BULL. AUSTRAL. MATH. SOC. 06A75
VOL. 19 (1978), 417-435.

On loop near-rings

D. Ramakotaiah and C. Santhakumari

A new class of algebraic systems known as loop near-rings are introduced, which includes near-rings and consequently rings. Different types of radicals are introduced in a loop near-ring N, which coincide with the Jacobson radical when N happens to be a ring, and several characterizations of these radicals are obtained.

Introduction

The notion of a loop near-ring arises out of an axiomatization of the algebraic systems of mappings of the additive loop G into G which fix the identity of G. Every near-domain (additively non-associative near-field) in the sense of Pilz [3, Definition 8.41] is a loop near-ring. We introduce a right quasi-regular element in a different way from the usual tradition, and this seems to define three types of right quasi-regular elements as there are three types of modular maximal right ideals.

This paper is divided into four sections. In §1, loop near-rings, loop near-ring loops are introduced and examples of such systems are presented. Right ideals, ideals, and modular right ideals are introduced in §2, and a characterization of the unique maximal ideal contained in a modular right ideal is obtained. In §3, N-loops of type V, V-primitive ideals, V-primitive loop near-rings, V-modular right ideals for $V = 0, 1, 2$, and various radicals are introduced and we characterize the ideals $J_V(N)$ in terms of the largest ideals of N contained in V-modular right ideals in N for $V = 0, 1, 2$. In §4, we introduce the notion of "right quasi-regular element of type V" for $V = 0, 1, 2$, which generalizes the notion of a right quasi-regular element as introduced

Received 10 October 1978.

in [4]. If N happens to be a ring all these three notions coincide with the notion of a right quasi-regular element as introduced in ring theory. Characterization of the radicals in terms of quasi-regular elements is obtained.

1. Fundamental definitions and simple consequences

For definitions of loops, subloops, normal subloops, see [2]. We begin this section with the following:

DEFINITION 1.1. A system $N = (N, +, \cdot, 0)$ is called a loop near-ring if the following conditions are satisfied:

(i) $(N, +, 0)$ is a loop which we denote by N^+ ;

(ii) $(N, \cdot)$ is a semigroup;

(iii) $a\cdot(b+c) = a\cdot b + a\cdot c$ for all a, b, c in N ;

(iv) $0\cdot a = 0$ for all a in N, where 0 is the identity of the loop N^+ .

For any a belonging to an additive loop, we shall denote the unique right and left additive inverses of a by a_r and a_l respectively.

Using Definition 1.1 (iii), it is easy to verify that $a\cdot 0 = 0$, $(a\cdot b)_r = a\cdot b_r$, $(a\cdot b)_l = a\cdot b_l$ for all a, b in N .

Throughout this paper N always stands for a loop near-ring. We abbreviate $(N, +, \cdot, 0)$ by N . The identity element of N^+ will be denoted by 0 . Multiplication in most cases will be indicated by justaposition; so we write nm instead of $n\cdot m$.

EXAMPLE 1.2. If G is an additive loop, then the set of all mappings of G into itself fixing the identity of G has the structure c a loop near-ring under addition and composition of mappings [2, p. 68].

EXAMPLE 1.3. Every near-domain (additively non-associative near-field) [3, Definition 8.41] is a loop near-ring.

EXAMPLE 1.4. Let $(G, +, \bar{0})$ be an additive loop, where $\bar{0}$ is the identity element of G . Define $ab = b$ for all $\bar{0} \neq a$ and b in G ; define $\bar{0}b = \bar{0}$. Then $(G, +, \bar{0})$ is a loop near-ring.

Of course every near-ring is a loop near-ring.

. Subloop near-rings, isomorphisms, and homomorphisms of loop near-rings are defined in the usual way. Left (right) identities, left (right) invertible elements, and nilpotent elements are defined as in near-rings.

We introduce the notion of N-loops, N-loop homomorphisms in the usual way.

DEFINITION 1.5. An additive loop $(G, +, \bar{0})$ is called an N-loop provided there exists a mapping $(g, n) \to gn$ of $G \times N \to G$ such that

(i) $g(n+m) = gn + gm$,

(ii) $g(nm) = (gn)m$ for all $g \in G$, $n, m \in N$.

Clearly N^+ is an N-loop. If $\bar{0}$ is the identity of the loop G , $g0 = \bar{0}$ and $\bar{0}n = (\bar{0}0)n = \bar{0}(0n) = \bar{0}0 = \bar{0}$ for all $n \in N$. Further $(gn)_r = gn_r$ and $(gn)_l = gn_l$.

We abbreviate $(G, +, \bar{0})$ by G . The identity element of an N-loop G will be denoted by $\bar{0}$.

EXAMPLE 1.6. If G is an additive loop and N is the loop near-ring of all mappings of G into itself fixing the identity element, then G has the structure of an N-loop.

If G is an N-loop and Δ and K are subsets of G and N respectively, then the set $\{\delta k \mid \delta \in \Delta, k \in K\}$ will be denoted by ΔK .

DEFINITION 1.7. A subloop Δ of an N-loop G is called an N-subloop of G if $\Delta N \subseteq \Delta$.

The N-subloops in N^+ are called N-loop modules of N .

