) = \ell(I) =: t > 0$. *If* $\mathscr{R}(I)$ *is* C-M *then the following conditions are fulfilled*:

(i) $H^i_M(G)_n = \begin{cases} H^i_m(R) & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}$ *and* $i < d = \dim R$,

(ii) *There exist elements* $z_1, \cdots, z_t \in I$ *such that* $I^t = (z_1, \cdots, z_t)I^{t-1}$,

(iii) depth $R \geq \dim R/I + 1$,

(iv) *R is normally Cohen-Macaulay along* I[1].

The following example shows that without any restriction on I Proposition 1.5 is false:

Let $R = k[[X]]$, where $X = (X_{ij})$ is the $n \times (n+1)$ matrix of indeterminates X_{ij} over a field k. Let $I = I_n(X)$ be the ideal generated by the n-minors. Then $\mathscr{R}(I)$ is C-M by C. Huneke [Hu], but I/I^2 is not C-M for $n \geq 2$ by J. Herzog [H].

§ 2. Proofs of Propositions 1.2-1.5

Proof of 1.2. Condition (iii) implies [H-O-2]

$$\mathrm{ht}(I) + \dim(R/I) = \dim R \tag{1}$$

and the existence of a minimal reduction $z_1, \cdots, z_t$ ($\underset{\sim}{z}$ for short) of I, where $t = \mathrm{ht}(I)$. Condition (i) tells us ([N-R], § 4, Theorem 2) that IR_p has no proper reduction for all $p \in \mathrm{Ass}(R/I)$. Hence we get

$$(\underset{\sim}{z})R_p = IR_p \qquad \text{for } p \in \mathrm{Ass}(R/I)\,. \tag{2}$$

Now, by the assumption (that all R_p are C-M for $p \neq m$), for each system of parameters $z_1, \cdots, z_t, z_{t+1}, \cdots, z_d$ of R ($d = \dim(R)$) there exists an $N > 0$ (which depends on the system of parameters) such that

3) i.e. I^n/I^{n+1} is an R/I-module of depth equal to $\dim R/I$ for $n \geq 0$.

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$$(z_1, \cdots, z_i): z_{i+1} \subseteqq (z_1, \cdots, z_i): m^N \qquad \text{for } 0 \leq i < d ,$$

where $z_0 = 0$ by convention. This means $0: z_1 \subset 0: m^N$ for $i = 0$. Hence we get $0: z_1 = 0$, since depth R is at least $\mathrm{ht}(I) + 1$. So z_1 is a regular element. Now, considering the ring R/z_1R and using the same argument as before we see that z_1, z_2 constitute a regular sequence, and so on. Finally we have that

$$z_1, \cdots, z_t \qquad \text{is a regular sequence in } R .$$

Case 1. If $m \in \mathrm{Ass}(R/I)$, we have $I = (\underset{\sim}{z})R$ by (2) and the proposition is proved in this case.

Case 2. If $m \notin \mathrm{Ass}(R/I)$ then $(\underset{\sim}{z})R_p$ is unmixed for all $p \in \mathrm{Ass}(R/I)$ since $(\underset{\sim}{z})R_p$ is an ideal of the principal class in the C-M ring R_p. From this and (2) we obtain that I is unmixed (see also [H-O-1], proof of Satz 1). Since $\underset{\sim}{z}R$ is a minimal reduction of I we know that I and $\underset{\sim}{z}R$ have the same minimal primes. If $p \in \mathrm{Ass}\, R/\underset{\sim}{z}R$ then $p \neq m$ by assumption (ii). Hence R_p is C-M and $\underset{\sim}{z}R_p$ is unmixed of height t. Since $pR_p \in \mathrm{Ass}_{R_p}(R_p/\underset{\sim}{z}R_p)$ we have $\mathrm{ht}\, p = t$. Therefore p is a minimal prime of $\underset{\sim}{z}R$ and hence of I. Hence

$$\mathrm{Ass}(R/\underset{\sim}{z}R) = \mathrm{Ass}(R/I) .$$

By (2) we have $I = \underset{\sim}{z}R$.

Proof of Proposition 1.3. Since R/I is C-M and I is generated by a regular sequence $z_1, \cdots, z_t$ (by Proposition 1.2) we get a regular sequence $z_1, \cdots, z_t, x_1, \cdots, x_r$, where $r = \dim R/I$. Hence depth $R = \dim R$ by formula (1).

