CHARACTERIZATIONS OF SUPPORTS OF BALAYAGED MEASURES

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Introduction

Beurling and Deny [1], [2] introduced Dirichlet spaces by generalizing the notion of "*energy*" in potential theory. They showed the existences of balayaged measures and condensor measures in the theory of Dirichlet spaces. The purpose of this paper is to characterize the supports of those measures. First we obtain the following result.

Let D be a Dirichlet space on a locally compact Hausdorff space X, and let F be a regular set in $X \times X$ containing the diagonal set of $X \times X$. Then the following two conditions are equivalent.

(1) *For any pure potential* u_μ *in D and any closed neighborhood ω of the support of μ, the support of the balayaged measure of μ to $\mathscr{C}\omega$ is contained in* $F_\omega \cap \overline{\mathscr{C}\omega}$.

(2) *The support of the singular measure of D is contained in F.*

In the above statement, F_ω means the projection of $(\omega \times X) \cap F$ to X.

Furthermore we shall examine a relation between the supports of condensor measures and the support of the singular measure of D.

Next we shall consider special Dirichlet spaces on the n-dimensional Euclidean space $R^n (n \geq 1)$. With a special Dirichlet space D associates a real valued negative definite function λ such that λ^{-1} is locally summable, and the norm in D is explicitly represented. By Levy-Khintchine's theorem, $\lambda(x)$ is as follows:

$$\lambda(x) = C + Q(x) + \int_{|y|>0} (1 - e^{2\pi i x \cdot y}) d\sigma_1(y),$$

where C is a non-negative constant, $Q(x)$ is a positive quardratic form and σ_1 is a positive symmetric measure in $R^n - \{0\}$. Then we shall obtain a more

precise result about the relation among the supports of balayaged measures, of condensor measures and of the above-mentioned σ_1.

1. Preliminaries on Dirichlet spaces

To begin with, we shall give the definition of Dirichlet spaces according to Beurling and Deny [2].

DEFINITION 1.[1]   *A transformation T on the complex plane $\mathfrak{C}$ into itself is called a normal contraction if the following conditions are satisfied:*

$$T(0) = 0 \ and \ |Tz_1 - Tz_2| \leq |z_1 - z_2|$$

for each couple z_1, z_2 in $\mathfrak{C}$.

Let X be a locally compact Hausdorff space.   Let C_K be the space of complex-valued continuous functions with compact support provided with the topology of uniform convergence.   We denote by C_K^+ the totality of positive functions in C_K.

DEFINITION 2.[2]   *Let ξ be a positive Radon measure in X which is everywhere dense (i.e., $\xi(\omega) > 0$ for each non-empty open set ω in X).   A Hilbert space $D = D(X, \xi)$ is called a ξ-Dirichlet space (simply, a Dirichlet space) if each element in D is a complex-valued function $u(x)$[3] which is locally summable for ξ and the following three conditions are satisfied:*

a)   For each compact subset K in X, there exists a positive number $A(K)$ such that

$$\int_K |u(x)| \, d\xi(x) \leq A(K) \|u\|$$

for any u in D.

b) $C_K \cap D$ is dense both in C_K and D.

c)   For any normal contraction T and any u in D,

$$Tu \in D \ and \ \|Tu\| \leq \|u\|.$$

Now we shall define potentials in the Dirichlet space D.

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[1]   Cf. [2], p. 209.
[2]   Cf. [2], p. 209.
[3]   We take the usual liberty of speaking of functions when actually equivalence classes are meant.

DEFINITION 3.[4]   *An element u in D is called a potential if there exists a Radon measure μ in X satisfying*

$$(u, v) = \int v(x) \, d\mu(x)$$

for any v in $C_K \cap D$.   We denote it by u_μ.   Especially if μ is positive, u_μ is called a pure potential.

The space of linear combinations of pure potentials is dense in D.   This definition of potentials leads to a kernel-free potential theory.   Next we shall give the definition of the spectrum of an element in D.

Given an element u in D, there exists the greatest open set ω having the following property:

$$(u, v) = 0$$

for any v in $C_K \cap D$ with support in ω.

DEFINITION 4.[5]   *The complementary set of such an open set is called the spectrum of u, denoted by $\mathfrak{S}(u)$.*

Obviously $\mathfrak{S}(u_\mu) = S_\mu$[6].

The next four theorems obtained by Beurling and Deny [2] are essential.

BEURLING AND DENY'S CONDENSOR THEOREM[7].   *Let ω_1 and ω_0 be two open sets in X with disjoint closures, ω_1 being relatively compact.   Then there exists a potential u_μ generated by a real measure μ such that*

(C. 1) $0 \leq u_\mu(x) \leq 1 \ p.p. \ in \ X$[8],

(C. 2) $u_\mu(x) = 1 \ p.p. \ in \ \omega_1 \ and \ u_\mu(x) = 0 \ p.p. \ in \ \omega_0$,

(C. 3) μ^+ *is supported by* $\overline{\omega_1}$ *and* μ^- *is supported by* $\overline{\omega_0}$.

We shall say that the measure μ is a condensor measure associated with ω_1 and ω_0.   Let ω_0 be the empty set.   Then we get the equilibrium theorem.

BEURLING AND DENY'S BALAYAGE THEOREM[9].   *For a pure potential u_μ and an*

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[4] Cf. [2], p. 209.

[5] Cf. [2], p. 214.

[6] S_μ is the support of μ.

[7] Cf. [2], p. 210 and [4], p. 4.

[8] A property is said to hold *p.p.* in a subset E in X if the property holds in E except a set which is locally of ξ-measure 0.

[9] Cf. [2], p. 210 and [4], p. 6.

open set ω, there exists a pure potential $u_{\mu'}$ satisfying the following conditions:

(B. 1)   *The measure μ' is supported by $\bar{\omega}$ and $\int d\mu' \leq \int d\mu$.*

(B. 2) $u_{\mu'}(x) = u_{\mu}(x)$ *p.p. in* ω.

(B. 3) $u_{\mu'}(x) \leq u_{\mu}(x)$ *p.p. in* X.

We shall $u_{\mu'}$ the balayaged potential of u_{μ} to ω and μ' the balayaged measure of μ to ω.

BEURLING AND DENY'S REFINEMENT THEOREM[10]. *For each u in D, there exists a function u^* defined p.p. in X satisfying the following conditions:*

(R. 1) $u^*(x) = u(x)$ *p.p. in X and $u^*(x)$ vanishes outside some σ-compact set.*

(R. 2) u^* *is measurable with respect to the measure μ associated with any pure potential u_{μ} and*

$$(u, u_{\mu}) = \int u^* d\mu.$$

We shall call u^* the refinement of u.

BEURLING AND DENY'S REPRESENTATION THEOREM[11]. *For a Dirichlet space D, there exist a positive measure ν in X, a positive Hermitian form $N(f, g)$ on $C_K \cap D$ and a positive symmetric measure σ in $X \times X - \delta$ (δ is the diagonal set of $X \times X$) such that*

$$(f, g) = \int f\bar{g}\,d\nu + N(f, g) + \iint (f(x) - f(y))(\bar{g}(x) - \bar{g}(y))d\sigma(x, y)$$

for any f, g in $C_K \cap D$.   Here $N(f, g)$ has the following local character: if g is constant in some neighborhood of the support S_f of f, then $N(f, g)$ vanishes.

We shall call the measure σ the singular measure of D.   With respect to the measure ν, we have the following

Remark. Let $(\omega_{\alpha})_{\alpha \in I}$ be an increaing net of relatively compact open sets tending to X, and let ν_{α} be the equilibrium measure of ω_{α}.   Then the net $(\nu_{\alpha})_{\alpha \in I}$ converges vaguely to ν.

For the proof, see [2], p. 212.   We shall call the measure ν the equilibrium measure of X.

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[10] Cf. [2], p. 210.
[11] Cf. [2], pp. 211-213.

