A GENERALIZATION OF THE CAUCHY PRINCIPAL VALUE

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1. Introduction. If $a < u < b$ and $n > 0$ then

$$\int_a^b \frac{f(x)\,dx}{(x-u)^{n+1}} \tag{1}$$

is a so-called improper integral owing to the infinity in the integrand at $x = u$. When $n = 0$ we have associated with (1) the well-known Cauchy principal value, namely

$$\lim_{\epsilon \to +0} \left\{ \int_a^{u-\epsilon} \frac{f(x)\,dx}{(x-u)} + \int_{u+\epsilon}^b \frac{f(x)\,dx}{(x-u)} \right\}. \tag{2}$$

Hadamard (**1**, p. 117 et seq.) derives from an improper integral an expression which he calls its finite part and which, as he shows, possesses many important properties. For given $f(x)$ this finite part is obtained by constructing a function $g(x)$ so that the following limit exists (**1**, pp. 136 and 138):

$$\lim_{t \to u-0} \int_a^t \left\{ \frac{f(x) - g(x)}{(x-u)^m} \right\} dx. \tag{3}$$

Hadamard confines himself to the case when $m = n + \frac{1}{2}$ and n is a positive integer. In this paper we shall use Hadamard's idea to define a principal value for (1) in the case when n, in (1), is a positive integer greater than zero. When $n = 0$ the definition will reduce to (2).

The principal value so defined enables us to generalize several well-known theorems. We shall illustrate this generalization later by discussing the Hilbert transform (**4**, p. 120 (5.1.11)) and the Plemelj formulae (**2**, p. 42 (17.2)).

2. The principal value of (1). For the rest of this paper n will always denote a positive integer or zero; $f^i(x)$ will denote the ith derivative of $f(x)$ with respect to x and the principal value of (1) will be indicated by means of a prefix P before the integral sign.

For integration along the real axis with $a < u < b$ the principal value of (1) is defined as follows:

$$P\int_a^b \frac{f(x)\,dx}{(x-u)^{n+1}} = \lim_{\epsilon \to +0} \left\{ \int_a^{u-\epsilon} \frac{f(x)\,dx}{(x-u)^{n+1}} + \int_{u+\epsilon}^b \frac{f(x)\,dx}{(x-u)^{1+n}} - H_n(u,\epsilon) \right\} \tag{4}$$

where

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(5) $$H_0(u, \epsilon) = 0, \quad n = 0,$$

(6) $$H_n(u, \epsilon) = \sum_{i=0}^{n-1} \frac{f^i(u)}{i!} \left\{ \frac{1 - (-1)^{n-i}}{(n-i)\,\epsilon^{n-i}} \right\}, \qquad n > 0.$$

When $n = 0$ this principal value evidently reduces to (2).

THEOREM 1. *If* (i) $f(x)$ *possess derivatives up to order* n *in* (a, b) *and* (ii) $f^n(x)$ *satisfies a Lipschitz (or Hölder) condition, namely*

(7) $$|f^n(x_1) - f^n(x_2)| \leqslant A|x_1 - x_2|^\mu$$

whenever x_1 *and* x_2 *both lie in* (a, b), A *is a constant and* $0 < \mu \leqslant 1$, *then the limit on the right hand side of* (4) *exists.*

Proof. Consider the expression E given by

(8) $$E = \left\{ f(x) - f(u) - \frac{(x-u)}{1!} f^1(u) - \ldots - \frac{(x-u)^{n-1}}{(n-1)!} f^{n-1}(u) \right\} \frac{1}{(x-u)^{n+1}}.$$

From (i) and the mean value theorem, we see that

(9) $$E = \frac{f^n(t)}{n!\,(x-u)},$$

where t lies between x and u. On writing

(10) $$E = \left\{ \frac{f^n(t) - f^n(u)}{n!\,(x-u)} \right\} + \frac{f^n(u)}{n!\,(x-u)} = E_1 + E_2$$

we see from (ii) that, since t lies between x and u,

(11) $$|E_1| \leqslant A|x - u|^{\mu - 1}, \qquad -1 < \mu - 1 \leqslant 0.$$

Consequently it follows from the usual theory of the Cauchy principal value (**2**, chap. 2) that

$$\lim_{\epsilon \to +0} \left\{ \int_a^{u-\epsilon} + \int_{u+\epsilon}^b \right\} E\,dx$$

exists.

