n-ANR'S FOR CERTAIN NORMAL SPACES

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Introduction. For various classes Q of metric spaces, there are several well-known results characterizing the local n-connectivity of a metric space in terms of n-ANR(Q)'s. Specifically, we have in mind the results of Kuratowski **(13**, p. 265**)** and Kodama **(10**, p. 79**)**. The main purpose of this paper will be to obtain similar results along these lines for non-metric classes Q. In the last part of the paper we specify Q to be the class of totally normal spaces and characterize the local n-connectivity of an n-dimensional separable metric space in terms of ANR(Q)'s.

Preliminaries. All spaces are assumed to be Hausdorff. By the term dim X we shall mean the covering dimension of X. Let Q be a class of topological spaces and n a non-negative integer. A topological space X is called an n-AR(Q) (n-ANR(Q)) if

(a) $X \in Q$ and

(b) whenever $Z \in Q$ and X is embedded as a closed subset of Z with dim $(Z - X) \leqslant n$, then X is a retract of Z (X is a retract of some neighbourhood of X in Z).

If we drop the dimension requirement in (b), X is simply called an AR(Q) (ANR(Q)). An example of Borsuk **(2**, p. 179**)** serves to show that a space can be an n-ANR(Q) for each n and yet fail to be an ANR(Q). A space X is called an n-ES(Q) (n-NES(Q)) if whenever $Y \in Q$, C is a closed subset of Y with dim $(Y - C) \leqslant n$, and $f\colon C \to X$ is a continuous mapping, then f has a continuous extension over Y (over some neighbourhood of C in Y) with respect to X. Again, if the dimension requirement is dropped, X is simply called an ES(Q) (NES(Q)). We define a space X to be locally connected in dimension n if for every point $x \in X$ and every neighbourhood U of x, there exists a neighbourhood $V \subset U$ of x such that every continuous map $f\colon S^n \to V$ of the n-sphere into V extends to a continuous map $g\colon E^{n+1} \to U$ of the $(n+1)$-cell into U. A space is said to be locally n-connected (i.e. LC^n) if it is locally connected in dimension q for every $q \leqslant n$. It is known **(13**, p. 287**)** that a locally contractible space is LC^n for every n. Finally, a metric space X is called an absolute G_δ if whenever X is embedded in a metric space M, X is a G_δ set in M.

Our main objective is to prove the following theorem:

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THEOREM 1. *Let X be a separable metric space and an absolute G_δ. Let Q be any class of normal spaces containing the class of all metric spaces. Then the following statements are equivalent:*

(a) *X is* LC^{n-1}.

(b) *X is an* n-NES(Q).

(c) *X is an* n-ANR(Q).

(d) *If* $Y \in Q$, $\dim Y \leqslant n$, *and C is a closed subset of Y, then any continuous map* $f: C \to X$ *has a continuous extension* $F: V \to X$ *over some neighbourhood V of C in Y with respect to X.*

Remarks. For a separable metric space X, Theorem 1 has been proved for the class Q of separable metric spaces by Kuratowski in (**13**, p. 265). In (**10**, p. 79), Kodama later generalized these results to the case where X is a metric space and Q is the class of all metric spaces. Using the concept of an adjunction space and the results of Katětov in (**9**), McCandless in (**14**, p. 193) and (**15**, p. 205) proved Theorem 1 for the case where X is separable metric and Q is the non-metric class of perfectly normal spaces. Before we prove Theorem 1, some preliminary results are in order.

THEOREM 2 (Katětov). *Let C be a closed set of type G_δ in a normal space X such that* $\dim (X - C) \leqslant n$. *Let Y be a separable metric space and* $f: C \to Y$ *a continuous map. Then there exists a separable metric space S such that Y is closed in S with* $\dim (S - Y) \leqslant n$ *and a continuous extension* f^* *of f over X with respect to S.*

For the proof see (**9**, p. 510).

Remark. This result of Katětov is a generalization of Kuratowski's classical theorem in (**12**, p. 217).

LEMMA 1. *A separable Banach space is an* ES(*normal*).

