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LINEAR COMBINATIONS OF UNIVALENT FUNCTIONS WITH COMPLEX COEFFICIENTS

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Let U be the class of all normalized analytic functions

$$f(z) = z + a_2z^2 + \ldots + a_nz^n + \ldots,$$

where $z \in E = \{z : |z| < 1\}$ and f is univalent in E. Let K denote the sub-class of U consisting of those members that map E onto a convex domain. MacGregor [**2**] showed that if $f_1 \in K$ and $f_2 \in K$ and if

$$F(z) = \lambda f_1(z) + (1 - \lambda)f_2(z), \tag{1}$$

then $F \notin K$ when λ is real and $0 < \lambda < 1$, and the radius of univalency and starlikeness for F is $1/\sqrt{2}$.

In this paper, we examine the expression (1) when $f_1 \in K, f_2 \in K$ and λ is a complex constant and find the radius of starlikeness for such a linear combination of complex functions with complex coefficients. Interest in such a problem is sparked by examples of such functions as

$$f_1(z) = \frac{z}{1 - \eta z}$$

and

$$f_2(z) = \frac{1}{\eta - \xi} \cdot \log \frac{1 - \eta z}{1 - \xi z},$$

where $|\eta| = |\xi| = 1$ and $\eta \neq \xi$. If

$$F(z) = \frac{\eta + \xi}{\eta - \xi} \cdot f_1(z) + \frac{-2\xi}{\eta - \xi} \cdot f_2(z), \tag{2}$$

then by direct calculation F is close-to-convex (and thus univalent) in E relative to f_1.

The method used by MacGregor for the case when the coefficients are real did not lend itself to the more general problem when the coefficients are complex. An approach used by Labelle and Rahman [**1**] to find the radius of convexity for the arithmetic mean of two convex functions is used here to prove:

THEOREM 1. *Let $f_1 \in K$ and $f_2 \in K$ and*

$$F(z) = \lambda f_1(z) + (1 - \lambda)f_2(z), \tag{3}$$

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where $z \in E$ *and* $0 \leqq \alpha = \arg[\lambda/(1-\lambda)] < \pi$. *Then* $\operatorname{Re}\{zF'(z)/F(z)\} > 0$ *if* $|z| < (\cos \alpha/4 - \sin \alpha/4)/\sqrt{2}$, *and the bound is best possible.*

In addition, we use the same techniques to prove

THEOREM 2. *If* f_1, f_2 *and* F *are as in Theorem* 1, *then*

$$\operatorname{Re}\{1 + zF''(z)/F'(z)\} > 0$$

if $|z| < R_c$, *where* R_c *is the least positive value of* r *that satisfies the equation:*

$$1 - 2\sec(\alpha/2 + 2\sin^{-1}r) + r^2 = 0. \tag{4}$$

We note that if λ is real ($\alpha = 0$), then the result of Theorem 1 agrees with the bound found by MacGregor, and Theorem 2, for $\lambda = 1/2$, coincides with the findings of Labelle and Rahman. The bound in Theorem 2 is not best possible as noted by Labelle and Rahman when $\lambda = 1/2$.

The proofs of these theorems rest upon several lemmas which we state and prove at the outset.

LEMMA 1. *If* $|w - a| < d$ *where* a *and* d *are real and* $a > d \geqq 0$, *and* w_0 *is a given complex number, then*

$$\operatorname{Re} ww_0 \geqq |w_0| \, [a \cos(\arg w_0) - d]. \tag{5}$$

This lemma was used by Labelle and Rahman and its verification is done as a direct calculation.

LEMMA 2. *If* $|u - a| \leqq d$ *and* $|v - a| \leqq d$ *where* a *and* d *are real and* $a > d \geqq 0$, *and*

$$w = u\frac{1}{1 + Ae^{i\alpha}} + v\frac{1}{1 + A^{-1}e^{-i\alpha}},$$

where A *is real and* $A > 0$ *and* $\alpha \in [0, \pi)$, *then*

$$\operatorname{Re} w \geqq a - d \sec \alpha/2. \tag{7}$$

Proof. Since

$$\frac{1}{|1 + Ae^{i\alpha}|} = \frac{1}{(1 + 2A\cos\alpha + A^2)^{1/2}},$$

$$\frac{1}{|1 + A^{-1}e^{-i\alpha}|} = \frac{A}{(1 + 2A\cos\alpha + A^2)^{1/2}},$$

$$\cos\left[\arg\left(\frac{1}{1 + Ae^{i\alpha}}\right)\right] = \frac{1 + A\cos\alpha}{(1 + 2A\cos\alpha + A^2)^{1/2}}$$

and

$$\cos\left[\arg\left(\frac{1}{1 + A^{-1}e^{-i\alpha}}\right)\right] = \frac{\cos\alpha + A}{(1 + 2A\cos\alpha + A^2)^{1/2}},$$