DEFINITION 1.8. Suppose G and G' are N-loops. A mapping $f : G \to G'$ is called an N-loop homomorphism provided

(1) $f(x+y) = f(x) + f(y)$ for all x, y in G ,

(2) $f(xn) = f(x)n$ for all x in G and n in N .

An N-loop homomorphism f of G into G' is called an N-loop isomorphism if f is a bijection of G into G' .

EXAMPLE 1.9. Let G be an N-loop and let $g \in G$. Then the mapping $n \to gn$ is an N-loop homomorphism of N^+ into G .

DEFINITION 1.10. The kernel of an N-loop homomorphism of an N-loop

G is called an N-loop kernel of G .

We now obtain necessary and sufficient conditions for a nonempty subset of an N-loop G to be an N-loop kernel of G .

THEOREM 1.11. *A nonempty subset K of an N-loop G is an N-loop kernel of G if and only if*

(i) $(K, +)$ *is a normal subloop of* G ,

(ii) $(g+k)n + gn_r \in K$ *for all* $g \in G$, $k \in K$, *and* $n \in N$.

Proof. Let $\phi : G \to G'$ be an N-loop homomorphism and let $K = \ker(\phi)$. Then K is a normal subloop of G [2, p. 60]. Let $g \in G$, $k \in K$, and $n \in N$. Consider

$$\phi\big((g+k)n+gn_r\big) = \phi\big((g+k)n\big) + \phi\big(gn_r\big) = \phi(g+k)n + \phi(g)n_r$$
$$= \phi(g)n + \phi(g)n_r = \phi(g)0 = \bar{0}' ,$$

where $\bar{0}'$ is the identity of G' . Therefore $(g+k)n + gn_r \in K$ for all $g \in G$, $k \in K$, and $n \in N$. Therefore K satisfies conditions *(i)* and *(ii)*. Conversely let G be an N-loop and K be a nonempty subset of G , satisfying *(i)* and *(ii)*. We wish to show that $G|K$ has the structure of an N-loop. For $g + K$, $g' + K$ in $G|K$, define $(g+K) + (g'+K) = (g+g') + K$. Then it can be shown that $G|K$ has the structure of a loop [2, p. 61]. Put $(g+K)n = gn + K$. Suppose $g + K = g' + K$, $g \in g' + K$. Then $g = g' + k$, where $k \in K$. Now $gn = (g'+k)n$. Therefore $gn + g'n_r = (g'+k)n + g'n_r \in K$. Hence, $\big(gn+g'n_r\big) + K = \big(g'n+g'n_r\big) + K$. Since cancellation laws hold good in a loop and since $G|K$ is a loop we have $(gn+K) = g'n + K$. Hence the map $(g+K, n) \to (g+K)n$ of $G|K \times N \to G|K$ is well defined. Let $g+K \in G|K$ and $n, m \in N$. Then

$$(g+K)(n+m) = g(n+m) + K = (gn+gm) + K = (gn+K) + (gm+K) = (g+K)n + (g+K)m$$

and

$$(g+K)nm = g(nm) + K = (gn)m + K = (gn+K)m = \big((g+K)n\big)m .$$

Therefore $G|K$ has the structure of an N-loop. Now the mapping $\bar{\phi} : x \to x+K$ is an N-loop homomorphism of G onto $G|K$, $x \in \ker(\bar{\phi})$ if and only if $\bar{\phi}(x) = \bar{0}$ (where $\bar{0}$ is the identity of $G|K$), and $\bar{\phi}(x) = \bar{0}$

if and only if $x + K = K$, and $x + K = K$ if and only if $x \in K$. Hence $K = \ker(\overline{\Phi})$. Hence K is an N-loop kernel of an N-loop G.

In a similar way it can be shown that a nonempty subset K of G is an N-loop kernel of G, if and only if $(K, +)$ is a normal subloop of $(G, +)$ and $gn_l + (g+k)n \in K$ for all $g \in G$, $k \in K$, and $n \in N$.

The factor loop of an N-loop G by an N-loop kernel K of G is denoted by $G - K$.

REMARK 1.12. By Theorem 1.11, it can be easily shown that every N-loop kernel of G is an N-subloop of G.

We now have the following:

THEOREM 1.13. *Let $h : G \to G'$ be an N-loop epimorphism. Then h induces a one-to-one correspondence between the N-subloops (N-loop kernels) of G containing* $\ker(h)$ *and the N-subloops (N-loop kernels) of G' by $K\ (\subseteq G) \to h(K)$.*

Proof. If K is a subloop (normal subloop) of G then $h(K)$ is a subloop (normal subloop) of G'. Conversely if K' is a subloop (normal subloop) of G' then $h^{-1}(K')$ is a subloop (normal subloop) of G [2, iv, Lemma 1.6]. The rest of the proof would follow in the usual way and hence is omitted.

THEOREM 1.14. *The intersection of any family of N-loop kernels of an N-loop G is an N-loop kernel of G.*

Proof. Let $\{K_\alpha\}_{\alpha \in A}$ be a family of N-loop kernels of an N-loop G. By [2, iv, Theorem 1.2], $\bigcap_{\alpha \in A} K_\alpha$ is an N-loop kernel of G.