Remark. Note that condition (i) of Proposition 1.2 means for a prime ideal $I = p$ that R_p is regular. The purely technical conditions "R_p is C-M for all $p \neq m$" and "depth $R \geq \mathrm{ht}(I) + 1$" imply that $z_1, \cdots, z_t$ is a regular sequence in R, generating the ideal I. Hence in the case of a regular ring R_p we have the implication: if

(i) $\mathrm{ht}(p) = \ell(p)$ and

(ii) R_q is C-M for $q \neq m$ and depth $R \geq \mathrm{ht}(p) + 1$,

then R is normally flat along p.

QUESTION 1. How far is normal flatness of R along p from these two conditions (i), (ii) in the general case?

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QUESTION 2. Is there an example such that R is not C-M, $\mathrm{ht}(p) = \ell(p) = 2$ and $\mathscr{R}(p)$ is C-M?

In [I] an example is given with $\mathrm{ht}(p) = \ell(p) = 3$ instead of $\mathrm{ht}(p) = \ell(p) = 2$.

For the proof of Proposition 1.4 it is enough to show by [HSV], Lemma 3.15, p. 66 and Lemma 3.8, p. 117 the following statement.

PROPOSITION 2.1 (S. Ikeda). *Let (R, m) be a local ring with $\ell(H^i_m(R)) < \infty$ for $i < d = \dim R$ and let I be an ideal such that $t = \mathrm{ht}(I) > 0$ and I^n/I^{n+1} is* C-M *of depth equal to* $\dim R/I$ *for all $n \geq 0$. Then the following conditions are equivalent*:

(i) *$\mathscr{R}(I)$ is* C-M,

(ii) a) $H^i_M(G)_n = \begin{cases} H^i_m(R) & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}$ *and* $i < d$,

b) $H^d_M(G)_n = 0$ *for* $n \geq 0$,

where $G = \mathrm{gr}_I(R)$ *and* $M = m \oplus \sum_{n>0} I^n$.

For the proof of this proposition we need the following two lemmas. The following result was first obtained in [V].

LEMMA 2.2. *Let $(a_1, \cdots, a_t)$ be a minimal reduction of an ideal I with* $\mathrm{ht}(I) = \ell(I) = t > 0$ *and let $b_1, \cdots, b_s$ be a system of parameters with respect to I*[4]. *Then the sequence*

$$\{a_1, a_2 - a_1X, \cdots, a_t - a_{t-1}X, a_tX, b_1, \cdots, b_s\}$$

in the Rees-algebra $R[IX] \simeq \mathscr{R}(I)$, X an indeterminate, forms a system of parameters of $\mathscr{R}(I)_M$.

Proof. Let $P = \sqrt{(a_1, a_2 - a_1X, \cdots, b_s)}$. Since $a_1 \in P$ we can prove that all $a_i \in P$ by induction on i: If $i \geq 2$ we have $a_i\ (a_i - a_{i-1}X) = a_i^2 - a_{i-1}(a_iX) \in P$. Since $a_iX \in \mathscr{R}(I)$ and since $a_{i-1} \in P$ by induction hypothesis, we get $a_i \in P$. Hence $(a_1, \cdots, a_t, b_1, \cdots, b_s)\mathscr{R}(I) \subset P$. Note furthermore that the ideal $(a_1, \cdots, a_t, b_1, \cdots, b_s)$ is m-primary. We have $I^n = (a_1, \cdots, a_t)I^{n-1}$ for some $n > 0$. Take any $a \in I$. Then we have $a^n = \sum_{i=1}^t a_ix_i$ for some $x_i \in I^{n-1}$. Now $(aX)^n = \sum_{i=1}^t a_iXx_iX^{n-1} \in P$ because $a_iX \in P$ and $x_iX^{n-1} \in \mathscr{R}(I)$, i.e. $M \supseteq P \supseteq m\mathscr{R}(I) + IX\mathscr{R}(I) = M$. Therefore $a_1, a_2 - a_1X, \cdots, b_s$ form a system of parameters of $\mathscr{R}(I)_M$.

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4) i.e. the images of $b_1, \cdots, b_s$ in R/I form a system of parameters of R/I.