Denote the part of (4) inside the brackets $\{\}$ by R. Then, apart from functions of u, we see that

(12) $$R - \left\{ \int_a^{u-\epsilon} + \int_{u+\epsilon}^b \right\} E\,dx$$

contains only negative powers of $x - u$ in its integrand. On performing the integrations we find that (12) is independent of ϵ and it therefore follows that

$$\lim_{\epsilon \to +0} R$$

must also exist. This completes the proof of Theorem 1.

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The definition (4) is easily extended to the case when the integration is taken along the arc of a plane curve. The variables are then all complex, a and b correspond to the end points of the arc of integration, x is any point on this arc and u is a fixed point on it. Draw a circle centre u and radius $\epsilon(>0)$ so small that the arc of integration is cut in two points only, $u - \epsilon_1$ between a and u and $u + \epsilon_2$ between u and b. The principal value is then obtained by making the following changes in the right hand side of (4): (i) in the first integral replace $u - \epsilon$ by $u - \epsilon_1$, (ii) in the second integral replace $u + \epsilon$ by $u + \epsilon_2$ and (iii) in $H_n(u, \epsilon)$ replace

$$\left\{\frac{1-(-1)^{n-i}}{(n-i)\,\epsilon^{n-i}}\right\} \text{ by } \frac{1}{(n-i)}\left\{\frac{1}{\epsilon_1^{n-i}} - \frac{(-1)^{n-i}}{\epsilon_2^{n-i}}\right\}. \tag{13}$$

Theorem 1 holds for this definition also if the interval (a, b) is replaced by the arc of integration from a to b.

3. Some properties of the principal value (4).

THEOREM 2. *With the same conditions as in Theorem 1 and* $0 < m \leqslant n$, $n \geqslant 1$, *we have*

$$P\int_a^b \frac{f(x)\,dx}{(x-u)^{n+1}} = \sum_{i=0}^{m-1} \frac{(n-i-1)!}{n!}\left\{\frac{f^i(a)}{(a-u)^{n-i}} - \frac{f^i(b)}{(b-u)^{n-i}}\right\} + \frac{(n-m)!}{n!} P\int_a^b \frac{f^m(x)\,dx}{(x-u)^{n-m+1}}. \tag{14}$$

Proof. Denote the part of (4) inside the brackets $\{\}$ by R. Denote the result of replacing $f(u)$ by $f^1(u)$ and n by $n-1$ in the right hand side of (6) by $K_{n-1}(u, \epsilon)$. On integrating the two integrals in R by parts we have

$$R = \frac{f(a)}{n(a-u)^n} - \frac{f(b)}{n(b-u)^n} - \frac{f(u-\epsilon)}{n(-\epsilon)^n} + \frac{f(u+\epsilon)}{n(\epsilon)^n} - H_n(u,\epsilon) + \frac{1}{n}K_{n-1}(u,\epsilon) + \frac{1}{n}\left\{\int_a^{u-\epsilon}\frac{f^1(x)\,dx}{(x-u)^n} + \int_{u+\epsilon}^b \frac{f^1(x)\,dx}{(x-u)^n} - K_{n-1}(u,\epsilon)\right\}. \tag{15}$$

Denote the sum of the 3rd, 4th, 5th, and 6th terms on the right by S. From condition (i) we may expand $f(u-\epsilon)$ and $f(u+\epsilon)$, by the mean value theorem, in powers of ϵ as far as the nth derivative of $f(u)$. It is then easily found that all the coefficients of the various powers of ϵ vanish, so that S is equal to the remainder terms only. Consequently

$$S = \frac{-f^n(t_1) + f^n(t_2)}{n(n!)}, \tag{16}$$

where t_1 lies between $u - \epsilon$ and u, while t_2 lies between u and $u + \epsilon$. From the Hölder condition it follows immediately that $S \to 0$ as $\epsilon \to +0$. On letting $\epsilon \to +0$ in (15) we see from (4) and the definition of $K_{n-1}(u, \epsilon)$ that

$$P\int_a^b \frac{f(x)\,dx}{(x-u)^{n+1}} = \frac{f(a)}{n(a-u)^n} - \frac{f(b)}{n(b-u)^n} + \frac{1}{n}P\int_a^b \frac{f^1(x)\,dx}{(x-u)^n}. \tag{17}$$

This is evidently (14) with $m = 1$. On applying (17) to the principal value on the right hand side of (17) we establish (14) for the case $m = 2$. By continuing this process m times we establish (14) for every integral value of $m \leqslant n$.