LEMMA 1.15. *The set S of all N-loop kernels of an N-loop G form a commutative semigroup under addition.*

Proof. Let A and B be N-loop kernels of an N-loop G: $A + B = \{a+b \mid a \in A, b \in B\}$. Now A and B are normal subloops of G (Theorem 1.11). Since the set of all normal subloops of an additive loop form a commutative semigroup under addition [2, iv, Theorem 1.4], $A + B$ is a normal subloop of G and $A + B = B + A$; further $(A+B) + C = A + (B+C)$ for all $A, B, C \in S$. We wish to show that

$\big(g+(a+b)\big)n + gn_r \in A + B$ for all $g \in G$, $n \in N$, and $a+b \in A+B$. Since B is a normal subloop of G, $g + (a+B) = (g+a) + B$. But $g + (a+b) \in g + (a+B)$. Hence $g + (a+b) = (g+a) + b'$, where $b' \in B$. Since B is an N-loop kernel of G and $b' \in B$,

$$\{((g+a)+b')n+(g+a)n_r\} + B = B = \overline{0} + B = \{(g+a)n+(g+a)n_r\} + B .$$

Since $G - B$ is a loop, $\big((g+a)+b'\big)n + B = (g+a)n + B$. Now $\{\big(g+(a+b)\big)n+gn_r\} + B = \{\big((g+a)+b'\big)n+gn_r\} + B = \big((g+a)n+gn_r\big) + B = a' + B$, where $a' = (g+a)n + gn_r \in A$, since $a \in A$ and A is an N-loop kernel of G. Therefore $\big(g+(a+b)\big)n + gn_r \in A + B$. Hence $A + B$ is an N-loop kernel of G. Therefore the set of all N-loop kernels of an N-loop G is a commutative semigroup under addition.

LEMMA 1.16. *If G is an N-loop, then for every $g \in G$, $gN = \{gn \mid n \in N\}$ is an N-subloop of G.*

Proof. Let $gn, gn' \in gN$. Then $gn + gn' = g(n+n') \in gN$ and $g0 = \overline{0} \in gN$. Since $gn, gn' \in G$ and G is a loop, there exist unique elements $x, y \in G$ such that $gn = gn' + x = y + gn'$. Further there exist unique elements $m, m' \in N$ such that $n = n' + m = m' + n'$. Hence $gn = gn' + gm = gm' + gn'$. Since x and y are unique, $gm = x$ and $gm' = y$. Therefore $x, y \in gN$. Hence gN is a subloop of G.

2. Modular right ideals

In this section we introduce the notion of a modular right ideal in a loop near-ring and obtain a characterization of the unique maximal ideal contained in a modular right ideal.

DEFINITION 2.1. By a right ideal of a loop near-ring N we mean an N-loop kernel of N^+ as an N-loop.

In view of Theorem 1.11, a nonempty subset L of a loop near-ring N is a right ideal of N if and only if $(L, +)$ is a normal subloop of N^+ and $(x+n)m + nm_r \in L$ for all $x \in L$, $n, m \in N$. Further, if L is a right ideal of N, then $LN \subseteq L$.

Nil and nilpotent right ideals in N are defined in the usual way.

https://doi.org/10.1017/S0004972700008959 Published online by Cambridge University Press

DEFINITION 2.2. A right ideal L of N is called an ideal of N if $NL \subseteq L$.

REMARK 2.3. If P is an ideal of N then $N|P$ is a loop near-ring in which $(a+P)(b+P) = ab + P$ for all $a + P$, $b + P$ in $N|P$.

LEMMA 2.4. *If L and Q are two ideals of N, then $L + Q$ is an ideal of N.*

Proof. Let L and Q be two ideals of N. By Lemma 1.15, $L + Q$ is a right ideal of N. Let $x+y \in L+Q$. For every $n \in N$, $n(x+y) = nx + ny \in L + Q$. Hence $L + Q$ is an ideal of N.

We now introduce the notion of a modular right ideal in a loop near-ring.

DEFINITION 2.5. A right ideal L of N is said to be a modular right ideal of N if there exists an element $e \in N$ such that $n + en_r \in L$ for all $n \in N$. e is said to be a left identity modulo L.

LEMMA 2.6. *A right ideal L of N is a modular right ideal if and only if there exists an element e in N such that $en_l + n \in L$ for all $n \in N$.*

The proof of this lemma is easy and will be omitted.

LEMMA 2.7. *If L is a proper modular right ideal with e as a left identity modulo L then $e \notin L$.*

Proof. Suppose $e \in L$. Then $en \in L$ for all $n \in N$. Since e is a left identity modulo L, $n + en_r \in L$ for all $n \in N$. Then

$$(n+en_r) + L = L = 0 + L = (en+en_r) + L .$$

Since $N^+ - L$ is a loop, $n + L = en + L$. Since $en \in L$, $en + L = L$. Therefore $n + L = L$. Hence $n \in L$. Then $L = N$, a contradiction. Therefore $e \notin L$.

LEMMA 2.8. *Every proper modular right ideal can be extended to a maximal modular right ideal.*

The proof of this lemma would follow in the usual way.

We now characterize the unique maximal ideal contained in a modular right ideal. For this we require the following notation.

Let L be a modular right ideal of N. We denote the set $\{a \in N \mid Na \subseteq L\}$ by $(L : N)$.

THEOREM 2.9. *If* L *is a modular right ideal of* N *then* $(L : N)$ *is an ideal in* N *and it is the largest ideal contained in* L.