Proof. In (**1**, p. 18), Arens showed that a Banach space is an AR (fully normal). Since a Banach space is metrizable, it is an AR(metric). Separability and completeness ensure that it is an ES(normal) by a theorem of Michael in (**16**, p. 793).

LEMMA 2. *Let X be a normal space, C a closed subset of X with* $\dim (X - C) \leqslant n$. *Let Y be a complete separable metric space and* $f: C \to Y$ *a continuous map. Then there exists a closed G_δ set C^* of X with* $C \subset C^*$, $\dim(X - C^*) \leqslant n$, *and a continuous extension F of f over C^* with respect to Y.*

Proof. By well-known results of Kuratowski (**11**, p. 543) and Wojdyslawski (**19**, p. 186), we may assume that Y is embedded in a separable Banach space B. By Lemma 1, B is an ES(normal). Hence the map $f: C \to Y$ has a continuous extension $g: X \to B$. Since Y is complete, Y is an absolute G_δ.

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Therefore

$$Y = \bigcap_{i=1}^{\infty} G_i$$

where G_i is open in B for each i.

We now use the normality of X to construct the closed G_δ set C^* of X such that $C \subset C^*$.

For each i, $g^{-1}(G_i)$ is open in X and $C \subset g^{-1}(G_i)$. Since X is normal, there exist maps $\phi_i: X \to I = [0, 1]$ such that $\phi_i(C) = 0$ and $\phi_i(X - g^{-1}(G_i)) = 1$. For each i, let $C_i = \phi_i^{-1}(0)$. Then each C_i is a closed G_δ set of X such that $C \subset C_i \subset g^{-1}(G_i)$. If we let

$$C^* = \bigcap_{i=1}^{\infty} C^i,$$

then C^* is a closed G_δ set of X and $C \subset C^*$.

We now show that $\dim(X - C^*) \leqslant n$. Since C^* is a closed G_δ set of X, $X - C^*$ is an open F_σ set of X, i.e.,

$$X - C^* = \bigcup_{j=1}^{\infty} F_j,$$

where F_j is closed in X for each j. By the normality of X, there exists K_j open in X such that $F_j \subset K_j \subset \mathrm{Cl}(K_j) \subset X - C^*$. It is easily seen that $\mathrm{Cl}(K_j)$ is normal in $X - C^*$ for each j. Moreover, since

$$\bigcup_{j=1}^{\infty} K_j = X - C^*,$$

$\{\mathrm{Cl}(K_i)\}$ is a sequence of closed sets of $X - C^*$ whose interiors cover $X - C^*$. Hence, by a lemma of Dowker (**4**, p. 475), $X - C^*$ is normal.

By hypothesis, $\dim(X - C) \leqslant n$. Since each F_j is closed in $X - C^*$ and hence in $X - C$, we have by Čech (**3**, p. 280) that $\dim F_j \leqslant n$ for each j. Čech's sum theorem for normal spaces (**3**, p. 291) gives $\dim(X - C^*) \leqslant n$.

Finally, we claim that $F = g|C^*$ is the desired continuous extension of f with respect to Y. Clearly F is a continuous extension of f over C^* by definition of g. We need only check that $g(C^*) \subset Y$, but this is easily verified by the definition of C^*.

We are now in a position to prove Theorem 1. Recall that X is separable metric and an absolute G_δ, and Q is *any* class of normal spaces containing the class of all metric spaces.

Proof of Theorem 1. (a) implies (b): We must show that X is an n-NES(Q). Let $Y \in Q$, C a closed subset of Y with $\dim(Y - C) \leqslant n$; and $f: C \to X$ a continuous map. We must find an extension of f over some neighbourhood of C in Y with respect to X.

By hypothesis, X is homeomorphic to a complete separable metric space, say X'. Let $h: X \to X'$ be the homeomorphism. Then we may use Lemma 2 on the map $h \circ f: C \to X'$ to get a closed G_δ set C^* of Y with $C \subset C^*$, $\dim(Y - C^*) \leqslant n$, and a continuous extension $F: C^* \to X'$ of $h \circ f$. We can now apply Theorem 2 of Katětov to obtain a separable metric space S such that X' is closed in S, $\dim(S - X') \leqslant n$, and a continuous extension g of F over Y with respect to S. That is, $g: Y \to S$ and $g|C^* = F$.