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LEMMA 2.3. *Let I be an m-primary ideal and let $(a_1, \cdots, a_d)$ be a minimal reduction of I, where $d = \dim R$. If $I^d = (a_1, \cdots, a_d)I^{d-1}$ then we have*

$$H^d_M(G)_n = 0 \qquad \textit{for } n \geq 0 .$$

Proof. Let $a_i^* = \mathrm{In}_I(a_i) \in I/I^2$ the initial form of a_i with respect to I. Since the ideal $(a_1^*, \cdots, a_d^*)$ in G is primary[5] to the maximal homogeneous ideal of G there exists an exact sequence (see [R], p. 78, Proposition 2.3)

$$\bigoplus_{i=1}^{d} G_{a_1^* \cdots \check{a}_i^* \cdots a_d^*} \xrightarrow{\varphi} G_{a_1^* \cdots a_d^*} \longrightarrow H^d_M(G) \longrightarrow 0 ,$$

where φ is given by

$$\varphi((f_1, \cdots, f_d)) = \sum_{i=1}^{d} (-1)^i \frac{f_i}{1} \qquad \text{for } f_i \in G_{a_1^* \cdots \check{a}_i^* \cdots a_d^*} .$$

Pick $x \in H^d_M(G)_n$, $n \geq 0$, and assume that x is represented by

$$\frac{f}{(a_1^* a_2^* \cdots a_d^*)^k} \in (G_{a_1^* \cdots a_d^*})_n ,$$

where $f \in G$ is homogeneous of degree $kd + n$. If $k = 0$, then f is of course in the image of φ. If $k \geq 1$, then by applying the assumption we get

$$\begin{aligned} I^{kd+n} &= (\underline{a})^{(k-1)d+1+n} I^{d-1} \\ &= (a_1^k, \cdots, a_d^k)(a_1, \cdots, a_d)^{(k-1)(d-1)+n} I^{d-1} \\ &= (a_1^k, \cdots, a_d^k) I^{k(d-1)+n} . \end{aligned}$$

Hence f can be written in the form

$$f = \sum_{i=1}^{d} a_i^{*k} g_i \qquad \text{where } g_i \in G_{k(d-1)+n} .$$

Therefore $f/(a_1^* \cdots a_d^*)^k$ is in the image of φ.

Proof of Proposition 2.1. Since $\ell(H^i_m(R)) < \infty$ for $i < d = \dim R$ we have

$$\dim R/p + \mathrm{ht}(p) = \dim R \qquad \text{for all } p \in \mathrm{Spec}(R) ,$$

hence in particular $\dim R/I + \mathrm{ht}(I) = \dim R$ (see [S-T-C]).

5) The elements of a minimal reduction induce a system of parameters $\tilde{a}_d^*, \cdots, \tilde{a}_d^*$, in the ring $\mathrm{gr}_I R \otimes_R R/m$, hence $a_1^*, \cdots, a_d^*$ form a system of parameters in $\mathrm{gr}_I R$.

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Furthermore the C-M property of I^n/I^{n+1} implies that $\mathrm{ht}(I) = \ell(I) =: t$ by [H-O-2].

First we prove (i) $\Rightarrow$ (ii). Consider the exact sequences

$$(1)\qquad \begin{array}{l} 0 \longrightarrow \mathscr{R}(I)_+ \longrightarrow \mathscr{R}(I) \longrightarrow R \longrightarrow 0 \quad \text{and} \\ 0 \longrightarrow \mathscr{R}(I)_+(1) \longrightarrow \mathscr{R}(I) \longrightarrow G \longrightarrow 0^{6)} . \end{array}$$

We get the following exact sequences.

$$(1')\qquad \begin{array}{l} \to H^i_M(\mathscr{R}(I)) \to H^i_m(R) \to H^{i+1}_M(\mathscr{R}(I)_+) \to H^{i+1}_M(\mathscr{R}(I)) \to \cdots \quad \text{and} \\ \to H^i_M(\mathscr{R}(I)) \to H^i_M(G) \to H^{i+1}_M(\mathscr{R}(I)_+)(1) \to H^{i+1}_M(\mathscr{R}(I)) \to \cdots . \end{array}$$

Since $\mathscr{R}(I)$ is C-M by assumption (i) of Proposition 2.1 we obtain for $i < d$

$$H^i_m(R) \simeq H^{i+1}_M(\mathscr{R}(I)_+) \quad \text{and} \quad H^i_M(G) \simeq H^{i+1}_M(\mathscr{R}(I)_+)(1) .$$

Hence we get (ii), a) in Proposition 2.1.