We break this theorem into several lemmas and prove one after the other.

LEMMA 2.9.1. *If* L *is a modular right ideal of* N, *then* $(L : N) \subseteq L$.

Proof. Let $a \in (L : N)$ and let e be a left identity modulo L. Since $a \in (L : N)$, $Na \subseteq L$. Then $ea \in L$. Since e is a left identity modulo L, $a + ea_r \in L$ for all $a \in N$. Then

$$(a+ea_r) + L = L = (ea+ea_r) + L .$$

Since $N^+ - L$ is a loop, $a + L = ea + L$. Since $ea \in L$, $ea + L = L$. Hence $a + L = L$. Therefore $a \in L$. Hence $(L : N) \subseteq L$.

LEMMA 2.9.2. $(L : N)$ *is a subloop of* N^+.

Proof. Clearly $0 \in (L : N)$. Let $n, n' \in (L : N)$. Since $(L : N) \subseteq L$, $n, n' \in L$. Now for each $m \in N$, $m(n+n') = mn + mn' \in L$ and hence $n + n' \in (L : N)$. Since L is a subloop of N^+, there exist unique elements a, a' in L such that $n' = n + a = a' + n$. Then for any $m \in N$, $mn' = mn + ma = ma' + mn$. Since L is a normal subloop of N^+ and since $mn, mn' \in L$, we have

$$L = L + mn' = L + (mn+ma) = (L+mn) + ma = L + ma .$$

Then $ma \in L$ for all $m \in N$. Hence $a \in (L : N)$. Further

$$L = mn' + L = (ma'+mn) + L = ma' + (mn+L) = ma' + L .$$

Hence $ma' \in L$ for all $m \in N$. Therefore $a' \in (L : N)$. Hence $(L : N)$ is a subloop of N^+.

LEMMA 2.9.3. $(L : N)$ *is a normal subloop of* N^+.

Proof. By Lemma 2.9.2, $(L : N)$ is a subloop of N^+. Let $a \in (L : N)$ and $n \in N$. We wish to show that $n + (L : N) = (L : N) + n$. Since $a \in (L : N)$, $a \in L$, and since L is a normal subloop of N^+, $n + L = L + n$. But $n+a \in n+L$. Hence $n + a = b + n$, where $b \in L$.

https://doi.org/10.1017/S0004972700008959 Published online by Cambridge University Press

Then for any $r \in N$, $rn + ra = rb + rn$. Since L is a normal subloop of N^+ and since $ra \in L$, $(rn+ra) + L = rn + (ra+L) = rn + L$. Hence $(rb+rn) + L = rn + L$. Since $N^+ - L$ is a loop, $rb + L = L$. Hence $rb \in L$ for all $r \in N$. Therefore $b \in (L : N)$. Hence $n + a = b + n$ where $b \in (L : N)$. Therefore $n + (L : N) \subseteq (L : N) + n$. By a similar argument it can be shown that $(L : N) + n \subseteq n + (L : N)$. Therefore $n + (L : N) = (L : N) + n$. Let $n, m \in N$ and $a \in (L : N)$. We wish to show that $(n+m) + (L : N) = n + \big(m+(L : N)\big)$. Since $a \in (L : N)$, $a \in L$, and since L is a normal subloop of N^+, $(n+m) + L = n + (m+L)$ for all $n, m \in N$. Now $(n+m) + a \in (n+m) + L$. Hence $(n+m) + a = n + (m+b)$ where $b \in L$. We show that $b \in (L : N)$. For every $r \in N$, $\big((rn+rm)+ra\big) + L = \big(rn+(rm+rb)\big) + L$. Since L is a normal subloop of N^+ and since $ra \in L$,

$$\big((rn+rm)+ra\big) + L = (rn+rm) + (ra+L) = (rn+rm) + L .$$

Therefore

$$\big(rn+(rm+rb)\big) + L = \big((rn+rm)+ra\big) + L = (rn+rm) + L .$$

Since $N^+ - L$ is a loop, $(rm+rb) + L = rm + L$. Hence $rb + L = L$. Then $rb \in L$ for all $r \in N$. Hence $b \in (L : N)$. Therefore $(n+m) + (L : N) \subseteq n + \big(m+(L : N)\big)$. The other inclusion is also true. Hence $(n+m) + (L : N) = n + \big(m+(L : N)\big)$. By a similar argument it can be shown that $(L : N) + (m+n) = \big((L : N)+m\big) + n$ for all $m, n \in N$. Therefore $(L : N)$ is a normal subloop of N^+.

LEMMA 2.9.4. *$(L : N)$ is an ideal of N.*

Proof. $(L : N)$ is a normal subloop of N^+ by Lemma 2.9.3. Let $a \in (L : N)$, $n, n' \in N$. Now for every $m \in N$,

$$m\{(a+n)n'+nn'_r\} = (ma+mn)n' + (mn)n'_r \in L ,$$

since $ma \in L$ and L is a right ideal of N. Therefore $(a+n)n' + nn'_r \in (L : N)$. Let $a \in (L : N)$ and $n \in N$; then $N(na) = (Nn)a \subseteq Na \subseteq L$. Therefore $na \in (L : N)$ for all $n \in N$. Hence $(L : N)$ is an ideal of N.