Since X is LC^{n-1}, X' is LC^{n-1} and so by Kuratowski **(13**, p. 265**)** we have that X' is an n-ANR(separable metric). Hence there exists a retraction $r: U \to X'$ where U is some neighbourhood of X' in S. Let $V = g^{-1}(U)$. Then V is a neighbourhood of C^* and hence of C in Y. Clearly, the composition $h^{-1} \circ r \circ g \mid V: V \to X$ is the desired continuous extension of f over a neighbourhood of C in Y with respect to X.

(b) implies (c). Since Q contains the class of all metric spaces, $X \in Q$ and so this implication follows from a fundamental property of NES(Q)'s **(6**, 2.11, p. 318**)**.

(c) implies (d). By hypothesis, X is separable metric. Since X is an n-ANR(Q), a fundamental property of ANR(Q)'s gives the result that X is an n-ANR(separable metric) **(6**, 2.13, p. 318**)**. By a result of Kuratowski in **(13**, p. 265**)**, X is LC^{n-1}. Thus the same argument as in "(a) implies (b)" goes through except that we observe that Lemma 2 remains true if dim $(X - C) \leqslant n$ is replaced by $\dim X \leqslant n$.

(d) implies (a). This implication follows easily from Kuratowski's result **(13**, p. 265**)**, for if (d) holds, then it holds a fortiori if "$Y \in Q$" is replaced by "Y is a separable metric."

A space X is said to be C^n if every continuous map of the i-sphere into X can be extended to a continuous map of the $(i + 1)$-cell into X for $i \leqslant n$. By slightly modifying the arguments in the proof of Theorem 1, the following theorem is easily proved:

THEOREM 3. *Let X be a separable metric space and an absolute G_δ. Let Q be any class of normal spaces containing the class of metric spaces. Then the following statements are equivalent:*

(a′) *X is LC^{n-1} and C^{n-1}.*

(b′) *X is an n-ES(Q).*

(c′) *X is an n-AR(Q).*

(d′) *If $Y \in Q$, $\dim Y \leqslant n$, and C is a closed subset of Y, then any continuous map $f: C \to X$ has a continuous extension $F: Y \to X$.*

In **(4**, p. 267**)**, Dowker introduced a new class of normal spaces called totally normal spaces. A normal space X is called totally normal if every open set G of X is the union of a collection $\{G_\lambda\}$, locally finite in G, of open F_σ sets of X. We recall that a perfectly normal space is a normal space in which every closed set is of type G_δ, and a completely normal space is a normal space in which every subspace is normal. To clarify the relationship between these classes of normal spaces, Dowker shows that every perfectly normal space is totally normal and every totally normal space is completely normal (**4**, pp. 273–6). As stated in the Introduction, our aim in the final part of this paper is to specify Q to be the class of totally normal spaces and prove the following theorem:

THEOREM 4. *Let X be an n-dimensional separable metric space. Then X is an* ANR(*totally normal*) *if and only if X is* LC^n *and an absolute G_δ.*

Before giving a proof, some observations are in order. An example of Hanner in (**7**, p. 381) and a result of Michael in (**16**, p. 793) serve to show that not every metric ANR(perfectly normal) is an absolute G_δ. We can, however, prove the following lemma:

LEMMA 3. *Every metric* ANR(*totally normal*) *is an absolute G_δ.*

Proof. Let Y be a metric ANR(totally normal) and let Y be embedded in a metric space M. As in (**6**, p. 333), we construct a new space Z as follows: the points of Z are the points of M, and U is open in Z if and only if $U = O \cup A$, where O is open in M and $A \subset Z - Y$. It is easily verified that Z is Hausdorff. We first wish to show that Z is totally normal.