2. Some lemmas

First we consider the following subspaces of D. We put, for a non-empty open set ω in X,

$$D_\omega^{(1)} = \overline{\{u \in D \; ; \; \mathfrak{S}(u) \subset \omega\}},$$
$$D_\omega^{(2)} = \overline{\{f \in C_K \cap D \; ; \; S_f \subset \omega\}},$$

and for a closed set F in X,

$$D_F^{(1)} = \{u \in D \; ; \; \mathfrak{S}(u) \subset F\},$$
$$D_F^{(3)} = \{u \in D \; ; \; u^*(x) = 0 \; p.p.p. \; \text{on} \; F\}^{12)}.$$

LEMMA 1. *For any open set ω in X, $D_\omega^{(2)} = D_{\mathscr{C}\omega}^{(3)}$.*

Proof. For any u in $D_\omega^{(2)}$, there exists a sequence (u_n) in $C_K \cap D_\omega^{(2)}$ such that $u_n \to u$ strongly in D as $n \to \infty$. Then for any pure potential u_μ such that $S_\mu \subset \mathscr{C}\omega$,

$$(u_n, u_\mu) = \int u_n \, d\mu = 0,$$

and hence

$$(u, u_\mu) = \int u^* \, d\mu = 0.$$

That is, $u^*(x) = 0 \; p.p.p$ on $\mathscr{C}\omega$. Consequently $D_\omega^{(2)} \subset D_{\mathscr{C}\omega}^{(3)}$.

Conversely, let u be in $D_{\mathscr{C}\omega}^{(1)}$. By Beurling and Deny's theorem [13], there exists a sequence (u_{μ_n}) such that u_{μ_n} is a linear combination of pure potentials, μ_n is supported by $\mathscr{C}\omega$ and $u_{\mu_n} \to u$ strongly in D as $n \to \infty$. Then for any v in $D_{\mathscr{C}\omega}^{(3)}$,

$$(v, u_{\mu_n}) = \int v^* \, d\mu_n = 0,$$

and hence $(v, u) = 0$. This means that $D_{\mathscr{C}\omega}^{(3)\perp} \supset D_{\mathscr{C}\omega}^{(1)}$. Consequently $D_\omega^{(2)} \supset D_{\mathscr{C}\omega}^{(3)}$, since we have $D_\omega^{(2)\perp} = D_{\mathscr{C}\omega}^{(1)}$. This completes the proof.

LEMMA 2. *An element u in D is a pure potential if and only if there exists a sequence (f_n) of positive bounded measurable functions with compact support such that the sequence (u_{f_n}) of pure potentials converges strongly to u as $n \to \infty$.*

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[12] A property is said to hold $p.p.p.$ on a subset E in X if the property holds μ-$p.p.$ for any pure potential u_μ in D such that $S_\mu \subset E$.

[13] Cf. [2], p. 214.

For the proof, see Lemma 2 in [2] and [5], p. 2.

LEMMA 3. *For an element u in D and a real valued continuous function φ, suppose that $u(x) \geq \varphi(x)$ (resp. $u(x) \leq \varphi(x)$) p.p. in an open set ω. Then $u^*(x) \geq \varphi(x)$ (resp. $u^*(x) \leq \varphi(x)$) p.p.p. in ω.*

Proof. Let u_μ be a pure potential such that $S_\mu \subset \omega$. It is sufficient to prove that

$$(u, u_\mu) = \int u^* d\mu \geq \int \varphi d\mu.$$

First suppose that S_μ is compact in ω. By Lemma 2, there exist positive bounded measurable functions f_n with compact support such that (u_{f_n}) converges strongly to u_μ in D. We take another relatively compact open set ω_1 such that

$$S_\mu \subset \omega_1 \subset \bar{\omega}_1 \subset \omega.$$

Put $f_{n,1} =$ the restriction of f_n to ω_1 and $f_{n,2} =$ the restriction of f_n to $\mathscr{C}\omega_1$. Then the sets $(u_{f_{n,1}})$ and $(u_{f_{n,2}})$ are bounded in D. Hence we may assume that there exist u_1 and u_2 in D such that

$$u_{f_{n,1}} \to u_1 \text{ and } u_{f_{n,2}} \to u_2$$

weakly in D, respectively. Since the measure f_n converges vaguely to μ, the measure $f_{n,1}$ converges vaguely to μ, so that for any f in $C_K \cap D$,

$$(u_1, f) = \int \bar{f} d\mu = (u_\mu, f).$$

By the denseness of $C_K \cap D$ in D, $u_1 = u_\mu$ and $u_2 = 0$. Now let u be the function in our lemma. Then by our assumption,

$$(u_{f_{n,1}}, u) = \int u(x) f_{n,1}(x) d\xi(x) \geq \int \varphi(x) f_{n,1}(x) d\xi(x),$$

ω_1 being relatively compact, we obtain

$$\int \varphi(x) f_{n,1}(x) d\xi(x) \to \int \varphi d\mu$$

as $n \to \infty$. Consequently

$$\int u^* d\mu \geq \int \varphi d\mu.$$

When S_μ is non-compact, the above result implies that for any restriction μ_K

of μ to a compact set K, $u^*(x) \geq \varphi(x)$ μ_K-$p.p.$ in X. Hence $u^*(x) \geq \varphi(x)$ μ-$p.p.$ in X. This completes the proof.

Let u_μ be a pure potential in D and ω be an open set in X. Put

$$E_{u_\mu, \omega} = \{v \in D \; ; \; Rev(x) \geq u_\mu(x) \; p.p. \; in \; \omega\}.$$

Beurling and Deny [2] showed that the balayaged potential $u_{\mu'}$ of u_μ to ω is a unique element which minimizes the norm in $E_{u_\mu, \omega}$.

LEMMA 3. *Let Pru_μ be the projection of a pure potential to $D_\omega^{(1)}$. Then $Pru_\mu = u_{\mu'}$.*

Proof. Since Pru_μ is the projection of u_μ to $D_\omega^{(1)}$,

$$(u, Pru_\mu) = (u, u_\mu)$$

for any u in $D_\omega^{(1)}$. Hence $Pru_\mu(x) = u_\mu(x)$ $p.p.$ in ω. Therefore

$$Pru_\mu \in E_{u_\mu, \omega}.$$

Consequently $\|u_{\mu'}\| \leq \|Pru_\mu\|$. On the other hand, by Beurling and Deny's theorem[14], there exists a sequence (u_n) of linear combinations of pure potentials such that $\mathfrak{S}(u_n) \subset \omega$ and (u_n) converges strongly to Pru_μ. Similarly as lemma 2, $u_\mu^*(x) = u_{\mu'}^*(x)$ $p.p.p.$ in ω. Hence

$$(u_n, u_{\mu'}) = (u_n, u_\mu) \to (Pru_\mu, u_\mu) = \|Pru_\mu\|^2.$$

Therefore

$$\|u_{\mu'}\| \|Pru_\mu\| \geq (Pru_\mu, u_{\mu'}) = \|Pru_\mu\|^2,$$

i.e., $\|u_\mu\| \geq \|Pru_\mu\|$. Consequently $u_{\mu'} = Pru_\mu$. This completes the proof.

Similarly as above, let u_μ be a potential in D such that $u_{|\mu|}$ in D. Then

$$Pru_\mu = u_{\mu_1'} - u_{\mu_2'} + i(u_{\mu_3'} - u_{\mu_4'}),$$

where $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ and μ_j is a positive measure for $j = 1, 2, 3, 4$.

LEMMA 5. *Let u and v be positive elements in D such that $S_u \cap S_v = \emptyset$ [15] and S_u or S_v is compact in X. Then $(u, v) \leq 0$.*

Proof. Assume that S_u is compact. There exists two open sets ω_1 and ω_2

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[14] Cf. [2], p. 214.
[15] S_u is the complement of the greatest open set in all open sets G that $u(x) = 0$ $p.p.$ in G.

such that $S_u \subset \omega_1$, $S_v \subset \omega_2$ and $\bar{\omega}_1 \cap \bar{\omega}_2 = \emptyset$. By Lemma 1, there exist two sequences (f_n) and (g_n) of functions in $C_K \cap D$ such that

$$S_{f_n} \subset \omega_1, \quad S_{g_n} \subset \omega_2$$

and the sequences (f_n) and (g_n) converge strongly to u and v, respectively. We may assume that $f_n(x) \geq 0$ and $g_n(x) \geq 0$ in X for each n. Because the sequences (Tf_n) and (Tg_n) converge strongly to $Tu = u$ and $Tv = v$, respectively, where T is the following normal contraction: $T(z) = sup\,(Rez, 0)$ for each z in $\mathfrak{C}$. By Beurling and Deny's Representation Theorem,

$$(f_n, g_n) = \iint (f_n(x) - f_n(y))(g_n(y))\,d\sigma(x, y)$$

$$= -2 \iint f_n(x) g_n(y)\,d\sigma(x, y) \leq 0.$$

Consequently

$$(u, v) = \lim_{n \to \infty} (f_n, g_n) \leq 0.$$

This completes the proof.