Theorem 2 is also true for complex integration:

THEOREM 2A. *If conditions* (i) *and* (ii) *of Theorem 1 hold with* $a = -\infty$ *and* $b = +\infty$, (iii) *for large* $|x|, f^m(x) = O(x^{n-m-p})$ $(p > 0, m \leqslant n)$, *where O is the Landau order symbol and* (iv) $f^i(x)/x^{n-i} \to 0$ $(i = 0, 1, \ldots m-1)$ *when* $x \to \infty$ *or* $x \to -\infty$, *then*

$$P\int_{-\infty}^{\infty} \frac{f(x)\,dx}{(x-u)^{n+1}} = \frac{(n-m)!}{n!} P\int_{-\infty}^{\infty} \frac{f^m(x)\,dx}{(x-u)^{n-m+1}} \qquad (m \leqslant n). \tag{18}$$

Proof. From (iii) the integrals in (14) converge when $a = -\infty$ and $b = +\infty$ and so, from (i) and (ii), (14) is true with $-\infty$ and ∞ as the limits of integration. From (iv) all the terms in the summation sign of (14) vanish for such infinite limits and so (14) reduces to (18).

THEOREM 2B. *If* $f(z)$ *is one valued and analytic in a domain which includes the simple closed Jordan curve C and its interior then*

$$P\int_c \frac{f(z)\,dz}{(z-u)^{n+1}} = \frac{(n-m)!}{n!} P\int_c \frac{f^m(z)\,dz}{(z-u)^{n-m+1}} \qquad (m \leqslant n), \tag{19}$$

where the integrals are taken once round C and u is a fixed point on C.

Proof. Since $f(z)$ is analytic it possesses derivatives of all orders, each derivative satisfying a Lipschitz condition. Hence (14), with integration along the contour C, is true when $f(z)$ is analytic. Since C is closed the end points coincide, i.e. $b = a$. Hence since $f(z)$ and its derivatives are one valued it follows that the terms in the summation sign of (14) cancel in pairs, leaving us with (19).

4. An extension of the Hilbert transform.

THEOREM 3. *If the conditions of Theorem 2A hold, and* (v) *for large* $|x|$ $f^n(x) = O(x^{-\frac{1}{2}-p})$ *for* $p > 0$, *and*

$$g(u) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f(x)\,dx}{(x-u)^{n+1}}, \tag{20}$$

where u is real, then

$$g(x) \in L^2(-\infty, \infty): \tag{20.1}$$

$$f^n(u) = -\frac{n!}{\pi} P\int_{-\infty}^{\infty} \frac{g(x)\,dx}{(x-u)}, \tag{21}$$

and

$$\int_{-\infty}^{\infty} \{g(x)\}^2\,dx = \frac{1}{(n!)^2}\int_{-\infty}^{\infty} \{f^n(x)\}^2\,dx. \tag{22}$$

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Proof. Since Theorem 2A holds we may use (18). On using (18) with $m = n$ it follows that (20) can be written in the form

$$g(u) = \frac{1}{\pi(n!)} P \int_{-\infty}^{\infty} \frac{f^n(x)\,dx}{(x-u)}. \tag{23}$$

On replacing x by $u - t$ in the range $-\infty < x < u - \epsilon$ and x by $u + t$ in the range $u + \epsilon < x < \infty$, (23) becomes

$$g(u) = \frac{1}{\pi(n!)} \lim_{\epsilon \to +0} \int_{\epsilon}^{\infty} \frac{f^n(u+t) - f^n(u-t)}{t}\,dt. \tag{24}$$

From (v) $f^n(x) \in L^2(-\infty, \infty)$ and so we may apply Hilbert's transform theorem **(4**, Theorem 91, p. 122**)** to (24). The truth of (20.1), (21) and (22), including the existence of the principal value on the right of (21), then follows immediately.

Evidently we may look upon (20) as an integral equation with $g(x)$ as a known and $f(x)$ as an unknown function. The solution is then given by (21). Theorem 3 can also be established under a different set of conditions if we use M. Riesz's version of the Hilbert Transform **(4**, p. 132**)**.