LEMMA 2.9.5. *$(L : N)$ is the largest ideal of N contained in L.*

Proof. Let P be an ideal of N contained in L. Let $p \in P$. Then $Np \subseteq P \subseteq L$. Hence $p \in (L : N)$. Therefore $P \subseteq (L : N)$. Hence

https://doi.org/10.1017/S0004972700008959 Published online by Cambridge University Press

$(L : N)$ is the largest ideal of N contained in L .

Proof of Theorem 2.9. The proof of this theorem follows from Lemmas 2.9.1 to 2.9.5.

3. A characterization of the ideals $J_V(N)$

Let G be an N-loop. If Δ is a nonempty subset of G then the set $A(\Delta) = \{n \in N \mid gn = \overline{0}$ for all $g \in \Delta\}$ is called the annihilating set of Δ in N .

LEMMA 3.1. *If G is an N-loop and $g \in G$, then $A(g) = \{n \in N \mid gn = \overline{0}\}$ is an N-loop kernel of N^+ .*

Proof. The mapping $f : n \to gn$ is an N-loop homomorphism of N^+ into G and hence $\ker(f) = A(g)$ is a right ideal of N .

We remark that if G is an N-loop, then $A(G) = \bigcap_{g \in G} A(g)$ is an ideal in N and $A(G)$ is called the annihilating ideal of G in N .

We introduce various types of N-loops as in near-rings. Let G be an N-loop not equal to $\{\overline{0}\}$.

DEFINITION 3.2. An element $g \in G$ is called an N-generator of G if $gN = G$.

DEFINITION 3.3. G is said to be a faithful N-loop if $A(G) = (0)$.

DEFINITION 3.4. G is said to be an irreducible N-loop provided G has no nontrivial N-loop kernels.

DEFINITION 3.5. An N-loop G is said to be a minimal N-loop provided G has only the trivial N-subloops $(\overline{0})$ and G .

DEFINITION 3.6. An irreducible N-loop G with a generator g is called an N-loop of type 0 .

DEFINITION 3.7. An N-loop of type 0 is called an N-loop of type 1 if for each $g \in G$ either $gN = (\overline{0})$ or $gN = G$.

DEFINITION 3.8. An N-loop G is said to be an N-loop of type 2 if G is minimal and $GN \neq (\overline{0})$.

LEMMA 3.9. *If G is a faithful N-loop then N is isomorphic to a loop near-ring of zero fixing mappings of G into itself, and we can* *identify* $n \in N$ *with the mapping* $G \to G : g \to gn$.

The proof is easy and will be omitted.

LEMMA 3.10. *Let* G *be a faithful* N*-loop with an* N*-generator* g . *Then* N *is a near-ring if and only if* G *is a group.*

Proof. Suppose N is a near-ring. It is enough to show that '+' in G is associative. Now $G = gN$. Let $x, y, z \in G$. Then $x = gn_1$, $y = gn_2$, $z = gn_3$, where $n_1, n_2, n_3 \in N$. Since N^+ is associative,

$$x + (y+z) = gn_1 + (gn_2+gn_3) = g(n_1+(n_2+n_3)) = g((n_1+n_2)+n_3)$$
$$= (gn_1+gn_2) + gn_3 = (x+y) + z .$$

Hence G is a group. Conversely suppose that G is a group. Let $x, y, z \in N$ such that $(x+y) + z \neq x + (y+z)$. Since G is faithful, there exists a $g \in G$ such that $g((x+y)+z) \neq g(x+(y+z))$, for otherwise, $g((x+y)+z) = g(x+(y+z))$ for all $g \in G$. Then

$$((x+y)+z) + (x+(y+z))_r \in A(G) = (0) .$$

Therefore

$$((x+y)+z) + (x+(y+z))_r = 0 = (x+(y+z)) + (x+(y+z))_r .$$

Then $(x+y) + z = x + (y+z)$, which is not true. Therefore, for some $g \in G$, $g((x+y)+z) \neq g(x+(y+z))$. Then $(gx+gy) + gz \neq gx + (gy+gz)$, which contradicts that G is a group. Therefore, for all x, y, z in N , $(x+y) + z = x + (y+z)$. Hence N^+ is associative and consequently N is a near-ring.

LEMMA 3.11. *Every* N*-loop of type* 2 *is an* N*-loop of type* 1 *and hence an* N*-loop of type* 0 .

Proof. The proof of this lemma will follow as in the case of near-rings (see [*1*]).

COROLLARY 3.12. *If* N *contains a unity element and* G *is an* N*-loop of type* 1 , *then* G *is an* N*-loop of type* 2 .

The proof of this corollary will follow as in the case of near-rings (see [*1*]).

EXAMPLE 3.13. Let $G = \{1, 2, 3, 4, 5, 6\}$. Addition in G is

https://doi.org/10.1017/S0004972700008959 Published online by Cambridge University Press

defined as shown below:

+	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	3	4	5
3	3	4	5	2	6	1
4	4	5	1	6	2	3
5	5	6	4	1	3	2
6	6	3	2	5	1	4 .

Then $(G, +)$ is a loop with identity 1 and G can be generated by any one of 3, 4, 5, 6 [2, p. 58]. $H = \{1, 2\}$ is the only subloop of G which is different from $\{1\}$ and G.

