PRIMARY DECOMPOSITIONS OVER DOMAINS

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Throughout, R denotes a commutative domain with 1, and Q ($\neq R$) its field of quotients, which is viewed here as an R-module. The symbol K will stand for the R-module Q/R, while R^* denotes the multiplicative monoid $R\setminus 0$.

As customary, R_P will denote the localization of R at the prime ideal P, and $M_P = R_P \otimes_R M$ the localization of the R-module M at P. More generally, for a submonoid S of R^*, let R_S denote the localization of the domain R at S and $M_S = R_S \otimes_R M$ the localization of the R-module M at S. Note that M_S is an *S-torsion-free* R-module (i.e. no non-zero element of M is annihilated by any $s \in S$) which is *S-divisible* in the sense that $sM_S = M_S$ for each $s \in S$. Moreover, M_S is an R_S-module in the natural way.

We are interested in the S-torsion modules: M is *S-torsion* if every $x \in M$ is annihilated by some $s \in S$. For an R-module M, $S(M)$ will denote the set of elements of M annihilated by some $s \in S$; it is a submodule of the torsion submodule of M. From the definition it is evident that $S(M/S(M)) = 0$, i.e. $M/S(M)$ is S-torsion-free. We will say that the S-torsion modules admit *primary decompositions* if every S-torsion module M is the direct sum of its "P-components" M_P where P runs over the maximal ideals of R, $M = \oplus_P M_P$. Matlis **[2]** has shown that all torsion R-modules admit primary decompositions if and only if R is an h-local domain.

Recall that a domain R is said to be *h-local* if it satisfies the following two conditions (Matlis **[2]**):

(i) each non-zero element of R is contained but in a finite number of maximal ideals of R;

(ii) each non-zero prime ideal of R is contained in only one maximal ideal; equivalently, $R_P \otimes_R R_{P'} = Q$ for every pair P, P' of distinct maximal ideals of R.

The aim of this note is to generalize the mentioned result of Matlis by characterizing, for arbitrary domains R, the submonoids S of R^* for which the S-torsion modules admit primary decompositions. We shall show that a necessary and sufficient condition for this is that the following two conditions (analogous to (i) and (ii)) are satisfied by S:

(i*) each element of S is contained but in a finite number of maximal ideals of R, and

(ii*) each prime ideal of R which contains an element of S is contained in only one maximal ideal of R.

We shall see that, for every domain R, there is a largest monoid T in R^* which enjoys properties (i*) and (ii*). This T is uniquely determined by R and is distinguished by the property that, for a submonoid S of R^*, the S-torsion R-modules admit primary decompositions if and only if S is contained in T. Consequently, in every domain there is always a unique largest S-torsion theory which admits primary decompositions.

1. Monoids satisfying condition (i*). A submonoid S of R^* defines a torsion theory in the category of R-modules where the torsion class consists of all S-torsion modules and the torsion-free class consists of the S-torsion-free modules (as defined above). It is clear that there is no loss of generality in assuming that S is saturated in the sense that

$ab \in S (a, b \in R)$ implies $a, b \in S$. Then S will contain all the units of R. The complement $R \setminus S$ is the set union of those prime ideals of R that are disjoint from S.

The following lemma is well known, we prove it for the sake of completeness and easy reference. Note that $S(K) = R_S/R$; in fact, only the inclusion $\leq$ requires a proof. If $x + R \in S(K)$ for $x \in Q$, then $sx = r \in R$ for some $s \in S$, and so $x = r/s \in R_S$.

LEMMA 1. *If M is a torsion R-module, then*

$$S(M) = \mathrm{Tor}_1^R(S(K), M) \quad \text{and} \quad M_S = R_S \otimes_R M = \mathrm{Tor}_1^R(K/S(K), M).$$

Proof. The exact sequence $0 \to R \to R_S \to R_S/R \to 0$ induces the exact sequence

$$0 = \mathrm{Tor}_1^R(R_S, M) \to \mathrm{Tor}_1^R(R_S/R, M) \to M \to R_S \otimes_R M \to R_S/R \otimes_R M \to 0 \qquad (1)$$

for every R-module M. Similarly, from the exact sequence $0 \to S(K) \to K \to K/S(K) \to 0$ we derive the exactness of the sequence

$$\ldots \to \mathrm{Tor}_1^R(S(K), M) \to \mathrm{Tor}_1^R(K, M) = M \to \mathrm{Tor}_1^R(K/S(K), M) \to S(K) \otimes_R M \to K \otimes_R M = 0$$

provided M is a torsion R-module. The maps are natural everywhere, so a simple comparison shows that $M_S = R_S \otimes_R M = \mathrm{Tor}_1^R(K/S(K), M)$. As $S(M)$ is the kernel of the localization map $M \to M_S = R_S \otimes_R M$, we obtain $\mathrm{Tor}_1^R(R_S/R, M) = S(M)$. □

We continue with an easy (and basically well-known) lemma.