then, by Lemma 1,

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$$\operatorname{Re} w \geqq \frac{1}{(1 + 2A \cos \alpha + A^2)^{1/2}} \left[a \cdot \frac{1 + A \cos \alpha}{(1 + 2A \cos \alpha + A^2)^{1/2}} - d \right]$$
$$+ \frac{A}{(1 + 2A \cos \alpha + A^2)^{1/2}} \left[a \cdot \frac{\cos \alpha + A}{(1 + 2A \cos \alpha + A^2)^{1/2}} - d \right],$$

or

$$\operatorname{Re} w \geqq a - d \cdot \frac{1 + A}{(1 + 2A \cos \alpha + A^2)^{1/2}}$$
$$\geqq a - d \cdot \frac{2}{(2 + 2 \cos \alpha)^{1/2}};$$

i.e., $\operatorname{Re} w \geqq a - d \sec \alpha/2$.

LEMMA 3. *If* $\operatorname{Re} P(z) > 0$ *for* $|z| < \rho < 1$ *and* $P(0) = 1$, *then*

$$\left| P(z) - \frac{1 + r^2/\rho^2}{1 - r^2/\rho^2} \right| \leqq \frac{2r/\rho}{1 - r^2/\rho^2}$$

for $|z| \leqq r < \rho$.

This lemma is proved by writing $P(z) = (1 + w(z))/(1 - w(z))$ where $|w(z)| < 1$ and $w(0) = 0$ and then noting that $|w(z)| \leqq |z|/\rho$ as a natural extension of Schwarz's Lemma for $|z| < \rho < 1$.

Proof of Theorem 1. From equation (3),

$$\frac{zF'(z)}{F(z)} = \frac{\lambda z f_1'(z) + (1 - \lambda) z f_2'(z)}{\lambda f_1(z) + (1 - \lambda) f_2(z)}$$
$$= \frac{z f_1'(z)}{f_1(z)} \cdot \left[1 + \left(\frac{\lambda}{1 - \lambda} \cdot \frac{f_1(z)}{f_2(z)} \right)^{-1} \right]^{-1}$$
$$+ \frac{z f_2'(z)}{f_2(z)} \cdot \left[1 + \frac{\lambda}{1 - \lambda} \cdot \frac{f_1(z)}{f_2(z)} \right]^{-1}$$

Now, because f_1 and f_2 are convex, then from [**3**; **4**]

$$\left| \frac{z f_i'(z)}{f_i(z)} - \frac{1}{1 - r^2} \right| \leqq \frac{r}{1 - r^2}, \tag{8}$$

and $|\arg(f_i(z)/z)| \leqq \sin^{-1} r$ for $i = 1, 2$ and $|z| \leqq r$. Thus, from Lemmas 1 and 2,

$$\operatorname{Re} \frac{zF'(z)}{F(z)} \geqq \frac{1}{1 - r^2} - \frac{r}{1 - r^2} \sec \beta/2,$$

where

$$\beta = \arg \left[\frac{\lambda}{1 - \lambda} \cdot \frac{f_1(z)}{f_2(z)} \right] = \alpha + \arg \left[\frac{f_1(z)}{z} \right] - \arg \left[\frac{f_2(z)}{z} \right].$$

Now, by (8), $|\beta| \leqq \alpha + 2\sin^{-1}r$. If $r < \cos\alpha/2$ then $0 \leqq \beta < \pi$, and

$$\sec\beta/2 \leqq \sec(\alpha/2 + \sin^{-1} r)$$
$$= \frac{1}{(1-r^2)^{1/2}\cos\alpha/2 - r\sin\alpha/2}.$$

Consequently, $\mathrm{Re}(zF'(z)/F(z)) > 0$ if

$$\frac{1}{1-r^2} - \frac{r}{1-r^2}\frac{1}{(1-r^2)^{1/2}\cos\alpha/2 - r\sin\alpha/2} > 0$$

and

$$r < \cos\alpha/2$$

or

$$r < \min_{\alpha}\left[\left(\frac{1-\sin\alpha/2}{2}\right)^{1/2}, \cos\alpha/2\right]$$
$$= \left(\frac{1-\sin\alpha/2}{2}\right)^{1/2}$$
$$= \frac{\cos\alpha/4 - \sin\alpha/4}{\sqrt{2}},$$

and the proof of the theorem is complete.

We note that the bound in Theorem 1 is best possible; i.e.,

$$[(1-\sin\alpha/2)/2]^{1/2}$$

is the radius of starlikeness for the set of all functions represented as in equation (3). If we let

$$f_1(z) = \frac{z}{1-ze^{i\gamma}},$$
$$f_2(z) = \frac{z}{1-ze^{-i\gamma}},$$
$$\frac{\lambda}{1-\lambda} = e^{i\alpha},$$

and

$$\gamma = \frac{\pi+\alpha}{4},$$

then

$$F'(z) = \frac{\lambda}{(1-ze^{i\gamma})^2} + \frac{(1-\lambda)}{(1-ze^{-i\gamma})^2}.$$

Now since

$$\lambda = \frac{1+i\tan\alpha/2}{2}, \quad 0 < \alpha < \pi,$$

we find that $F'(z_0) = 0$ when

$$z_0 = \left(\frac{1 - \sin \alpha/2}{2}\right)^{1/2}$$
$$= \frac{\cos \alpha/4 - \sin \alpha/4}{\sqrt{2}}.$$

Thus, F as represented as a linear combination of normalized convex functions with the given restriction on the coefficients is not univalent for $|z| < r$ if $r > |z_0|$.