In (**4**, p. 273), Dowker shows that every hereditarily paracompact space is totally normal. Hence it is sufficient to show that Z is hereditarily paracompact. In view of a well-known result of Stone (**17**, p. 977), it is sufficient to show that any subspace Z' of Z is fully normal.

Let $O' = \{O'_\lambda\}_{\lambda \in \Lambda}$ be an open cover of Z'. We must show that O' has an open star refinement. There exist sets O_λ, open in Z, such that $O'_\lambda = O_\lambda \cap Z'$ for each $\lambda \in \Lambda$. By the way we topologized Z, we have $O_\lambda = U_\lambda \cup A_\lambda$ where U_λ is open in M and $A_\lambda \subset Z - Y$ for each $\lambda \in \Lambda$

Let $U = \bigcup_{\lambda \in \Lambda} U_\lambda$. We now use an argument similar to that of Hanner in (**6**, p. 333). Since U is metric, U is fully normal (**18**, p. 53). Let $\{V_\gamma\}_{\gamma \in \Gamma}$ be an open star refinement of $\{U_\lambda\}_{\lambda \in \Lambda}$. Then, clearly, $\{V_\gamma \cap Z'\}_{\gamma \in \Gamma}$ is an open star refinement of $\{U_\lambda \cap Z'\}_{\lambda \in \Lambda}$. Using the fact that each $O'_\lambda = (U_\gamma \cup A_\lambda) \cap Z'$ and that O' covers Z', a routine calculation shows that $Z' \cap Y \subset U \cap Z'$.

We define a covering G of Z' as follows. Let G be the collection $\{V_\gamma \cap Z'\}_{\gamma \in \Gamma}$ together with the points of $Z' - U$. Observe that since Y is closed in Z, each point of $Z' - U$, being contained in $Z - Y$, is open in Z and hence open in Z'. It is not too difficult to see that G is an open star refinement of O'. Clearly each member of G is open in Z'. Let g be any member of G. We need only consider two cases:

(i) $g = V_\gamma \cap Z'$ for some $\gamma \in \Gamma$. Then there exists a $\lambda \in \Lambda$ such that $U_\lambda \cap Z'$ contains the $\mathrm{St}(g, G)$ ($\mathrm{St}(g, G)$ is the union of all members of G which meet g). Observe that no point of $Z' - U$ meets any $V_\gamma \cap Z'$. Now since $O'_\lambda = (U_\lambda \cup A_\lambda) \cap Z'$, O'_λ contains the $\mathrm{St}(g, G)$.

(ii) $g =$ a point p of $Z' - U$. Since O' covers Z', there exists a $\lambda \in \Lambda$ such that $p \in O'_\lambda$. But since $\mathrm{St}(p, G) = p$, we have that O'_λ contains $\mathrm{St}(g, G)$.

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We have shown that Z' is fully normal. By the sequence of remarks made in the beginning of the proof, Z is totally normal. We now proceed to show that Y is an absolute G_δ.

Since Y is closed in Z and Y is an ANR(totally normal), there exists a retraction $r: V \to Y$ where V is a neighbourhood of Y in Z. Let

$$V_i = \{z \in V; d(z, r(z)) < 1/i\},$$

where d is the metric on M. Then it is clear that for each i, V_i is open in Z and

$$Y = \bigcap_{i=1}^{\infty} V_i.$$

Hence Y is a G_δ set in Z. Now $V_i = O_i \cup A_i$ where O_i is open in M and $A_i \subset Z - Y$ for each i. Clearly

$$Y = \bigcap_{i=1}^{\infty} O_i$$

and so Y is a G_δ set in M. This completes the proof that Y is an absolute G_δ.

The following corollary is a result of Iseki (**8**, p. 571):

COROLLARY 1. *Every metric* ANR*(completely normal) is an absolute* G_δ.

Proof. Every metric ANR(completely normal) is an ANR(totally normal). By Lemma 3, it is an absolute G_δ.

It should be pointed out here that although Hanner (**6**, p. 333) proved Lemma 3 for the class of fully normal (= paracompact) spaces, there exists an example of a paracompact space which is not totally normal (**4**, p. 277). Hence Lemma 3 is not a direct consequence of Hanner's result.