By what we have mentioned at the beginning of the proof, there exists $a_1, \cdots, a_t \in I$ such that $I^n = (a_1, \cdots, a_t)I^{n-1}$ for some $n > 0$. By assumption $\mathscr{R}(I)$ is C-M. Therefore, by Lemma 2.2, $a_1, a_2 - a_1X, \cdots, a_t - a_{t-1}X, a_tX$ is an $\mathscr{R}(I)_M$-sequence[7]. Then we can use the same argument as in [I], p. 8 to show that

$$(2)\qquad I^t = (a_1, \cdots, a_t)I^{t-1} .$$

(The idea is to consider for any $a \in I^t$ the following congruence mod $(a_2 - a_1X, \cdots, a_tX)$:

$$a_1aX^t \equiv a_2aX^{t-1} \equiv \cdots \equiv a_taX \equiv 0 ,$$

hence $aX^t \in (a_2 - a_1X, \cdots, a_tX)\mathscr{R}(I)_M$ since $a_1, a_2 - a_1X, \cdots, a_tX$ is a regular sequence in $\mathscr{R}(I)_M$. Therefore we find an equation in $\mathscr{R}(I)$ of the form

$$raX^t = (a_2 - a_1X)f_1 + \cdots + a_tXf_t , \qquad r \notin M .$$

Comparing the coefficients of X^t in this equation we obtain (2).)

Since I^n/I^{n+1} is C-M for $n \geq 0$ we find elements $b_1, \cdots, b_s \in m$ ($s = \dim R/I$) forming a regular sequence on I^n/I^{n+1} for all $n \geq 0$.

[Any system $b_1, \cdots, b_s$ of parameters with respect to I is a system of parameters for each I^n/I^{n+1} since $\dim I^n/I^{n+1} = \dim R/I$, hence $b_1, \cdots, b_s$ is a regular sequence.]

6) $\mathscr{R}(I)_+ = \oplus_{n>0} I^n$ and $\mathscr{R}(I)_+(1)$ is the module with the degree shifted by 1.
7) $\mathscr{R}(I)$ is identified with $R[IX]$.

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Therefore $b_1, \cdots, b_t$ ($\underline{b}$ for short) is a G-sequence. Set

$$\bar{G} = \mathrm{gr}_{I+\underline{b}R/\underline{b}R}(R/\underline{b}R) \simeq G/\underline{b}G .$$

By Lemma 2.3 we see that

$$(3) \qquad H_M^t(\bar{G})_n = 0 \qquad \text{for } n \geq 0 .$$

Let $G_0 = G$ and $G_i = G/(b_1, \cdots, b_i)G$ for $i = 1, \cdots, s$. For $0 \leq i \leq s$ we have the exact sequence

$$0 \longrightarrow G_i \xrightarrow{b_{i+1}} G_i \longrightarrow G_{i+1} \longrightarrow 0 .$$

From this exact sequence we get an exact sequence

$$(4) \qquad \longrightarrow H_M^{j-1}(G_i) \longrightarrow H_M^{j-1}(G_{i+1}) \longrightarrow H_M^j(G_i) \xrightarrow{b_{i+1}} H_M^j(G_i) \longrightarrow \quad \text{for } 0 \leq i \leq s .$$

By ii), a) we have $H_M^j(G_i)_n = 0$ for $n \neq -1$, $j < d-1$ and $0 \leq i \leq s$. Assume that $H_M^{d-i-1}(G_{i+1})_n = 0$ for $n \geq 0$. Then the exact sequence (4) shows that b_{i+1} is a non-zero-divisor on $H_M^{d-i}(G_i)_n$ for $n \geq 0$. Let $n \geq 0$ and $x \in H_M^{d-i}(G_i)_n$. Since $b_{i+1} \in M$ we have $b_{i+1}^k x = 0$ for sufficiently large $k > 0$. Therefore $x = 0$ and we have $H_M^{d-i}(G_i)_n = 0$ for $n \geq 0$. Since $\bar{G} = G_s$ we see that $H_M^{d-i}(G_i)_n = 0$ for $n \geq 0$ and $0 \leq i \leq s$ by induction on s and (3).