Define

$$N_0 = \{f : G \to G \mid 1f = 1\} ,$$

$$N_1 = \{f : G \to G \mid 1f = 1, Hf \subseteq H\} ,$$

$$N_2 = \{f : G \to G \mid 1f = 1, Hf = \{1\}\} .$$

Then it is easy to verify that

(1) G is an N_0-loop of type 2 ,

(2) G is an N_1-loop of type 0 but not of type 1 ,

(3) G is an N_2-loop of type 1 but G is not an N_2-loop of type 2 .

LEMMA 3.14. *Let G be an N-loop and let P be an ideal of N such that $P \subseteq A(G)$. Then G has the structure of an $N|P$-loop.*

Proof. Define $g(n+P) = gn$. Suppose $n + P = n' + P$. Then $n = n' + p$ where $p \in P$. Now $gn = g(n'+p) = gn' + gp = gn'$. Hence the mapping $(g, n+P) \to g(n+P)$ of $G \times N|P \to G$ is well defined. It can be easily verified that G has the structure of an $N|P$-loop.

LEMMA 3.15. *Let P be an ideal of N and let G be an $N|P$-loop. Then G has the structure of an N-loop and $P \subseteq A(G)$.*

Proof. Let $g \in G$ and $n \in N$. Define $gn = g(n+P)$. Then it can be easily verified that G has the structure of an N-loop and $P \subseteq A(G)$.

COROLLARY 3.16. *Let G be an N-loop and let P be an ideal of N such that $P \subseteq A(G)$. Then the N-loop kernels of G are the same as the $N|P$-loop kernels of G.*

The proof is easy and will be omitted.

We are now in a position to introduce various radicals for loop near-rings as in the case of near-rings.

DEFINITION 3.17. $J_V(N)$ is defined as the intersection of all annihilating ideals of N-loops of type V in N for $V = 0, 1, 2$. In case N possesses no N-loops of type V then $J_V(N)$ is defined as N itself.

DEFINITION 3.18. $D(N)$ is defined as the intersection of all modular maximal right ideals of N. In case N has no modular maximal right ideals, $D(N)$ is defined as N itself.

DEFINITION 3.19. A loop near-ring N is said to be a V-primitive loop near-ring if there exists an N-loop G of type V such that $A(G) = (0)$.

DEFINITION 3.20. An ideal P of N is called a V-primitive ideal provided $N|P$ is a V-primitive loop near-ring.

COROLLARY 3.21. *An ideal P of N is V-primitive if and only if there exists an N-loop G of type V with $A(G) = P$.*

The proof is easy and will be omitted.

We remark that $J_V(N)$ is the intersection of all V-primitive ideals of N for $V = 0, 1, 2$.

COROLLARY 3.22. *If L is a right ideal in N, then $(L : N) = (0 : N^+ - L)$ where 0 is the identity of the loop $N^+ - L$.*

Proof. $\alpha \in (L : N)$ if and only if $N\alpha \subseteq L$, if and only if $(N^+ - L)\alpha = 0$, if and only if $\alpha \in (0 : N^+ - L)$.

DEFINITION 3.23. A modular right ideal L of N is said to be a V-modular right ideal provided $N^+ - L$ is an N-loop of type V.

We observe that a 0-modular right ideal is a modular maximal right ideal and a 2-modular right ideal is a maximal N-subloop of N^+. Hence

https://doi.org/10.1017/S0004972700008959 Published online by Cambridge University Press

$D(N)$ is the intersection of all 0-modular right ideals.

We now characterize V-primitive ideals of a loop near-ring in terms of V-modular right ideals.

LEMMA 3.24. *L is a V-modular right ideal of N if and only if $(L : N)$ is a V-primitive ideal of N.*

Proof. L is a V-modular right ideal if and only if $N^+ - L$ is an N-loop of type V, if and only if $(0 : N^+-L)$ is a V-primitive ideal. Since $(0 : N^+-L) = (L : N)$, $(L : N)$ is a V-primitive ideal.

LEMMA 3.25. *An ideal P of N is a V-primitive ideal if and only if $P = (L : N)$, where L is a V-modular right ideal of N.*

Proof. If $P = (L : N)$, where L is a V-modular right ideal of N, then, by Lemma 3.24, P is a V-primitive ideal of N. Suppose that P is a V-primitive ideal of N. Then there exists an N-loop G of type V such that $P = A(G)$. Let g be an N-generator of G. Then $G = gN$. Now the mapping $\phi : N^+ \to G$ defined by $\phi(n) = gn$ is an N-loop homomorphism of N^+ onto G. Let $L = \ker(\phi)$ and let $g = ge$ for some $e \in N$. Now for every $n \in N$,

$$g\left(n+en_r\right) = gn + g\left(en_r\right) = gn + (ge)n_r = gn + gn_r = g\left(n+n_r\right) = \bar{0} .$$

Therefore $n + en_r \in L$ for all $n \in N$. Therefore L is a modular right ideal with e as a left identity modulo L. Since G is of type V, $N^+ - L$ is an N-loop of type V. Therefore L is a V-modular right ideal. Now $P = A(G) = (0 : N^+-L) = (L : N)$.

COROLLARY 3.26. *$J_V(N) = \bigcap_L (L : N)$ where L ranges over all V-modular right ideals.*

Proof. $J_V(N) = \bigcap_P P$ where P ranges over all V-primitive ideals of N. Since P is a V-primitive ideal if and only if $P = (L : N)$, where L is a V-modular right ideal of N, $J(N) = \bigcap_L (L : N)$, where L ranges over all V-modular right ideals of N.