5. An extension of the Plemelj formulae. Let A denote an arc in the complex z plane generated by points $z = x + iy$ where $x = x(t)$ and $y = y(t)$ are continuous single valued functions of the real variable t. We shall assume that there is a unique tangent at each point of the arc and we shall denote the end points by E_1 and E_2. On describing A from E_1 to E_2 we can divide the neighbourhood of each point u on A (other than E_1 and E_2) into two areas, a left hand area and a right hand area with a small segment of A, containing u, as a boundary between the two areas. Certain functions $h(v)$ possess the following property. Let u be a point on the arc A, other than one of the end points, then as $v \to u$ the function $h(v)$ tends to a unique limit provided that the path to u lies *entirely* either in the left hand area or in the right hand area. If this is the case then the limit from the left hand area will be denoted by $h^+(u)$ and that from the right hand area by $h^-(u)$. We exclude paths which approach u ultimately along the tangent to A.

When A becomes a closed contour C then C is described so that its exterior, containing the point at infinity, is the right hand area.

Let $F(v)$ be defined by the equation

$$F(v) = \frac{1}{2\pi i} \int_A \frac{f(z)\,dz}{(z-v)^{n+1}}, \tag{25}$$

where the integration is taken along the arc A.

THEOREM 4. *If* (i) *for all points of A, except possibly the end points, $f(z)$ possess derivatives up to order n,* (ii) *$f(z)$ satisfies a Hölder condition on A*

(see inequality (7)) *and u is a point on A other than one of the end points, then $F^+(u)$ and $F^-(u)$ both exist and*

$$F^+(u) = \frac{1}{2(n!)} f^n(u) + \frac{1}{2\pi i} P \int_A \frac{f(z)\,dz}{(z-u)^{n+1}}, \tag{26}$$

$$F^-(u) = -\frac{1}{2(n!)} f^n(u) + \frac{1}{2\pi i} P \int_A \frac{f(z)\,dz}{(z-u)^{n+1}}. \tag{27}$$

Proof. With v not a point on A integrate (25) n times by parts. We obtain

$$F(v) = G(v) + \frac{1}{2\pi i(n!)} \int_A \frac{f^n(z)\,dz}{(z-v)}, \tag{28}$$

where

$$G(v) = \frac{1}{2\pi i}\sum_{i=0}^{n-1} \frac{(n-i-1)!}{n!}\left\{\frac{f^i(a)}{(a-v)^{n-i}} - \frac{f^i(b)}{(b-v)^{n-i}}\right\}. \tag{29}$$

The conditions assumed above ensure the truth of Theorems 1 and 2 for the case of integration along the arc A. Hence when $v = u$, where u is a point on A, the integral on the right of (25) has a principal value. Again on taking the case when $m = n$ in (14), multiplying by $1/(2\pi i)$ and subtracting from (28) we obtain

$$\begin{aligned} F(v) &- \frac{1}{2\pi i} P \int_A \frac{f(z)\,dz}{(z-u)^{n+1}} \\ &= G(v) - G(u) + \frac{1}{2\pi i(n!)}\left\{\int_A \frac{f^n(z)\,dz}{(z-v)} - P\int_A \frac{f^n(z)\,dz}{(z-u)}\right\}. \end{aligned} \tag{30}$$

Now let

$$h(v) = \frac{1}{2\pi i}\int_A \frac{f^n(z)\,dz}{(z-v)}. \tag{31}$$

Then if u is a point on A, since $f^n(z)$ satisfies a Hölder condition, it is known that $h^+(u)$ and $h^-(u)$ exist (**2**, §16, p. 37). Again as $v \to u$ we have $\{G(v) - G(u)\} \to 0$. Hence on making $v \to u$ through the left hand area it follows from (30) that $F^+(u)$ exists and that

$$F^+(u) - \frac{1}{2\pi i} P \int_A \frac{f(z)\,dz}{(z-u)^{n+1}} = \frac{1}{n!}\left\{h^+(u) - \frac{1}{2\pi i} P \int_A \frac{f^n(z)\,dz}{(z-u)}\right\}. \tag{32}$$

From the first of the Plemelj formulae given by Muskhelishvili (**2**, p. 42 (17.2)) we see that the right hand side of (32) is equal to $f^n(u)/2(n!)$ which establishes the truth of (26).

Similarly, by using the second of the Plemelj formulae just cited we can establish the fact that $F^-(u)$ exists and also the truth of (27).

When we place $n = 0$ in (26) and (27) they reduce to the Plemelj formulae.

The theorem still holds if A is a simple closed contour. It can be extended to the case when the path of integration is the real axis, from $-\infty$ to ∞, if suitable conditions are imposed upon the derivatives of $f(x)$ in order to make the integrals converge, for example conditions (iii) and (iv) of Theorem 2A.