LEMMA 2. *For any R-module M, there is an embedding of M in the direct product $M^* = \Pi_P M_P$ of the localizations of M where P runs over all maximal ideals of R.*

Proof. There is a homomorphism $\phi: M \to \Pi_P M_P$ acting as $\phi(x) = (\ldots, 1 \otimes x, \ldots)$ $(x \in M)$ where the coordinate $1 \otimes x$ at the place corresponding to the maximal ideal P is computed in $R_P \otimes_R M$. It is well known (and easy to see) that ϕ is monic. □

We can now verify the following lemma.

LEMMA 3. *For a monoid S the following hypotheses are equivalent*:
(a) *$S(K)$ embeds in the direct sum $\oplus_P S(K)_P$ of its localizations at maximal ideals P*;
(b) *for every R-module M, $S(M)$ can be embedded in the direct sum $\oplus_P S(M)_P$*;
(c) *S satisfies condition* (i*).

Proof. Let ϕ be defined as in the preceding proof with $M = K$. Note that the Pth coordinate of $\phi(x)$ $(x \in S(K))$ vanishes if and only if $\mathrm{Ann}\, x \not\subset P$. In fact, if $\mathrm{Ann}\, x \not\subset P$, then the Pth coordinate of $\phi(x)$ is zero, because $a + R_P = a + 1/r + R_P = (ra + 1)/r + R_p = R_P$ for any representative $a \in Q$ of the coset x and for any $r \in \mathrm{Ann}(x + R) \setminus P$. Furthermore, $a + R_P = R_P$ means $a \in R_P$, so there is a $t \notin P$ with $ta \in R$; thus $t \in \mathrm{Ann}\, x \not\subset P$. Thus it is evident that the image of an element $x \in K$ under ϕ belongs to the direct sum $S(K)^* = \oplus_P S(K)_P$ if and only if its annihilator ideal $\mathrm{Ann}\, x = \{r \in R \mid rx = 0\}$ is contained but in a finite number of maximal ideals. Since $\mathrm{Ann}(s^{-1} + R) = sR$, it follows that (a) and (c) are equivalent.

Clearly, (a) is a special case of (b). But (a) implies (b), since if (a) holds, then by Lemma 1 we have $S(M) \leq \oplus_P \mathrm{Tor}_1^R(S(K)_P, M)$ where the summands are the P-components $S(M)_P$. In fact, the exact sequence $0 \to R \to R_S \to S(K) \to 0$ implies

$0 \to R_P \to (R_S)_P \to S(K)_P \to 0$ whence we obtain the exact sequence $0 \to \mathrm{Tor}_1^R(S(K)_P, M) \to M_P \to (R_S)_P \otimes_R M = (M_P)_S \to S(K)_P \otimes_R M \to 0$. □

It is straightforward to check that the set

$$T_1 = \{t \in R^* \mid t \text{ is contained but in a finite number of maximal ideals of } R\}$$

is a submonoid of R^*. Consequently, a monoid $S \le R^*$ satisfies (i*) if and only if it is a submonoid of T_1.

2. Monoids satisfying (ii*). Next we wish to concentrate on submonoids $S \le R^*$ satisfying condition (ii*). We start with the following lemma.

LEMMA 4. *The following conditions on a submonoid S of R^* are equivalent*:

(a) *for every pair P, P' of distinct maximal ideals, the tensor product $R_P \otimes_R R_{P'}$ is S-divisible*;

(b) *for every pair P, P' of distinct maximal ideals, we have $R_S \le R_P \otimes_R R_{P'}$*;

(c) *for every pair P, P' of distinct maximal ideals, the prime ideals contained in $P \cap P'$ are disjoint from S*;

(d) *S satisfies condition* (ii*).

Proof. (a)$\Leftrightarrow$(b) Clearly, $R_P \otimes_R R_{P'}$ is S-divisible if and only if $R_P \otimes_R R_{P'} \otimes_R R_S = R_P \otimes_R R_{P'}$ which holds exactly if $R_S \le R_P \otimes_R R_{P'}$; here we have identified $R_P \otimes_R R_{P'}$ with a submodule of Q.

(a)$\Leftrightarrow$(c) The tensor product $R_P \otimes_R R_{P'}$ is the localization of R at the saturated semigroup $S(P, P')$ generated by $R \setminus P \cup R \setminus P' = R \setminus (P \cap P')$. Thus it is S-divisible exactly if $S \subseteq S(P, P')$; equivalently, exactly if every prime ideal of R disjoint from $S(P, P')$ is disjoint from S. But a prime ideal is disjoint from $S(P, P')$ if and only if it is contained in $P \cap P'$.