(ii) $\Rightarrow$ (i). Since $b_1, \cdots, b_s$ is a G-sequence (by the general assumption of Proposition 2.1) we have [V-V].

$$(b_1, \cdots, b_s) \cap I^n = (b_1, \cdots, b_s)I^n \qquad \text{for } n \geq 0 .$$

Hence

$$\mathscr{R}(I + \underline{b}R/\underline{b}R) \simeq \bigoplus_{n\geq 0} I^n/\underline{b}R \cap I^n = \bigoplus_{n\geq 0} I^n/\underline{b}I^n = \mathscr{R}(I)/\underline{b}\mathscr{R}(I) .$$

Since b_1 is regular on G we have

$$b_1R \cap I^n = b_1I^n .$$

Hence we have

$$\mathrm{gr}_{I+b_1R/b_1R}(R/b_1R) \simeq G/b_1G$$

and

$$\mathscr{R}(I + b_1R/b_1R) \simeq \mathscr{R}(I)/b_1\mathscr{R}(I) .$$

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Since b_1 is not a zero-divisor on $\mathscr{R}(I)$ we can conclude that $\underline{b}$ is an $\mathscr{R}(I)$-sequence by induction on s. Therefore it is sufficient to prove that $\bar{\mathscr{R}} \overset{\text{def}}{:=} \mathscr{R}(I + \underline{b}R/\underline{b}R)$ is C-M.

From the exact sequence (4) and ii), a) we have for $i < t$.

$$H_M^i(\bar{G})_n = 0 \ (n \neq -1) \quad \text{and} \quad \ell(H_M^i(\bar{G})) < \infty \tag{5}$$

by induction on s.

But (5) implies by [G, (3.1)]

$$\ell(H_M^i(\bar{\mathscr{R}})) < \infty \qquad \text{for } i \leq t .$$

Similarly for $i = t$ we have (see Lemma 2.3)

$$H_M^t(\bar{G})_n = 0 \qquad \text{for } n \geq 0 . \tag{6}$$

From the analogous exact sequence (1′) corresponding to $\bar{\mathscr{R}}$, we have for $i < t$ isomorphisms

$$\begin{aligned} H_M^i(\bar{\mathscr{R}}_+)_\nu &\xrightarrow{\sim} H_M^i(\bar{\mathscr{R}})_\nu , \qquad && \nu \neq 0 \\ H_M^i(\bar{\mathscr{R}}_+)_{\nu+1} &\xrightarrow{\sim} H_M^i(\bar{\mathscr{R}})_\nu , \qquad && \nu \neq -1 . \end{aligned}$$

Since $\ell(H_M^i(\bar{\mathscr{R}})) < \infty$, we know already that $H_M^i(\bar{\mathscr{R}})_\nu = 0$ for $\nu \gg 0$ or $\nu \ll 0$ (and $i \leq t$). Therefore we have

$$H_M^i(\bar{\mathscr{R}}) = 0 \qquad \text{for } i < t .$$

Now it remains to prove that $H_M^t(\bar{\mathscr{R}}) = 0$:

By (5) and (6) we have isomorphisms

$$\begin{aligned} H_M^t(\bar{\mathscr{R}}_+)_\nu &\xrightarrow{\sim} H_M^t(\bar{\mathscr{R}})_\nu \qquad && \nu \neq 0 , \\ H_M^t(\bar{\mathscr{R}}_+)_{\nu+1} &\xrightarrow{\sim} H_M^t(\bar{\mathscr{R}})_\nu \qquad && \nu \geq 0 \end{aligned}$$

and injective homomorphisms

$$H_M^t(\bar{\mathscr{R}}_+)_{\nu+1} \hookrightarrow H_M^t(\bar{\mathscr{R}})_\nu \qquad \nu \leq -2 .$$

Since $H_M^t(\bar{\mathscr{R}})_n = 0$ for $n \gg 0$ or $n \ll 0$ one can conclude that $H_M^t(\bar{\mathscr{R}}) = 0$.

Hence $\bar{\mathscr{R}}$ is C-M as required.

Proof of Proposition 1.5. We have already shown (i) and (ii) in the course of the proof of Proposition 2.1.

Let $(z_1, \cdots, z_t)$ be a minimal reduction of I and $b_1, \cdots, b_s$ a system of parameters with respect to I.