The following results will follow in a similar way as in the case of near-rings (see [4]).

THEOREM 3.27. *$J_V(N)$ is the intersection of all V-modular right ideals L in N for $V = 1, 2$.*

LEMMA 3.28. *If $\{L_\alpha \mid \alpha \in \Delta\}$ is a family of right ideals of N, then $\bigcap_{\alpha\in\Delta} (L_\alpha : N) = \left(\bigcap_{\alpha\in\Delta} L_\alpha : N\right)$.*

THEOREM 3.29. *$(D(N) : N) = J_0(N)$.*

COROLLARY 3.30. *$J_0(N)$ is the largest ideal of N contained in $D(N)$.*

DEFINITION 3.31. An ideal L of N is said to be a modular ideal if and only if L is a modular right ideal.

THEOREM 3.32. *Any modular maximal ideal L of N is a 0-primitive ideal.*

4. Quasi-regular elements of type V

The notion of a quasi-regular element in near-rings has been introduced by various authors in different ways. However in the case of loop near-rings we introduce three types of quasi-regular elements.

DEFINITION 4.1. An element z of a loop near-ring N is called a *right quasi-regular* element of type V if there is no V-modular right ideal containing all elements of the form $x + zx_r$, $x \in N$.

We remark that every right quasi-regular element of type 0 is a right quasi-regular element of type 1, and every right quasi-regular element of type 1 is a right quasi-regular element of type 2.

DEFINITION 4.2. A right ideal (loop module) L of N is a quasi-regular right ideal (loop module) of type V if every element of L is a right quasi-regular element of type V, and an ideal L of N is called a quasi-regular ideal of type V provided L is a quasi-regular right ideal of type V.

We remark that a quasi-regular right ideal of type 0 is a quasi-regular right ideal of type 1, and a quasi-regular right ideal of type 1 is a quasi-regular right ideal of type 2.

By Corollary 2.8 it will follow that a left identity modulo a proper modular right ideal L can not be a right quasi-regular element of type 0 .

LEMMA 4.3. *An element* z *of* N *is a right quasi-regular element of type* 0 *if and only if the minimal right ideal containing all elements of the form* $x+zx_r$, $x \in N$, *coincides with* N .

The proof is easy and will be omitted.

Now we prove the following important lemma.

LEMMA 4.4. *Any nilpotent element of* N *is a right quasi-regular element of type* V , $V = 0, 1, 2$.

Proof. Let z be a nilpotent element of N and $z^n = 0$ where n is a positive integer. Let L be a V-modular right ideal containing all elements of the form $x+zx_r$, $x \in N$. Now for each $x \in N$, $x+zx_r$, $zx+z(zx)_r$, ..., $z^{n-1}x+z(z^{n-1}x)_r$ belong to L . Hence $x+zx_r$, $zx+z^2x_r$, ..., $z^{n-1}x+z^nx_r$ belong to L . Since $x+zx_r \in L$, $(x+zx_r)+L = (zx+zx_r)+L$. Since $N^+ - L$ is a loop, we have $x+L = zx+L$. Since $zx+z^2x_r \in L$ and since L is a normal subloop of N^+ ,

$$\left(x+z^2x_r\right)+L = L+\left(x+z^2x_r\right) = (L+x)+z^2x_r$$
$$= (L+zx)+z^2x_r = L+\left(zx+z^2x_r\right) = L .$$

Therefore $x+z^2x_r \in L$ for all $x \in N$. Since $x+z^2x_r \in L$ and $z^2x+z^3x_r \in L$, we have $x+z^3x_r \in L$ for all $x \in N$. Proceeding in this way we finally get $x+z^nx_r \in L$ for all $x \in N$. Now $x+z^nx_r = x \in L$. Hence $L = N$, a contradiction. Hence there is no V-modular right ideal of N containing all elements of the form $x+zx_r$, $x \in N$. Therefore z is a right quasi-regular element of type V , $V = 0, 1, 2$.

COROLLARY 4.5. *Any nil right ideal (loop module) of* N *is a quasi-regular right ideal of type* V, $V = 0, 1, 2$.

The proof of this is a direct consequence of Lemma 4.4.

We now characterize the ideals $J_V(N)$ in terms of right quasi-regular elements of type V.

LEMMA 4.6. $J_V(N)$ *is a quasi-regular ideal of type* V, $V = 0, 1, 2$.

Proof. If $J_V(N) = N$, there is nothing to prove. Suppose $J_V(N) \neq N$. Let z be an element of $J_V(N)$, and assume that z is not a right quasi-regular element of type V. Then there exists a V-modular right ideal, say L, such that L contains all elements of the form $x + zx_r$, $x \in N$. Since $z \in J_V(N)$, z belongs to every V-modular right ideal and in particular $z \in L$. So $zx \in L$ for all $x \in N$. Since $x + zx_r \in L$, $(x+zx_r) + L = (zx+zx_r) + L$. Since $N^+ - L$ is a loop, $x + L = zx + L$. Therefore $x + L = L$. Hence $x \in L$. Then $L = N$, a contradiction. Therefore z is a right quasi-regular element of type V. Hence $J_V(N)$ is a quasi-regular ideal of type V.