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We now see that $F(v)$, as defined in (25) with v complex, possesses the following properties: (i) if v is not on A it is an analytic function of v, (ii) for large v it is $O(v^{-n-1})$ and (iii) the arc A is a line of discontinuity. In fact from (26) and (27), when u is on A we have

$$F^{+}(u) - F^{-}(u) = \frac{1}{(n!)} f^{n}(u). \tag{33}$$

Again, with u on A, $F(u)$ is undefined, but if the conditions of Theorem 4 are satisfied we may define $F(u)$ to be equal to the principal value of the integral on the right of (25).

If a is one of the end points of A and $f(z)$ has a zero of order r at $z = a$ then it is not difficult to see that for $r > n$ $F(a)$ exists and that $F^{+}(a) = F^{-}(a) = F(a)$. If $r \leqslant n$ then in general $F(v)$ has a singularity at $v = a$ which is the sum of a logarithmic singularity and a pole.

6. Two applications of (33). When $n = 0$, (33) reduces to a result which can be derived from the Cauchy principal value, a result which can be used to solve many important boundary problems in various branches of mathematical physics (**2**, chaps. 12 and 13; **3** pt. V). We now discuss briefly two such problems where we can use (33) in the more general case when $n > 0$ (n an integer).

Problem 1. Find a function $F(v)$ which (i) is analytic at all points v except for points on the arc A, (ii) for large v is $O(v^{-n-1})$ and (iii) for given $g(u)$, where u is on the arc A but is not one of its end points, we have

$$F^{+}(u) - F^{-}(u) = g(u). \tag{34}$$

To obtain a formal solution we first solve the differential equation

$$f^{n}(u) = (n!)\, g(u) \tag{35}$$

for $f(u)$ and then by an obvious substitution we can express (34) in the form (33). We then obtain as a formal solution of our problem

$$F(v) = \frac{1}{2\pi i}\int_{A} \frac{f(z)\, dz}{(z-v)^{n+1}}. \tag{36}$$

If $g(z)$ is an analytic function of z in a domain D which includes the arc A then there exists a solution of (35), $f(z)$ say, which is also analytic in D. Since $f(z)$ then satisfies the conditions of Theorem 4 it will follow that $F(v)$ is a solution of the problem. With a more prolonged discussion it is possible to show that a solution exists if $g(z)$ satisfies a Hölder condition along the arc A.

Problem 2. This is connected with the singular integral equation. For the case $n = 0$ the singular integral equation below has been studied in great detail by Muskhelishvili (**2**, chap. 6). For the general case $n \geqslant 1$ many new

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difficulties occur but in one case a solution can be obtained by means of a reduction to Problem 1. The equation in question is

$$g(u)f^n(u) + \frac{h(u)}{\pi i} P \int_A \frac{f(z)\,dz}{(z-u)^{n+1}} = k(u), \tag{37}$$

where $g(u)$, $h(u)$ and $k(u)$ are given along the arc A and $f(z)$ is to be determined.

A formal solution is obtained by assuming that a function $F(v)$ exists which is related to $f(z)$ as in (25) and for which (26) and (27) both hold. On adding and subtracting (26) and (27) we obtain both $f^n(u)$ and the integral in (37) in terms of $F^+(u)$ and $F^-(u)$. After an obvious division (37) is then transformed to

$$F^+(u) = \left\{\frac{g(u)\ (n!) - h(u)}{g(u)\ (n!) + h(u)}\right\} F^-(u) + \left\{\frac{k\ (u)}{g(u)\ (n!) + h(u)}\right\}. \tag{38}$$

This equation can be reduced to the same type of equation as is solved in Problem 1, namely (34) with functions transformed from $F^+(u)$, $F^-(u)$, $g(u)$, $h(u)$ and $k(u)$ of (38) by means of known operations. $F(v)$ can then be found and then, by using (33), $f(z)$ can be found by integration.

The most important part of this solution is the reduction of (38) to an equation of type (34), an equation which is solvable by means of the methods of Problem 1. This reduction does not depend upon the value of n and is therefore the same for the general value of n as for the case when $n = 0$. The details and the ingenious methods used by Muskhelishvili to effect this reduction when $n = 0$ can be found in (**2**, §47, p. 123). If the coefficient of $F^-(u)$ in (38) and the second term on the right hand side of (38) both satisfy Hölder conditions then the solution obtained by this method is valid.

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