DEFINITION 5. *Let A be an arbitrary symmetric set in the product space $X \times X$. The projection of the set $(\{x\} \times X) \cap A$ to X is called the x-section of A and denoted by A_x.*

Let F be a symmetric closed set in $X \times X$. Put

$$\mathfrak{F} = \{f(x)g(y) \; ; \; f, g \in C_K^+ \cap D, \; S_f \subset \mathscr{C}F_{S_g} \text{ and } S_g \subset \mathscr{C}F_{S_f}\},$$

where

$$F_{S_g} = \cup \{F_x \; ; \; x \in S_g\}.$$

Let $C_K(\mathscr{C}F)$ be the space of complex-valued continuous functions in $\mathscr{C}F$ with compact support provided with the topology of uniform convergence.

LEMMA 6. $\mathfrak{F}$ *is total*[16] *in* $C_K(\mathscr{C}F)$.

Proof. It is sufficient to prove that for each point in $\mathscr{C}F$, there exists a base of open neighborhoods of the point satisfying the following condition. For any couple of ω_1 and ω_2 of the base of open neighboods with

$$\omega_1 \subset \bar{\omega}_1 \subset \omega_2,$$

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[16] This means that the space of linear combinations in $\mathfrak{F}$ is dense in $C_K(\mathscr{C}F)$.

there exists a function $f(x)g(y)$ in $\mathfrak{F}$ such that $f(x)g(y) = 1$ in ω_1 and $f(x)g(y) = 0$ on $\mathscr{C}\omega_2$. We put

$$\mathfrak{N}(x, y) = \{ U(x) \times U(y) \; ; \; U(x) \times U(y) \subset \mathscr{C}F\},$$

where $U(x)$ and $U(x)$ are open neighborhoods of x and y, respectively.   Then $\mathfrak{N}(x, y)$ is a base of open neighborhoods of $(x, y) \in \mathscr{C}F$.   For a couple of $U(x_0) \times U(y_0)$ and $V(x_0) \times V(y_0)$ in $\mathfrak{N}(x_0, y_0)$ with

$$\overline{U(x_0)} \subset V(x_0) \quad \text{and} \quad \overline{U(y_0)} \subset V(y_0),$$

we construct a function $f(x)g(y)$ as above.   Since $C_K^+ \cap D$ is dense in $C_K^{+[17)}$, there exists functions f_1 and g_1 in $C_K^+ \cap D$ such that $f_1(x) \geq 0$ in $U(x_0)$, $f_1(x) = 0$ on $\mathscr{C}V(x_0)$, $g_1(x) \geq 1$ in $U(y_0)$ and $g_1(x) = 0$ on $\mathscr{C}V(y_0)$.   Let T be the projection of the complex plane to the closed interval $[0, 1]$.   Then T is a normal contraction.   Put

$$f(x) = Tf_1(x) \quad \text{and} \quad g(x) = Tg_1(x).$$

Then f and g is contained in $C_K^+ \cap D$,

$$f(x)g(y) = 1$$

in $U(x_0) \times U(y_0)$ and

$$f(x)g(y) = 0$$

on $\mathscr{C}(V(x_0) \times V(y_0))$.   Next we shall prove that

$$S_f \subset \mathscr{C}F_{S_g} \quad \text{and} \quad S_g \subset \mathscr{C}F_{S_f}.$$

It is sufficient to show that

$$F_{V(x_0)} \cap V(y_0) = \emptyset \quad \text{and} \quad F_{V(y_0)} \cap V(x_0) = \emptyset.$$

Suppose that $F_{V(x_0)} \cap V(y_0) \neq \emptyset$.   Then there exists a point x_1 in $V(x_0)$ such that $F_{x_1} \cap V(y_0) \neq \emptyset$.   Let x_2 be a point of $F_{x_1} \cap V(y_0)$.   Since

$$(x_1, x_2) \in \{x_1\} \times F_{x_1} \subset F$$

and

$$(x_1, x_2) \in V(x_0) \times V(y_0),$$
$$V(x_0) \times V(y_0) \cap F \neq \emptyset.$$

This contradicts our assumption and hence $F_{V(x_0)} \cap V(y_0) = \emptyset$. Similarly $F_{V(y_0)} \cap V(x_0) = \emptyset$. This completes the proof.

3. First main theorems

First we give the following definition.

DEFINITION 6. *A symmetric closed set F in $X \times X$ is said to be regular if the mapping: $x \in X \to F_x$ is continuous, that is, for any open set ω containg F_x, there exists a neighborhood $U(x)$ of x such that $F_y \subset \omega$ for any y in $U(x)$.*

One of our main theorems is the following

THEOREM 1. *Let D be a Dirichlet space with respect to X and ξ, and let F be a regular closed set in $X \times X$ such that for any x in X, the x-section F_x is compact and F contain the diagonal set δ of $X \times X$. Then the following two conitions are equivalent.*

(I. 1) *The singular measure σ of D is supported by F.*

(I. 2) *For a function u in D and a point x_0 in X, if there exists an open set ω such that $\omega \supset F_{x_0}$ and $u(x) = 0$ p.p. in ω, then $x_0 \notin \mathfrak{S}(u)$.*

Proof. First we prove the implication (I. 1) $\Longrightarrow$ (I. 2). We can take another open set ω_1 such that

$$F_{x_0} \subset \omega_1 \subset \bar{\omega}_1 \subset \omega,$$

because F_{x_0} is compact. By Lemmas 1 and 2, there exists a sequence (u_n) of $C_K \cap D$ such that $S_{u_n} \subset \mathscr{C}\bar{\omega}_1$ and (u_n) converges strongly to u in D as $n \to \infty$. Since F is regular, there exists an open neighborhood $V(x_0)$ of x_0 such that $F_y \subset \omega_1$ for any y in $V(x_0)$. For any f in $C_K \cap D$ with support in $V(x_0)$,

$$(u_n, f) = \iint (u_n(x) - u_n(y))(\bar{f}(x) - \bar{f}(y)) d\sigma(x, y)$$

$$= -2 \iint u_n(x) \bar{f}(y) d\sigma(x, y) = 0.$$

Here we used the symmetricity of σ. By this equality, it follows that $\mathfrak{S}(u_n) \subset \mathscr{C}V(x_0)$ for any n. Consequently

$$\mathfrak{S}(u) \subset \mathscr{C}V(x_0).$$

Next we prove the converse. Suppose that $F \not\supset S\sigma$. By Lemma 5, there exists an element $f(x)g(y)$ in $\mathfrak{F}$ such that

$$\iint f(x) g(y) \, d\sigma(x, y) > 0.$$

On the other hand by our assumption,

$$F_{S_g} \subset \mathscr{C} S_f, \text{ so that } S_g \subset \mathscr{C} \mathfrak{S}(f).$$

Therefore (f, g) vanishes.   This contradicts our assumption and the proof is complete.

Applying the above theorem, we obtain the following

THEOREM 2.   *Let D and F be same as in Theorem 1.   Then the condition* (I.1) *is satisfied in D if and only if the following condition* (II. 1) *is satisfied in D.*

(II. 1)   *For any pure potential u_μ in D and any closed neighborhood ω of S_μ, let $u_{\mu'}$ be the balayaged potential of u_μ to $\mathscr{C}\omega$.   Then*

$$S_{\mu'} \subset F\omega \cap \overline{\mathscr{C}\omega}.$$

Proof.   By Theorem 1, it is sufficient to prove that the condition (I. 2) and (II. 1) are equivalent.   First we prove the implication (I. 2) $\Rightarrow$ (II. 1).   For a pure potential u_μ and a closed neighborhood ω of S_μ, let $u_{\mu'}$ be the balayaged potential of u_μ to $\mathscr{C}\omega$.   It is sufficient to prove

$$\int f(x) \, d\mu' = 0$$

for any f in $C_K^+ \cap D$ with support in $\mathscr{C}F\omega$.   Since $f(x) = 0$ in some neighborhood of $F\omega$, $\mathfrak{S}(f) \subset \mathscr{C}\omega$ by the condition (I. 2).