(c)$\Leftrightarrow$(d) This equivalence is obvious. □

It is now easy to verify:

COROLLARY 5. *The set*

$$T_2 = \{t \in R^* \mid \text{any prime ideal of } R \text{ containing } t \text{ is contained in only one maximal ideal}\}$$

is a multiplicative submonoid in R^*.

Proof. (By default, the units of R belong to T_2.) By definition, T_2 satisfies condition (d) of Lemma 4 stated for S. From the proof of this lemma it is evident that $T_2 \subseteq S(P, P')$ for every pair P, P' of maximal ideals, thus $T_2 \subseteq \bigcap_{P \neq P'} S(P, P')$. Since every element in this intersection belongs to T_2, we have $T_2 = \bigcap_{P \neq P'} S(P, P')$. This proves that T_2 is indeed a monoid. □

LEMMA 6. *If S is a submonoid of T_2, then*

1) *for every S-torsion module M and maximal ideal P, the localization map $M \to M_P$ is surjective*;

2) *for every pair of S-torsion modules M, N, and for distinct maximal ideals P, P' we have*

$$\mathrm{Hom}_R(M_P, N_{P'}) = 0.$$

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Proof. 1) In view of the sequence (1) with $S=R\setminus P$, it suffices to show that under the stated hypotheses $R_P/R\otimes_R M=0$ holds. We prove that localizations of $R_P/R\otimes_R M$ vanish. Clearly, $R_{P'}\otimes_R R_P/R\otimes_R M=(R_{P'}\otimes_R R_P)/R_{P'}\otimes_R M$ which is obviously 0 whenever $P'=P$. If $P'\neq P$, then the first module in the last tensor product is S-divisible by Lemma 4, so it annihilates the S-torsion module M.

2) $H=\mathrm{Hom}_R(M_P,N_{P'})$ is both an R_P- and an $R_{P'}$-module, so it is an $R_P\otimes_R R_{P'}$-module, and hence S-divisible by Lemma 4. An S-divisible homomorphism annihilates S-torsion modules, and since by part 1) M_P is S-torsion, we must have $H=0$. □

COROLLARY 7. *If S is a submonoid of T_2, then every S-torsion module M is a subdirect product of its P-components M_P.*

Proof. This is an immediate consequence of Lemmas 2 and 6. □

3. Monoids satisfying conditions (i*) and (ii*). Set $\Sigma(P)=\bigcap_{P'\neq P}R_{P'}$ for a maximal ideal P of R where P' runs over all maximal ideals distinct from P.

LEMMA 8. *Let S be a monoid satisfying both* (i*) *and* (ii*). *Then for every maximal ideal P of R the following direct decomposition holds*:

$$(R_S)_P/R=R_P/R\oplus((R_S)_P\cap\sum(P))/R.$$

Proof. For the sake of brevity, we will write $A=R_S$. For a maximal ideal P, consider the homomorphism $\phi_P:A_P/R\to\oplus_{P'\neq P}(A_P/R)_{P'}$ defined similarly as in the proof of Lemma 2; we could replace the direct product by the direct sum as a result of condition (i*) (cf. Lemma 3). Evidently, an element of A_P/R is mapped upon 0 if and only if it belongs to $R_{P'}$ for every $P'\neq P$. Thus $\mathrm{Ker}\,\phi_P=(A_P\cap\sum(P))/R$, and so $A_P/(A_P\cap\sum(P))$ is isomorphic to a submodule of $\oplus_{P'\neq P}(A/R)_{P'}$. From condition (ii*) we obtain $R_P\otimes_R(A_P/R)_{P'}=0$ whence $R_P\otimes_R(\oplus_{P'\neq P}(A_P/R)_{P'})=0$, and so $R_P\otimes_R(A_P/(A\cap\sum(P))=0$. This implies $R_P\otimes_R(A_P\cap\sum(P))=A_P$ whence we derive that the submodules R_P/R and $(A_P\cap\sum(P))/R$ generate A_P/R. As the intersection of the last two submodules is obviously R/R, we arrive at the desired conclusion that A_P/R is the direct sum of its submodules R_P/R and $(A_P\cap\sum(P))/R$. □

THEOREM 9. *If the monoid $S\leq R^*$ satisfies conditions both* (i*) *and* (ii*), *then there is a direct decomposition*

$$R_S/R=\oplus_P(R_S/R)_P. \tag{2}$$

Proof. In view of the preceding lemma, $(A_p\cap\sum(P))/R$ is a summand of A_P/R. Manifestly, it is isomorphic to $A_P/R_P\cong(A/R)_P$, where as before, $A=R_S$. We can now imitate the proof of the implication 2)⇒3) in Matlis [**2**, Thm 8.5] to argue that for every finite set $\{P_1,\ldots,P_n\}$ of maximal ideals the submodules $(A_{P_i}\cap\sum(P_i))/R$ generate their direct sum in A/R, and this direct sum is a summand of A/R. It then follows that $A/R=\oplus_P(A_P\cap\sum(P))/R$ where the summands are nothing else than $(A/R)_P$. □