The following result is easily verified by using Lemma 3 and a result of Dowker (**5**, p. 507):

LEMMA 4. *Let Y be a separable metric space. Then Y is an* ANR*(totally normal) if and only if Y is an* ANR*(metric) and an absolute* G_δ.

THEOREM 4. *Let Y be an n-dimensional separable metric space. Then Y is an* ANR*(totally normal) if and only if Y is* LC^n *and an absolute* G_δ.

Proof. Let Y be LC^n and an absolute G_δ. Since $\dim Y = n$, a result of Kuratowski (**13**, p. 289) shows that Y is an ANR(separable metric) and hence an ANR(metric) (**6**, p. 333). Thus Y is an ANR(metric) and an absolute G_δ. But this is equivalent to Y being an ANR(totally normal) by Lemma 4.

Final remarks. In (**16**, p. 793), Michael characterizes an ANR(metric) in terms of ANR(Q)'s for various classes Q of normal spaces. We emphasize, however, that in view of Borsuk's example in (**2**, p. 179) of a compact metric space which is an n-ANR(metric) for each n but not an ANR(metric), we cannot avail ourselves of Michael's results in our proof of Theorem 1.

Since every AR(separable metric) is contractible and locally contractible (**13**, p. 287), Hanner's example in (**7**, p. 381) serves as an example of a space X which is C^n and LC^n for every n. By constructing a new space Z from X as in the proof of Lemma 3, it is not difficult to see that X is neither a 1-ANR (normal) nor a 1-AR(normal). However, this apparent contradiction to Theorem 1 is dispelled when we note that X is not an absolute G_δ. Clearly, then, our requirement that X be an absolute G_δ in Theorems 1 and 3 cannot be relaxed.

References

1. R. Arens, *Extension of functions on fully normal spaces*, Pacific J. Math., *2* (1951), 11–22.

2. K. Bursuk, *Sur un espace compact localement contractile qui n'est pas un rétracte absolu de voisinage*, Fund. Math., *35* (1948), 175–180.

3. E. Čech, *Contribution à la theorie de la dimension*, Cas. Mat. Fys., *62* (1933), 277–291.

4. C. H. Dowker, *Inductive dimension of completely normal spaces*, Quart. J. Math., *4* (1953), 267–281.

5. ——— *On a theorem of Hanner*, Ark. Mat., *2* (1952–54), 307–313.

6. O. Hanner, *Retraction and extension of mappings of metric and non-metric spaces*, Ark. Mat., *2* (1952–54), 315–360.

7. ——— *Solid spaces and absolute retracts*, Ark. Mat., *1* (1949–52), 375–382.

8. K. Iseki, *On a property of mappings of metric spaces*, Proc. Japan Acad., *30* (1954), 570–571.

9. M. Katětov, *On the dimension of non-separable spaces, II*, Czechoslovak Math. J., *6* (1956), 485–516.

10. Y. Kodama, *On LC^n metric spaces*, Prov. Japan Acad., *33* (1957), 79–83.

11. C. Kuratowski, *Quelques problèmes concernant les espaces métriques non séparables*, Fund. Math., *25* (9935), 534–545.

12. ——— *Topologie I* (Warszawa-Lwów, 1933).

13. ——— *Topologie II* (Warszawa-Wroclaw, 1950).

14. B. H. McCandless, *Retracts and extension spaces for perfectly normal spaces*, Michigan J. Math., *9* (1962), 193–197.

15. ——— *Retracts and extension spaces for perfectly normal spaces II*, Portugal. Math., *22* (1963), 205–207.

16. E. Michael, *Some extension theorems for continuous functions*, Pacific J. Math., *3* (1953), 789–806.

17. A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc., *54* (1948), 977–982.

18. J. W. Tukey, *Convergence and uniformity in topology*, (Princeton, 1940).

19. M. Wojdyslawski, *Rétractes absolus et hyperespaces des continus*, Fund. Math., *32* (1939), 184–192.

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