(1)   The case that S_μ and ω are compact.   We can take an open set ω_1 in X such that

$$\bar{\omega} \subset \omega_1 \subset \bar{\omega}_1 \subset \mathscr{C}\mathfrak{S}(f).$$

By Lemma 1, there exists a sequence (u_n) in $C_K \cap D$ supported by ω_1 such that (u_n) converges strongly to $u_\mu - u_{\mu'}$ in D, because

$$u_\mu^*(x) - u_{\mu'}^*(x) = 0$$

p.p.p. on $\mathscr{C}\omega_1$.   Obviously $(u_n, f) = 0$, so that

$$(u_\mu - u_{\mu'}, f) = 0.$$

That is,

$$(u_\mu - u_{\mu'},\, f) = \int f(x)\,d\mu - \int f(x)\,d\mu'$$

$$= -\int f(x)\,d\mu' = 0.$$

(2) The general case. Let K be a compact set in X. Let μ_K be the restriction of μ to K. We take an increasing net $(\omega_\alpha)_{\alpha \in I}$ of compact neighborhoods of S_{μ_K} such that $(\omega_\alpha)_{\alpha \in I} \nearrow \omega$. By the case (1),

$$S_{\mu'_{K,\alpha}} \subset F_{\omega_\alpha} \subset F_\omega,$$

where $\mu'_{K,\alpha}$ is the balayaged measure of μ_K to $\mathscr{C}\omega_\alpha$. Let μ'_K be the balayaged measure to $\mathscr{C}\omega$. Since the net $(u_{\mu'_{K,\alpha}})$ converges strongly to $u_{\mu'_K}$ in D,

$$S_{\mu'_K} \subset F_\omega \cap \overline{\mathscr{C}\omega}.$$

Next making K tend to X, we obtain that $(u_{\mu'_K})$ converges strongly to $u_{\mu'}$ in D. Consequently

$$S_{\mu'} \subset F_\omega \cap \overline{\mathscr{C}\omega}.$$

Now we prove the implication $(\mathrm{II}.\,1) \Longrightarrow (\mathrm{I}.\,2)$. For a function u in D, suppose that there exists a point x_0 such that $u(x) = 0$ $p.p.$ in some open set ω containing F_{x_0}. By the regularity of F, there exists an open neighborhood $V(x_0)$ such that $F_y \subset \omega$ for each y in $V(x_0)$. We take another neighborhood $V_1(x_0)$ such that

$$V_1(x_0) \subset \overline{V_1(x_0)} \subset V(x_0).$$

For a pure potential u_μ, let u' be the projection of u_μ to $D^{(1)}_{\mathscr{C}V_1(x_0)}$. If μ is supported by $V_1(x_0)$, $\mathfrak{S}(u')$ is contained in ω by Lemma 3. Hence

$$(u,\, u_\mu - u') = 0,$$

because we can take a sequence (u_n) in $C_K \cap D$ such that it converges strongly to u in D and

$$S_{u_n} \subset \mathscr{C}(\mathfrak{S}(u') \cup V_1(x_0)).$$

Since the space of linear combinations of such elements $u_\mu - u'$ is dense in $D^{(2)}_{V_1(x_0)}$, $(u, v) = 0$ for any v in $D^{(2)}_{V_1(x_0)}$. Therefore $x_0 \notin \mathfrak{S}(u)$. This completes the proof.

Especially when $F = \delta$, we obtain the following.

Theorem 2′. *Let D be the same as in Theorem 1. The following two conditions are equivalent.*

(II′. 1) *For any pure potential u_μ in D and any neighborhood ω of S_μ, let $u_{\mu'}$ be the balayaged potential of u_μ to $\mathscr{C}\omega$. Then*

$$S_{\mu'} \subset \partial\omega.$$

(II′. 2) *For any f, g in $C_K \cap D$,*

$$(f, g) = \int f\bar{g}\,d\nu + N(f, g).$$

Now we consider the relation between the supports of condensor measures and that of the singular measure of D.

Theorem 3. *Let D and F be the same as in Theorem 1. Let ν be the equilibrium measure of X. Then the following conditions are equivalent:*

(III. 1) *The singular measure σ of D is supported by F.*

(III. 2) *For any couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being relatively compact, let u_μ be the condensor potential with respect to ω_1 and ω_0. Then*

$$S_{\mu^+} \subset (F_{\mathscr{C}\omega_1} \cup S_\nu) \cap \bar{\omega}_1$$

and

$$S_{\mu^-} \subset F_{\mathscr{C}\omega_0} \cap \bar{\omega}_0.$$

Proof. First we shall prove the implication (III. 1) $\Longrightarrow$ (III. 2). By Theorem 1, it is obvious that

$$S_{\mu^-} \subset F_{\mathscr{C}\omega_0}.$$

That is,

$$S_{\mu^-} \subset F_{\mathscr{C}\omega_0} \cap \bar{\omega}_0.$$

On the other hand, for a function f with support in $\mathscr{C}(F_{\mathscr{C}\omega_1} \cup S_\nu) \cap \omega_1$, we prove that $(u_\mu, f) = 0$. It is obvious that the element u_μ is the element whose norm is minimum in the set $\overline{E_{0,1}}$, where

$$E_{0,1} = \{f \in C_K \cap D \,;\, f(x) \geq 1 \text{ in } \omega_1 \text{ and } f(x) \leq 0 \text{ in } \omega_0\}.$$

Hence there exists a sequence (f'_n) in $E_{0,1}$ such that (f'_n) converges strongly to u_μ in D. Put

$$f_n(x) = Tf'_n(x),$$

where T is the projection of $\mathfrak{C}$ to the closed interval $[0, 1]$. Then f_n is contained in $E_{0,1}$ and the sequence (f_n) converges strongly to $Tu_\mu = u_\mu$ in D. We have

$$(f, f_n) = \int f \bar{f}_n d\nu + N(f, f_n) + \iint (f(x) - f(y))(\bar{f}_n(x) - \bar{f}_n(y)) d\sigma(x, y)$$

$$= \iint (f(x) - f(y))(f_n(x) - f_n(y)) d\sigma(x, y)$$

$$= 2\iint f(x)(f_n(x) - f_n(y)) d\sigma(x, y)$$

$$= 2\iint f(x)(1 - f_n(y)) d\sigma(x, y) = 0,$$

because the support of the function $f(x)(1 - f_n(y))$ is contained in $\mathscr{C}F_{\mathscr{C}\omega_1} \times \mathscr{C}\omega_1$, and

$$\{\mathscr{C}F_{\mathscr{C}\omega_1} \times \mathscr{C}\omega_1\} \cap F = \emptyset.$$

Therefore

$$S_{\mu^+} \subset \mathscr{C}(F_{\mathscr{C}\omega_1} \cup S_\nu) \cap \bar{\omega}_1.$$

Next we prove the implication (III. 2) $\Rightarrow$ (III. 1). For a function f in $C_K \cap D$, suppose that there exists a point x_0 such that $f(x) = 0$ in some open set ω containing F_{x_0}. By the regularity of F, there exists an open neighborhood $V(x_0)$ of x_0 such that $F_y \subset \omega$ for each y in $V(x_0)$. We take a couple of open neighborhoods $U_1(x_0)$ and $U_0(x_0)$ of x_0 such that

$$\overline{U_1(x_0)} \subset U_0(x_0) \subset \overline{U_0(x_0)} \subset V(x_0).$$

Let u_μ be the condensor potential with respect to $U_1(x_0)$ and $\mathscr{C}\overline{U_0(x_0)}$. By our assumption,

$$(u_\mu, f) = 0.$$

For any g in $C_K^+ \cap D$ with support in $V(x_0)$, as we may assume that

$$U_1(x_0) \supset S_g,$$

there exists a positive constant M such that

$$g(x) \leq M u_\mu(x)$$

$p.p.$ in X. By Lemma 5,

$$0 = M(u_\mu, f) \leq (g, f) \leq 0.$$

Hence $(g, f) = 0$, i.e., $V(x_0) \subset \mathscr{C}\mathfrak{S}(f)$. Using the same method as in the proof of Theorem 2, we can prove that the condition (I. 2) is satisfied. This completes

the proof.

Similarly as Theorem 2′, putting $F = \delta$, we obtain the following

THEOREM 3′. *Let D be the same as in Theorem 1. Then the following two conditions are equivalent:*

(III′. 1) *For any f, g in $C_K \cap D$,*

$$(f, g) = N(f, g).$$

(III′. 2) *For any couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being relatively ompact, let u_μ be the condensor potential with respect to ω_1 and ω_0. Then*

$$S_{\mu^+} \subset \partial\omega_1 \ \text{and} \ S_{\mu^-} \subset \partial\omega_0.$$

In this theorem, we remark that if the condition (III′. 2) is satisfied, the equilibribrium measure of X vanishes.