Proof of Theorem 2. By direct calculation,

$$1 + \frac{zF''(z)}{F'(z)} = \left[1 + \frac{zf_1''(z)}{f_1'(z)}\right] \cdot \left[1\left(+ \frac{\lambda}{1-\lambda} \cdot \frac{f_1'(z)}{f_2'(z)}\right)^{-1}\right]^{-1}$$
$$+ \left[1 + \frac{zf_2''(z)}{f_2'(z)}\right] \cdot \left[1 + \frac{\lambda}{1-\lambda} \cdot \frac{f_1'(z)}{f_2'(z)}\right]^{-1}$$

Since f_1 and $f_2 \in K$, $\mathrm{Re}(1 + zf_i''(z)/f_i'(z)) > 0$, $i = 1, 2$, and

$$|\arg f_i'(z)| \leqq 2 \sin^{-1}|z|$$

[3; 4], then from Lemma 2

$$\mathrm{Re}\left[1 + \frac{zF''(z)}{F'(z)}\right] \geqq \frac{1 + r^2}{1 - r^2} - \frac{2r}{1 - r^2} \sec \sigma/2,$$

where

$$\sigma = \arg\left[\frac{\lambda}{1-\lambda} \cdot \frac{f_1'(z)}{f_2'(z)}\right].$$

Then $|\sigma| < \alpha + 4 \sin^{-1} r$, or $\sigma \in [0, \pi)$ if $r < \sin(\pi - \alpha)/4$. Thus, for the values of r, $\sec \sigma/2 \leqq \sec(\alpha/2 + 2 \sin^{-1} r)$, and $\mathrm{Re}(1 + zF''(z)/F'(z)) > 0$ if $r < \min_\alpha[\sin(\pi - \alpha)/4, R_c]$, where R_c is the least possible value of r satisfying the equation:

$$\frac{1 + r^2}{1 + r^2} - \frac{2r}{1 - r^2} \sec\,(\alpha/2 + 2 \sin^{-1} r) = 0,$$

or $1 - 2r \sec(\alpha/2 + 2 \sin^{-1} r) + r^2 = 0$, which was to be shown.

We observe that if $\lambda = 1/2(\alpha = 0)$, the bound (not best possible) is that found by Labelle and Rahman.

Using Lemma 3, we have finally

THEOREM 3. *If* $F(z) = \lambda f_1(z) + (1 - \lambda)f_2(z)$, *where* f_1 *and* f_2 *are normalized* ($f_i(0) = 0$; $f_i'(0) = 1$, $i = 1, 2$) *analytic univalent functions and* $0 \leqq \alpha = \arg(\lambda/(1 - \lambda)) < \pi$, *then* $\mathrm{Re}\, F'(z) > 0$ *when*

$$|z| < (\sec \alpha/2 - \tan \alpha/2) \sin \pi/8.$$

This result agrees with the result in [**2**] when $\alpha = 0$.

Proof of Theorem 3. It is well-known that $|\arg f_1'(z)| \leqq 4 \sin^{-1}|z|$ for $|z| < 1/\sqrt{2}$. Thus, $\operatorname{Re} f_i'(z) > 0$ for $|z| < \min(\sin \pi/8, 1/\sqrt{2}) = \sin \pi/8$. Consequently, by Lemma 3,

$$\left| f_i'(z) - \frac{1 + r^2/\sigma^2}{1 - r^2/\sigma^2} \right| \leqq \frac{2r/\sigma}{1 - r^2/\sigma^2},$$

where $\sigma = \sin \pi/8$. Then

$$F'(z) = \left[1 + \left(\frac{\lambda}{1-\lambda}\right)^{-1}\right]^{-1} \cdot f_1'(z) + \left[1 + \frac{\lambda}{1-\lambda}\right]^{-1} \cdot f_2'(z),$$

and

$$\operatorname{Re} F'(z) \geqq \frac{1 + r^2/\sigma^2}{1 - r^2/\sigma^2} - \frac{2r/\sigma}{1 - r^2/\sigma^2} \sec \alpha/2$$

by Lemma 2, or $\operatorname{Re} F'(z) > 0$ if $|z| < \sin \pi/8(\sec \alpha/2 - \tan \alpha/2)$, or F is univalent for $|z| < \sin \pi/8(\sec \alpha/2 - \tan \alpha/2)$.

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