THEOREM 4.7. $J_V(N)$ *is the largest quasi-regular right ideal of type* V, $V = 1, 2$.

Proof. By Lemma 4.6, $J_V(N)$ is a quasi-regular right ideal of type V. Now we shall show that $J_V(N)$ contains all the quasi-regular right ideals of type V, $V = 1, 2$. If $J_V(N) = N$ there is nothing to prove. Suppose $J_V(N) \neq N$. Let Q be a quasi-regular right ideal of type V, $V = 1, 2$. Suppose $Q \nsubseteq J_V(N)$. Then there exists a V-modular right ideal, $V = 1, 2$, say L, such that $Q \nsubseteq L$. Let e be a left identity modulo L. Now $N = L + Q$ and $e = n + s$ where $n \in L$, $s \in Q$. Since L is a right ideal, $ea + sa_r = (n+s)a + sa_r \in L$, for all $a \in N$. Since e is a left identity modulo L, $a + ea_r \in L$ for all $a \in N$. Therefore for each $a \in N$, $(a+ea_r) + L = (ea+ea_r) + L$. Since $N^+ - L$ is a loop, $a + L = ea + L$. Now $(a+sa_r) + L = (ea+sa_r) + L = L$. Therefore

https://doi.org/10.1017/S0004972700008959 Published online by Cambridge University Press

$a + sa_r \in L$ for all $a \in N$. So s can not be a right quasi-regular element of type ν, $\nu = 1, 2$. Since $s \in Q$, s is a right quasi-regular element of type ν, $\nu = 1, 2$; a contradiction. Therefore $Q \subseteq L$ and hence $Q \subseteq J_\nu(N)$, $\nu = 1, 2$.

COROLLARY 4.8. *$J_\nu(N)$ contains all nil right ideals of N, $\nu = 1, 2$.*

Proof. Since a nil right ideal is a quasi-regular right ideal of type ν, by Theorem 4.7, $J_\nu(N)$ contains all nil right ideals of N.

COROLLARY 4.9. *$J_\nu(N)$ contains all nilpotent right ideals of N, $\nu = 1, 2$.*

THEOREM 4.10. *$D(N)$ is the largest quasi-regular right ideal of type 0.*

Proof. Let z be an element of $D(N)$ and suppose L is a 0-modular right ideal containing all elements of the form $x + zx_r$, $x \in N$. Now $L \supseteq D(N)$ and hence $z \in L$. Since for each $x \in N$, $x + zx_r \in L$, $(x+zx_r) + L = (zx+zx_r) + L$. Since $N^+ - L$ is a loop, we have $x + L = zx + L$. Since $z \in L$, $zx \in L$. Therefore $x + L = L$. Hence $x \in L$. Then $L = N$, a contradiction. Therefore there is no 0-modular right ideal of N containing all elements of the form $x + zx_r$, $x \in N$. Hence z is a right quasi-regular element of type 0, and therefore $D(N)$ is a quasi-regular right ideal of type 0. Now we shall show that $D(N)$ contains all the quasi-regular right ideals of type 0. If $D(N) = N$ there is nothing to prove. Suppose $D(N) \neq N$. Let Q be any quasi-regular right ideal of type 0 and let L be any 0-modular right ideal. If $Q \nsubseteq L$, then $N = L + Q$. Now proceeding as in Theorem 4.7, we get a contradiction. Therefore $Q \subseteq L$ and hence $Q \subseteq D(N)$.

As there are near-rings where the radicals $J_\nu(N)$ and $D(N)$ are different, from Theorems 4.7 and 4.10 we observe that the three types of quasi-regular right ideals which we introduced are distinct.

COROLLARY 4.11. *$D(N)$ contains all nil right ideals.*

COROLLARY 4.12. *$D(N)$ contains all nilpotent right ideals.*

THEOREM 4.13. *$J_0(N)$ is the largest quasi-regular ideal of type* 0 .

Proof. By Lemma 4.6, $J_0(N)$ is a quasi-regular ideal of type 0 . Let L be any quasi-regular ideal of type 0 . Since a quasi-regular ideal of type 0 is a quasi-regular right ideal of type 0 , $L \subseteq D(N)$. Since $J_0(N)$ is the largest ideal contained in $D(N)$, it follows that $L \subseteq J_0(N)$.

COROLLARY 4.14. *$J_0(N)$ contains all the nil ideals of N .*

COROLLARY 4.15. *$J_0(N)$ contains all the nilpotent ideals of N .*

References

[1] Gerhard Betsch, "Struktursätze für Fastringe", (Dissertation, Universität zu Tübingen, 1963).

[2] Richard Hubert Bruck, *A survey of binary systems*, 2nd edition (Ergebnisse der Mathematik und ihrer Grenzgebiete, **20**. Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[3] Günter Pilz, *Near-rings. The theory and its applications* (North-Holland Mathematics Studies, **23**. North-Holland, Amsterdam, New York, Oxford, 1977).

[4] D. Ramakotaiah, "Radicals for near-rings", *Math. Z.* **97** (1967), 45-56.

Department of Mathematics,
Nagarjuna University,
Nagarjunanagar,
Andhra Pradesh,
India.

https://doi.org/10.1017/S0004972700008959 Published online by Cambridge University Press