4. Preliminaries on special Dirichlet spaces and some lemmas

First we define a negative definite function in Beurling's sense.

DEFINITION 7.[18] *A complex-valued continuous function $\lambda(x)$ on a locally compact abelian group X is called a negative definite function if the following form*

$$\sum_{i,\,j=1}^{n} \{\lambda(x_i) + \overline{\lambda(x_j)} - \lambda(x_i - x_j)\} \rho_i \bar{\rho_j}$$

is non-negative for each set of n points $x_1, x_2, \ldots, x_n$ in X and n complex numbers $\rho_1, \rho_2, \ldots, \rho_n$.

With respect to a negative definite function, the following result is known[19]. Let $\lambda(x)$ be a negative definite function on X. Then $(\lambda(x))^{-1}$ is a measure of positive type if it is locally summable for the Haar measure.

DEFINITION 8.[20] *A Dirichlet space $D = D(X, \xi)$ is said to be special if X is a locally compact abelian group and ξ is the Haar measure on X, the following condition being satisfied:*

d) If $U_x u$ is the function obtained from u in D by the translation $x \in X$ (i.e., $U_x u(y) = u(y - x)$), then $U_x u$ is in D and $\|U_x u\| = \|u\|$.

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[18] Cf. [2], p. 210, and [4], pp. 8–9.
[19] Cf. [4], p. 8.
[20] Cf. [2], p. 215, and [4], p. 9.

Beurling and Deny [2] obtained the following important theorem.

BEURLING AND DENY'S 2ND REPRESENTATION THEOREM[21]. *For a special Dirichlet space D on X, there exists a real-valued negative definite function $\lambda(\hat{x})$ on the dual group $\hat{X}$ of X such that $(\lambda(\hat{x}))^{-1}$ is locally summable and*

$$(\text{R}'.\ 1) \qquad\qquad \|u\|^2 = \int |\hat{u}(\hat{x})|^2 \lambda(\hat{x})\, d\hat{x}$$

for any u in $C_K \cap D$, where $\hat{u}$ is the Fourier transform of u.

Conversely, such a negative definite function $\lambda(x)$ on $\hat{X}$ defines by means of (R′. 1), a special Dirichlet space on X.

Hereafter in this section, we assume that X is a locally compact abelian group. By the above-mentioned result with respect to a negative definite function and by Bochner's theorem, there exists, for each special Dirichlet space on X, a positive measure κ on X such that the generalized Fourier transform is equal to $(\lambda(x))^{-1}$. This positive measure is called the associated convolution kernel (or simply, kernel) of D.

LEMMA 7. *Let D be a special Dirichlet space on X. For each potential u_μ in D such that $\int d|\mu| < +\infty$, it holds that*

$$u_\mu = \kappa * \mu.$$

Proof. By Beurling and Deny's 2nd Representation Theorem, the transformation $u \to \hat{u}$ of $C_K \cap D$ into $L^2(\lambda)$ is isometric. Since $C_K \cap D$ is dense in D, there exists an element $\tilde{u}_\mu(\hat{x})$ in $L^2(\lambda)$ such that

$$\|u_\mu\|^2 = \int |\tilde{u}_\mu(\hat{x})|^2 \lambda(\hat{x})\, d\hat{x}.$$

By the above equality and Parseval's formula,

$$\int \tilde{u}_\mu(\hat{x})\overline{\hat{\varphi}}(\hat{x})\, \lambda(\hat{x})\, d\hat{x} = (u_\mu,\ \varphi)$$

$$= \int \overline{\varphi}\, d\mu = \int \hat{\mu}(\hat{x})\overline{\hat{\varphi}}(\hat{x})\, d\hat{x}.$$

for any φ in $C_K \cap D$. Since the set $\{\hat{\varphi}\ ;\ \varphi \in C_K \cap D\}$ is dense in $C_K(\hat{X})$,[22]

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[21] Cf. [4], pp. 9–14, and [2], p. 215.
[22] Cf. [4], p. 10.

$$\tilde{u}_\mu(\hat{x}) = \frac{\hat{\mu}(\hat{x})}{\lambda(\hat{x})}$$

p.p. in $\hat{X}$. On the other hand $\kappa*\mu$ exists and

$$\widehat{\kappa*\mu}(\hat{x}) = \frac{\hat{\mu}(\hat{x})}{\lambda(\hat{x})}$$

p.p. in $\hat{X}$, since the total mass of $|\mu|$ is finite. Hence

$$\tilde{u}_\mu(\hat{x}) = \widehat{\kappa*\mu}(\hat{x})$$

p.p. in $\hat{X}$. By Parseval's formula,

$$\int \overline{\varphi}(x) u_\mu(x) dx = (u_\mu,\ u_\varphi) = \int \tilde{u}_\mu(\hat{x}) \overline{\tilde{u}}_\varphi(\hat{x}) \lambda(\hat{x}) d\hat{x}$$

$$= \int \widehat{\kappa*\mu}(\hat{x}) \overline{\hat{\varphi}}(\hat{x}) d\hat{x} = \int \overline{\varphi}(x) d(\kappa*\mu)$$

for any φ in C_K. Consequently $u_\mu = \kappa*\mu$. This completes the proof.

LEMMA 8. *Let D be a special Dirichlet space on X. Put*

$$P_0 = \left\{ u_\mu \in C_K \cap D\ ;\ \int d|\mu| < +\infty \right\}.$$

Then P_0 is total in C_K.

Proof. Similarly as the proof of Lemma 6, it is sufficient to prove that for any couple of open sets ω_1 and ω_2 with $\overline{\omega}_1 \subset \omega_2$, ω_1 being relatively compact, there exists an element u_μ in P_0 such that

(1) $u_\mu(x) = 1$ in ω_1,

(2) $S_{u_\mu} \subset \omega_2$.

We take other relatively compact open sets ω_1' and ω_2' such that

$$\overline{\omega}_1 \subset \omega_1' \subset \overline{\omega}_1' \subset \omega_2' \subset \overline{\omega}_2' \subset \omega_2.$$

Let u_ν be the condensor potential associated with ω_1' and $\mathscr{C}\overline{\omega}_2'$. Then there exists a function φ in C_K such that

$$u_\nu*\varphi = 1 \text{ in } \omega_1 \text{ and } S_{u_\nu*\varphi} \subset \omega_2.$$

It is known that $u_\nu*\varphi$ is contained in D.[23] By Lemma 7,

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[23] Cf. [4], p. 7.

$$u_\nu * \varphi = u_{\nu * \varphi}.$$

since the total mass of $|\nu|$ is finite, it holds that

$$\int |\nu * \varphi(x)| \, dx < +\infty.$$

Hence $u_\nu * \varphi$ belongs to P_0.   This completes the proof.

LEMMA 9 (UNICITY THEOREM).   *Let D be a special Dirichlet space on X, κ be the kernel of D and let μ_1, μ_2 be measures on X such that*

$$\int d|\mu_1| < +\infty \quad and \quad \int d|\mu_2| < +\infty.$$

If $\kappa * \mu_1 = \kappa * \mu_2$, then $\mu_1 = \mu_2$.

Proof. By Lemma 8, it is sufficient to prove the equality

$$\int \mu_\mu(x) \, d\mu_1(x) = \int u_\mu(x) \, d\mu_2(x)$$

for any u_μ in P_0.   Since κ is symmetric,

$$\int u_\mu(x) \, d\mu_1(x) = \kappa * \mu * \breve{\mu}_1(0) = (\kappa * \mu_1) * \breve{\mu}(0)$$

$$= (\kappa * \mu_2) * \breve{\mu}(0) = \kappa * \mu * \breve{\mu}_2(0) = \int u_\mu(x) \, d\mu_2(x).$$

This completes the proof.

Remark.   The above result is evident, when both u_{μ_1} and u_{μ_2} are in D.

LEMMA 10.   *Let D be a special Dirichlet space on X and let κ be the kernel of D.   For each x in X and each closed neighborhood ω of x, there exists a positive measure ε'_x such that*

(B'. 1)   *ε'_x is supported by $\overline{\mathscr{C}\omega}$ and $\int d\varepsilon'_x \leq 1$,*

(B'. 2)   *$\kappa * \varepsilon_x = \kappa * \varepsilon'_x$ as a measure in $\mathscr{C}\omega$,*

(B'. 3)   *$\kappa * \varepsilon_x \geq \kappa * \varepsilon'_x$.*

Here ε_x is a unit measure at x.