The decomposition of the preceding theorem yields:

COROLLARY 10. *If the monoid* $S \le R^*$ *satisfies conditions* (i*) *and* (ii*), *then every* S*-torsion* R*-module* M *decomposes as*

$$M = \oplus_P M_P.$$

Proof. Let M be an S-torsion R-module. Because of Lemma 1, we have $M = \mathrm{Tor}_1^R(A/R, M)$ which is—by Theorem 9—equal to $\oplus_P \mathrm{Tor}_1^R(A/R)_P, M)$. The exact sequence $0 \to R_P \to A_P \to A_P/R_P \to 0$ implies the exactness of the induced sequence $0 = \mathrm{Tor}_1^R(A_P, M) \to \mathrm{Tor}_1^R(A_P/R_P, M) \to M_P \to A_P \otimes_R M = 0$ whence $\mathrm{Tor}_1^R((A/R)_P, M) = M_P$, proving the assertion. □

Note that if (2) holds, then by Lemma 3, S satisfies (i*). Furthermore, the localization of $(R_S/R)_P$ at any maximal ideal $P' \neq P$ must be 0, thus $R_{P'} \otimes_R R_P \otimes_R R_S = R_{P'} \otimes_R R_P$, which implies $R_S \le R_{P'} \otimes_R R_P$. Hence, by Lemma 4, S satisfies (ii*). It is now clear that a monoid $S \le R^*$ satisfies both (i*) and (ii*) if and only if it is contained in the monoid $T = T_1 \cap T_2$. Consequently, we obtain our main result:

THEOREM 11. *In every domain* R, *there is a unique maximal monoid* $T \le R^*$ *such that the* T*-torsion* R*-modules admit primary decompositions.*

Furthermore, for a (saturated) submonoid S *of* R^*, *the following conditions are equivalent*:

(a) *the* S*-torsion* R*-modules admit primary decompositions*;

(b) S *satisfies conditions* (i*) *and* (ii*);

(c) S *is contained in* T. □

4. The case p.d.$R_S = 1$. If R is a Dedekind domain (i.e. a domain of global dimension 1), then the P-components of K are indecomposable. In the general case, this need not be true, but this favorable situation occurs when the projective dimension of the localization R_S (as an R-module) is 1. Indeed, we have:

THEOREM 12. *If* S *is a submonoid of* T *such that* p.d.$R_S \le 1$, *then* R_S/R *is the direct sum of its* P*-components which are all indecomposable and countably generated.*

Proof. In view of Theorem 11, only the claims concerning indecomposability and countable generation require proofs.

Let B/R be a summand of R_S/R where $R \le B \le R_S$. Because of [**1**, Thm 4.2], B must be a flat overring of R which is the intersection of the localizations R_P at maximal ideals P with $PB \neq B$. In the primary decomposition (2), the submodule R_P/R is the direct sum of the components $(R_S/R)_{P'}$ with $P' \neq P$. Therefore, B/R is the direct sum of certain P-components. We conclude that the P-components of R_S/R must be indecomposable.

In view of [**1**, Thm 3.2], p.d.$R_S \le 1$ implies that R_S/R is a direct sum of countably generated submodules. By [**1**, Prop. 4.1] all submodules of R_S/R are fully invariant, so the P-components must be direct sums of countably generated submodules. By indecomposability, they are themselves countably generated. □

The following examples exhibit various situations for the semigroup T.

EXAMPLE 1. If R is an h-local domain (in particular, a Dedekind domain), then the semigroup T is all of R^*.

EXAMPLE 2. In the polynomial ring $R = \mathbb{Z}[x]$ over the integers, every non-zero prime ideal which is not maximal is contained in infinitely many maximal ideals. No maximal ideal of R is principal whence it follows that the monoid T consists of the units of R.

EXAMPLE 3. (See McAdam [3]) Let R_0 be a complete discrete valuation domain with maximal ideal P. The ideals $I = PR_1 + xR_1$ and $J = PR_1 + (x+1)R_1$ are maximal ideals of the polynomial ring $R_1 = R_0[x]$ over R_0. The localization R of R_1 with respect to the semigroup $S = R_1 \setminus (I \cup J)$ is a domain with precisely two maximal ideals, viz. I_S and J_S. The only other non-zero prime ideal of R is PR which is contained in both I_S and J_S. In this case, T of R is nothing else than $R \setminus PR$.

EXAMPLE 4. Let R be a domain of Krull dimension 1. Then the monoid T consists of all the elements of R that are contained only in a finite number of maximal ideals.

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