By Lemma 2.2 $\{z_1, z_2 - z_1X, \cdots, z_t - z_{t-1}X, z_tX, b_1, \cdots, b_s\}$ is an $\mathscr{R}(I)_M$-

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sequence since $\mathscr{R}(I)$ is C-M by assumption. Now consider the exact sequence

$$0 \longrightarrow \frac{(z_1, z_1X)\mathscr{R}(I)}{z_1\mathscr{R}(I)} \longrightarrow \frac{\mathscr{R}(I)}{z_1\mathscr{R}(I)} \longrightarrow \frac{\mathscr{R}(I)}{(z_1, z_1X)\mathscr{R}(I)} \longrightarrow 0 .$$

We have

$$\frac{(z_1, z_1X)\mathscr{R}(I)}{z_1\mathscr{R}(I)} \simeq \frac{\mathscr{R}(I)}{(z_1\mathscr{R}(I): z_1X)}(-1) .$$

Since z_1 is also not a zero-divisor on R we have

$$(z_1\mathscr{R}(I): z_1X) = I\mathscr{R}(I) .$$

Hence we have the exact sequence

$$(1) \quad 0 \longrightarrow \mathrm{gr}_I(R)(-1) \longrightarrow \mathscr{R}(I)/z_1\mathscr{R}(I) \longrightarrow \mathscr{R}(I)/(z_1, z_1X)\mathscr{R}(I) \longrightarrow 0 .$$

To prove (iii) and (iv) we use induction on $s = \dim R/I$. If $s = 0$ then (iv) is clear and $\operatorname{depth} R \geq 1$ (z_1 is a non-zero-divisor in R). If $s > 0$ then z_1, b_1 is an $\mathscr{R}(I)_M$-sequence. By the exact sequence (1) b_1 is a non-zero-divisor on $\mathrm{gr}_I(R)$. Therefore $b_1R \cap I^n = b_1I^n$ for $n \geq 0$. Hence

$$\mathscr{R}(I + b_1R/b_1R) \simeq \mathscr{R}(I)/b_1\mathscr{R}(I)$$

is C-M since b_1 is a non-zero-divisior on $\mathscr{R}(I)$.

Let $\bar{R} = R/b_1R$ and $\bar{I} = I\bar{R}$. Since $\mathrm{gr}_{\bar{I}}(\bar{R}) = \mathrm{gr}_I(R)/b_1\,\mathrm{gr}_I(R)$ we have

$$\begin{aligned} \ell(\bar{I}) &= \dim \mathrm{gr}_{\bar{I}}(\bar{R})/m\,\mathrm{gr}_{\bar{I}}(\bar{R}) \\ &= \dim \mathrm{gr}_I(R)/m\,\mathrm{gr}_I(R) \\ &= \ell(I) . \end{aligned}$$

Let $p \in \operatorname{Spec}(R)$ be a minimal prime of (I, b_1) such that $\operatorname{ht}(p/b_1R) = \operatorname{ht}(\bar{I})$. Since b_1 is a non-zero-divisor on R/I we have $\operatorname{ht}(I) + 1 \leq \operatorname{ht}(p)$. Since b_1 is also a non-zero-divisor on R we see that

$$\operatorname{ht}(I) + 1 \leq \operatorname{ht}(p) = \operatorname{ht}(p/b_1R) + 1 = \operatorname{ht}(\bar{I}) + 1 .$$

Now we have

$$\operatorname{ht}(I) \leq \operatorname{ht}(\bar{I}) \leq \ell(\bar{I}) = \ell(I) = \operatorname{ht}(I) .$$

By induction hypothesis we have

$$\operatorname{depth} R/b_1R \geq \dim R/(I, b_1) + 1 = \dim R/I$$

and hence we have

$$\operatorname{depth} R \geq \dim R/I + 1 .$$

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Since

$$I^n + b_1R/I^{n+1} + b_1R \simeq I^n/I^{n+1} + b_1I^n \simeq (I^n/I^{n+1})/b_1(I^n/I^{n+1})$$

and since b_1 is a non-zero-divisor on I^n/I^{n+1} we have

$$\operatorname{depth} I^n/I^{n+1} = \operatorname{depth}(I^n + b_1R/I^{n+1} + b_1R) + 1 = \dim R/(I, b_1) + 1$$

by induction hypothesis. Hence

$$\operatorname{depth} I^n/I^{n+1} = \dim R/I \qquad \text{for } n \geq 0$$

as required.

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