Proof.   Without loss of generality we may assume that $x = 0$.   There exists a sequence (φ_n) in C_K such that

$$\varphi_n(x) \geq 0, \quad \int \varphi_n(x) \, dx = 1$$

and it converges vaguely to $\varepsilon(=\varepsilon_0)$ as $n \to \infty$. By Beurling and Deny's Balayage Theorem, there exists the balayaged measure φ_n' of the measure φ_n to $\mathscr{C}\omega$. For each n, the total mass of φ_n' being less than or equal to 1, the set $\{\varphi_n'\}$ is bounded in the vague topology. Hence we may assume that there exists a positive measure ε' to which (φ_n') tends vaguely. On the other hand since

$$\kappa * \varphi_n' \leq \kappa * \varphi_n$$

and the sequence $(\kappa * \varphi_n)$ converges vaguenely to κ, the set $\{\kappa * \varphi_n'\}$ is bounded in the vague topology. We may assume that there exists a positive measure η to which $(\kappa * \varphi_n')$ converges vaguely. Hereafter we prove that $\eta = \kappa * \varepsilon'$. Since the total mass of ε' is finite (≤ 1), $\kappa * \varepsilon'$ exists. It holds that

$$\kappa * \varphi * \overline{\check{\varphi}}(x) = (u_\varphi,\ U_x u_\varphi) = \int \frac{|\check{\varphi}(\hat{x})|^2}{\lambda(\hat{x})}\,(\hat{x}, x)\,d\hat{x}$$

for any φ in C_K, where $(\hat{x}, x)$ is a character of X. The function $\lambda^{-1}|\hat{\varphi}(\hat{x})|^2$ being integrable, $\kappa * \varphi * \overline{\check{\varphi}}(x)$ is 0 at infinity for each φ in C_K. Therefore

$$\int \varphi * \overline{\check{\varphi}}(x)\,d(\kappa * \varphi_n') = \int \kappa * \varphi * \overline{\check{\varphi}}(x)\,d\varphi_n' \to \int \kappa * \varphi * \overline{\check{\varphi}}(x)\,d\varepsilon',$$

because the boundedness of the total masses and the vague convergence to ε' of (φ_n') implies the convergence

$$\lim_{n \to \infty} \int f(x)\,d\varphi_n' = \int f(x)\,d\varepsilon'$$

for any complex-valued continuous function f with 0 at infinity. Hence

$$\int \varphi * \overline{\check{\varphi}}(x)\,d(\kappa * \varepsilon') = \int \varphi * \overline{\check{\varphi}}(x)\,d\eta.$$

The set $\{U_x \varphi * \overline{\check{\varphi}}\ ;\ \varphi \in C_K,\ x \in X\}$ being total in C_K, we have $\kappa * \varepsilon' = \eta$, which prove the conditions (B'.1), (B'.2) and (B'.3).

Remark. Similarly as above, for a positive measure μ with finite total mass and for an open set ω, there exists a positive measure μ' such that

(B''.1) μ' is supported by $\bar{\omega}$ and $\int d\mu' \leq \int d\mu$,

(B''.2) $\kappa * \mu' = \kappa * \mu$ as a measure in ω,

(B''.3) $\kappa * \mu' \leq \kappa * \mu$.

In this remark, the total mass of μ being finite, we obtain that the convolution $\kappa * \mu$ exists.

We call the measure μ' the balayaged measure of μ to ω, as same as in Beurling and Deny's Balayage Theorem.   It is an unknown problem if the balayaged measure μ to ω is unique when u_μ is not contained in D.

5. Second main theorems

In this section, we consider a special Dirichlet space on the n-dimensional Euclidean space $R^n (n \geq 1)$. Let $\lambda(x)$ be a real valued negative definite function in R^n such that λ^{-1} is locally summable.   Then by Levy-Khintchine's theorem, we have the following representation:

$$\lambda(x) = C + \sum_{i, j=1}^{n} a_{ij} x_i x_j + \int_{|y|>0} (1 - e^{2\pi i x \cdot y}) \, d\sigma_1(y),$$

where C is a non-negative constant, the second term is a positive quardratic form and σ_1 is a positive symmetric measure in $R^n - \{0\}$ such that

$$\int_{|x|<r} d\sigma_1 < +\infty \quad \text{and} \quad \int_{0<|x|<r} |x|^2 d\sigma_1(x) < +\infty$$

for any $r > 0$.

Similarly as Theorem 2, we obtain the following

THEOREM 4. *Let D be a special Dirichlet space on R^n and let $\lambda(x)$ be a negative definite function associated with D.   Given a symmetric (with respect to the origin) compact set K containing 0, the following conditions are equivalent.*

(IV. 1) *Let ε_r' be the balayaged measure of a unit measure ε at 0 to the outside of a closed ball $\overline{B}(0 ; r)$ with center at 0 and radius r.   Then*

$$S_{\varepsilon_r'} \subset (\overline{B}(0 ; r) + K) \cap \mathscr{C} B(0 ; r),^{[24]}$$

where $B(0 ; r)$ is the open ball with center at 0 and of radius r.

(IV. 2) $\sigma_1 = 0$ *as a measure in $\mathscr{C} K$.*

(IV. 3) *For each x in R^n and each closed neighborhood ω of x, let ε_x' be the balayaged measure of a unite measure ε_x at x to $\mathscr{C} \omega$.   Then*

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[24] Let A and B be subsets in R^n. $A+B$ means the set $\{x+y ; x \in A, y \in B\}$ and $A-B$ means the set $\{x-y ; x \in A, y \in B\}$.

$$S_{\varepsilon_x'} \subset (\omega + K) \cap \overline{\mathscr{C}\omega}.$$

Before the proof, we remark the following result. By Beurling and Deny's 2nd Representation Theorem, for any f, g in $C_K \cap D$,

$$(f, g) = \int \hat{f}(\hat{x}) \overline{\hat{g}}(\hat{x}) \lambda(\hat{x}) d\hat{x}.$$

By Parseval's formula,

$$(f, g) = C \int f(x) \overline{g}(x) dx + \sum_{i, j=1}^{n} a_{ij} \int \frac{\partial f}{\partial x_i} \frac{\overline{\partial g}}{\partial x_j} dx$$

$$+ \iint (f(x+y) - f(x))(\overline{g}(x+y) - \overline{g}(x)) d\sigma_1(y) dx$$

for any f, g in $C_k^2 \cap D$. Therefore we obtain that the singular measure σ of D is as follows:

$$\int f(x) g(y) d\sigma(x, y) = \iint f(x+y) g(x) d\sigma_1(y) dx$$

for any f, g in C_K such that

$$S_f \cap S_g = \emptyset.$$

Proof of Theorem 4. First we prove the implication $(\mathrm{IV}.1) \Rightarrow (\mathrm{IV}.2)$. Let κ be the kernal of D and $\sigma_m^{(1)}$ be the restriction of σ_1 to $\mathscr{C}B(0 ; m^{-1})$, where m is a positive number. Then $\sigma_m^{(1)}$ is a positive measure in R^n with finite total mass. By the definition of κ,

$$\hat{\kappa}(x) = (\lambda(x))^{-1}$$

$$= \left(C + \sum_{i, j=1}^{n} a_{ij} x_i x_j + \int_{|y|>0} (1 - e^{2\pi i x \cdot y}) d\sigma_1(y) \right)^{-1} \ p.p.,$$

where the symbol $\wedge$ denotes the Fourier transform. Hence

$$\hat{\kappa}(x) \left(C + \sum_{i, j=1}^{n} a_{ij} x_i x_j + \lim_{m \to \infty} \int (1 - e^{2\pi i x \cdot y}) d\sigma_m^{(1)}(y) \right) = 1 \ p.p.,$$

that is

$$\hat{\kappa}(x) \left(C + \sum_{i, j=1}^{n} a_{ij}' \frac{\widehat{\partial^2}}{\partial x_i \partial x_j}(x) + (\hat{\sigma}_m^{(1)}(0) - \hat{\sigma}_m^{(1)}(x)) \right) \to 1 \ p.p.$$

as $m \to \infty$, where

$$a_{ij}' = \frac{1}{4\pi^2} a_{ij}.$$

Put

$$\hat{\delta}_m(x) = \hat{\kappa}(x)\Big(C + \sum_{i,j=1}^{n} a'_{ij}\,\widehat{\frac{\partial^2}{\partial x_i \partial x_j}}\,(x) + (\hat{\sigma}_m^{(1)}(0) - \hat{\sigma}_m^{(1)}(x))\Big).$$

Then $\delta_m(x) \to 1$ $p.p.$ as $m \to \infty$. We shall show that $\delta_m \to 1$ in $\mathfrak{S}'$ [25]. For any f in $\mathfrak{S}$, [26]

$$|\hat{\delta}_m(x)f(x)| \leq C|\hat{\kappa}(x)||f(x)| + |\hat{\kappa}(x)|\Big|\sum_{i,j=1}^{n} a_{ij}x_i x_j\Big||f(x)|$$
$$+ |f(x)||\hat{\sigma}_m^{(1)}(0) - \hat{\sigma}_m^{(1)}(x)|$$
$$= C|\hat{\kappa}(x)||f(x)| + |\hat{\kappa}(x)|\Big|\sum_{i,j=1}^{n} a_{ij}x_i x_j\Big||f(x)| + 2|f(x)|\int \sin^2 x\cdot y\,d\sigma_m^{(1)}(y)$$
$$\leq C|\hat{\kappa}(x)||f(x)| + |\hat{\kappa}(x)|\Big|\sum_{i,j=1}^{n} a_{ij}x_i x_j\Big||f(x)|$$
$$+ C_1|\hat{\kappa}(x)||x|^2|f(x)| + C_2|\hat{\kappa}(x)||f(x)|,$$

where

$$C_1 = 2\int_{0<|y|<1} |y|^2 d\sigma_1(y), \text{ and } C_2 = 2\int_{y\geq 1} d\sigma_1.$$

The function $|\hat{\kappa}(x)|$ being slowly increasing, the last term of the above inequality is integrable. By Lebesgue's bounded convergence theorem,

$$\lim_{m\to\infty}\int \hat{\delta}_m(x)f(x)\,dx = \int f(x)\,dx$$

for any f in $\mathfrak{S}$. Evidently the set $\{\delta_m\}$ is bounded in $\mathfrak{S}'$. Hence

$$\hat{\delta}_m \to 1$$

in $\mathfrak{S}'$ as $m \to \infty$. Therefore the distribution

$$\delta_m = (C + \hat{\sigma}_m^{(1)}(0))\,\kappa + \sum_{i,j=1}^{n} a'_{ij}\frac{\partial^2}{\partial x_i \partial x_j} *\kappa - \kappa * \sigma_m^{(1)}$$

converges to ε in $\mathfrak{S}'$.

Now suppose that $\sigma_1 \neq 0$ in $\mathscr{C}K$. Then there exists an m_0 such that $\sigma_m^{(1)} \neq 0$ in $\mathscr{C}K$ for any $m \geq m_0$ and a function φ of class C^∞ with compact support such that $\varphi(x) \geq 0$, $S_\varphi \subset \mathscr{C}K$ and

$$\int \varphi(x)\,d\sigma_m^{(1)}(x) > 0$$

for any $m \geq m_0$. By the symmetricity of σ_1, there exists a positive number

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[25] $\mathfrak{S}'$ is the space of slowly increasing distributions.

[26] $\mathfrak{S}$ is the space of rapidly decreasing functions of class C^∞.

such that

$$\varphi * \sigma_{m_0}^{(1)}(x) > 0$$

on $\overline{B}(0 ; r)$ and

$$(\overline{B}(0 ; r) + K) \cap S_? = \emptyset.$$

By our assumption,

$$S_{\varepsilon_r'} \subset (\overline{B}(0 ; r) + K) \cap \overline{\mathscr{C}B(0 ; r)}.$$

On the other hand the following convolutions exist and

$$\delta_m * \varepsilon_r' = (C + \hat{\delta}_m^{(1)}(0)) \, \kappa * \varepsilon_r' + \sum_{i,j=1}^{n} a_{ij}' \frac{\partial^2}{\partial x_i \partial x_j} * \kappa * \varepsilon_r'$$
$$- \kappa * \varepsilon_r' * \sigma_m^{(1)},$$

because $\int d\varepsilon_r' \leq 1$. By the property (B'. 2) of the balayaged measure,

$$\delta_m * \varepsilon_r'(\varphi) = (C + \hat{\delta}_m^{(1)}(0)) \int \varphi d(\kappa * \varepsilon_r')$$

$$+ \sum_{i,j=1}^{n} a_{ij}' \int \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) d(\kappa * \check{\varepsilon}_r') + \int \varphi * \sigma_m^{(1)}(x) d(\kappa * \check{\varepsilon}_r')$$

$$= (C + \hat{\delta}_m^{(1)}(0)) \int \varphi d\kappa + \sum_{i,j=1}^{n} a_{ij}' \int \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) d\kappa$$

$$- \int \varphi * \sigma_m^{(1)}(x) d(\kappa * \check{\varepsilon}_r').$$

Hence

$$\delta_m * \varepsilon_r'(\varphi) - \delta_m(\varphi) = \int \varphi * \sigma_m^{(1)}(x) d(\kappa - \kappa * \check{\varepsilon}_r').$$

By the property (B'. 1) and (B'. 2) of the balayaged measure and Lemma 9,

$$\lim_{m \to \infty} \int \varphi * \sigma_m^{(1)}(x) d(\kappa - \kappa * \check{\varepsilon}_r') \geq \int \varphi * \sigma_{m_0}^{(1)}(x) d(\kappa - \kappa * \check{\varepsilon}_r') > 0.$$

on the other hand since $\delta_m \to \varepsilon$ in $\mathfrak{S}'$,

$$\delta_m * \varepsilon_r'(\varphi) - \delta_m(\varphi) \to 0$$

as $m \to \infty$. This is a contradiction.

Next we prove the implication (IV. 2) $\Rightarrow$ (IV. 3). Without loss of generality we may assume that $x = 0$. Let ω be the closed neighborhood of 0 and ε' be the balayaged measure of ε to $\mathscr{C}\omega$. By Lemma 10, we may assume that there exists a sequence (φ_n) in C_K such that it converges vaguely to ε, $\varphi_n(x) \geq 0$,

$$\dot{S}_{?n} \subset \omega$$

and the sequence (φ_n') of positive measures converges vaguely to ε', where φ_n' is the balayaged measure of the measure φ_n. By Theorem 2, $u_{\varphi_n'}$ being contained in D,

$$S_{\varphi_n'} \subset (\omega + K) \cap \overline{\mathscr{C}\omega}$$

for each n. Therefore

$$S_{\varepsilon'} \subset (\omega + K) \cap \overline{\mathscr{C}\omega}.$$

The implication (IV. 3) $\Rightarrow$ (IV. 1) is evident, and the proof is complete.

Similarly as Theorem 2', taking for K the set $\{0\}$, we immediately obtain the following

THEOREM 4'. *Let* D *be a special Dirichlet space on* R^n *and let* $\lambda(x)$ *be a negative definite function arrociated with* D. *The folllowing conditions are equivalent.*

(IV'. 1) *Let* ε_r' *be the same as in Theorem 4, (IV. 1). Then*

$$S_{\varepsilon_r'} \subset \partial B(0 \; ; \; r).$$

(IV'. 2) $\lambda(x) = C + \sum\limits_{i, j = 1}^{n} a_{ij} x_i x_j.$

(IV'. 3) *For each* x *in* R^n *and each closed neighborhood* ω *of* x, *let* ε_x' *be the same as in Theorem 4, (IV. 3). Then*

$$S_{\varepsilon_x'} \subset \partial\omega.$$

Next we shall consider relations between the supports of condensor measures and the support of σ_1. First we shall show the following two lemmas which we shall use later. Let C_K^∞ be the space of functions of class C^∞ with compact support.

LEMMA 11. *Let* D *be a special Dirichlet space on* R^n. *Then* $C_K^\infty \cap D$ *is dense both in* C_K *and* D.

This is evident from the result of Deny [4]. See p. 7.

LEMMA 12. *Let* D *be a special Dirichlet space on* R^n *and let* $\lambda(x)$ *be a negative definite function associated with* D *such that* $C = \lambda(0) \neq 0$. *Then for each increasing sequence* (ω_m) *of bounded open sets with* $\omega_m \to X$, *the equilibrium measure* μ_m *of* ω_m *converges vaguely to* C *as* $m \to \infty$,

Proof. By Lemma 8, for any φ in C_K, there exists a potential u_μ in P_0 such that

$$|\varphi(x)| \leq u_\mu(x).$$

Therefore

$$\int |\varphi(x)|\, d\mu_m \leq \int u_\mu(x)\, d\mu_m = (u_{\mu_{m}},\ u_\mu)$$

$$= \int u_{\mu_m}^*(x)\, d\mu \leq \int d|\mu|.$$

That is, the set μ_m is vaguely bounded.   Let μ_0 be a cluster point of $\{\mu_m\}$. Then for each u_μ in P_0,

$$\int u_\mu(x)\, d\mu_0 = \lim_{m \to \infty} \int u_\mu(x)\, d\mu_m = \lim_{m \to \infty} \int u_{\mu_m}^*(x)\, d\mu$$

$$= \int d\hat{\mu} = \hat{\mu}(0).$$

On the other hand since $(\lambda(x))^{-1}$ is finite continuous, the total mass of κ is finite.   It holds that

$$C \int u_\lambda(x)\, dx = C \int \kappa * \mu(x)\, dx = C\hat{\kappa}(0)\, \hat{\mu}(0) = \hat{\mu}(0).$$

Consequently

$$\int u_\mu(x)\, d\mu_0 = C \int u_\mu(x)\, dx.$$

Since P_0 is total in C_K, this shows that $\mu_0 = C$. This completes the proof.

THEOREM 5.   *Let D be a special Dirichlet space on R^n and let $\lambda(x)$ be the negative definite function associated with D.   Given a symmetric compact set K containing 0, the following three conditions are equivalent.*

(V. 1)   *For each couple of two open balls $B(0\ ;\ r)$ and $B(0\ ;\ R)(r < R \leq +\infty)$, let $\mu = \mu^+ - \mu^-$ be the condensor measure associated with $B(0\ ;\ r)$ and $\mathscr{C}\overline{B}(0\ ;\ R)$. Then*

$$S_{\mu^+} \subset (\mathscr{C}B(0\ ;\ r) + K) \cap \overline{B}(0\ ;\ r),$$
$$S_{\mu^-} \subset (\overline{B}(0\ ;\ R) + K) \cap \mathscr{C}B(0\ ;\ R).$$

(V. 2)   *$C = 0$ and $\sigma_1 = 0$ as a measure in $\mathscr{C}K$.*

(V. 3)   *For each couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being bounded, let $\mu = \mu^+ - \mu^-$ be the condensor measure associated with ω_1 and ω_0.   Then*

$$S_{\mu^+} \subset (\mathscr{C}\omega_1 + K) \cap \bar{\omega}_1,$$

$$S_{\mu^-} \subset (\mathscr{C}\omega_0 + K) \cap \bar{\omega}_0.$$

Proof. First we prove the implication (V. 1) $\Rightarrow$ (V. 2). Since K is compact, $C = \lambda(0) = 0$ by Lemma 12. For any f, g in $C_K \cap D$ with $S_f \cap S_g = \emptyset$,

$$(f, g) = -\frac{1}{2}\iint_{|y|>0} (f(x+y) - f(x))(\bar{g}(x+y) - \bar{g}(x))\, d\sigma_1(y)dx$$

$$= -\frac{1}{2}\iint_{|y|>0} (f(x+y)\bar{g}(x) + f(x)\bar{g}(x+y)\, dxd\sigma_1(y)$$

$$= -\frac{1}{2}\int_{|y|>0} (f*\bar{g}(x-y) + f*\bar{g}(y))\, d\sigma_1(y)$$

$$= -\int_{|y|>0} f*\bar{g}(y)\, d\sigma_1(y).$$

Suppose that $\sigma_1 \neq 0$ in $\mathscr{C}K$. Then there exists a positive number R_1 such that $\sigma_1 \neq 0$ in $\mathscr{C}(\bar{B}(0 ; R_1) + K)$. By Lemma 12, there exists a function φ in $C_K \cap D$ such that $\varphi(x) \geq 0$ in R^n,

$$S_\varphi \subset \mathscr{C}(\bar{B}(0 ; R_1) + K) \quad \text{and} \quad \int\varphi(x)\, d\sigma_1(x) > 0.$$

We take a positive number r and R such that

$$0 < R_1 - R < r < R < R_1.$$

Let u_μ be the condensor potential associated with $B(0 ; r)$ and $\mathscr{C}\bar{B}(0 ; R)$, and let ψ be a function in C_K such that $\psi(x) \geq 0$,

$$\int\psi(x)\, dx = 1 \quad \text{and} \quad S_\psi \subset B(0 ; R_1 - R).$$

The function $u_\mu*\psi$ is in $C_K^\infty \cap D$ and

$$S_{u_\mu*\psi} \subset \bar{B}(0 ; R_1) + K.$$

Hence

$$(u_\mu*\psi, \varphi) = \int\varphi(x)\,\mu*\psi(x)\, dx = 0.$$

On the other hand, since

$$u_\mu*\psi(x) = u_{\mu*\psi}(x) = 1 \quad \text{in} \quad B(0 ; r - R_1 + R),$$

we obtain

$$\int_{|x|>0} \varphi*u_\mu*\psi(x)\, d\sigma_1(x) > 0.$$

This is a contradiction.

By Theorem 3, the implication (V. 2) $\Rightarrow$ (V. 3) is evident.

The implication (V. 3) $\Rightarrow$ (V. 1) is evident and the proof is complete.

Similarly as Theorem 3', we obtain the following

THEOREM 5'. *Let D and $\lambda(x)$ be the same as in Theorem 5. Then the following three conditions are equivalent.*

(V'. 1) *For each couple of open balls $B(0 ; r)$ and $B(0 ; R)$ $(r < R \leq + \infty)$, let $\mu = \mu^+ - \mu^-$ be the condensor measure associated with $B(0 ; r)$ and $\mathscr{C}\overline{B}(0 ; R)$. Then*

$$S_{\mu^+} \subset \partial B(0 ; r) \ and \ S_{\mu^-} \subset \partial B(0 ; R).$$

(V'. 2)
$$\lambda(x) = \sum_{i, j = 1}^{n} a_{ij} x_i x_j.$$

(V'. 3) *For each couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being bounded, let $\mu = \mu^+ - \mu^-$ be the condensor measure associated with ω_1 and ω_0. Then*

$$S_{\mu^+} \subset \partial \omega_1 \ and \ S_{\mu^-} \subset \partial \omega_0.$$

We have immediately the following application.

PROPOSITION. *Let $\lambda(x)$ be a real valued negative definite function on R^n such that it is non-constant and $(\lambda(x))^{-1}$ is locally summable. Let D_α be the special Dirichlet space associated with a negative definite function $\lambda_\alpha(x) = (\lambda(x))^\alpha (0 < \alpha < 1)$[27]. Then there exists a symmetric positive measure σ_α in $R^n - \{0\}$ and a non-negative constant C_α such that*

$$\lambda_\alpha(x) = C_\alpha + \int_{|y| > 0} (1 - e^{2\pi i x \cdot y}) d\sigma_\alpha(y).$$

Furthermore, in D_α the following conditions are satisfied: (i) *Let ϵ'_x be the balayaged measure ϵ_x to the outside of a closed neighborhood ω of x. Then*

$$S_{\epsilon'_x} \neq \partial \omega.$$

(ii) *For each couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being bounded, let $\mu = \mu^+ - \mu^-$ be the condensor measure associated with ω_1 and ω_0. Then*

$$S_{\mu^+} \neq \partial \omega \ and \ S_{\mu^-} \neq \partial \omega_0.$$

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[27] When $\lambda(x)$ is negative definite, $(\lambda(x))^\alpha$ is negative definite for each $0 < \alpha < 1$. Cf. [4], p. 9.

Proof. Since $\lambda_\alpha(x)$ is negative definite, by Levy-Khintchine's theorem,

$$(\lambda(x))^\alpha = C_\alpha + \sum_{i,j=1}^n a_{ij}^{(\alpha)} x_i x_j + \int_{|y|>0} (1 - e^{2\pi i x \cdot y})\, d\sigma_\alpha(y).$$

Since $0 < \alpha < 1$,

$$\frac{(\lambda(x))^\alpha}{|x|^2} \to 0$$

as $|x| \to \infty$. Hence $a_{ij}^{(\alpha)} = 0$ for any i, j. The conditions (i) and (ii) are evidently satisfied because $\sigma_\alpha \neq 0$. This completes the proof.

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