ON THE STABILITY OF A SECOND-ORDER DIFFERENTIAL EQUATION

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1. Introduction

In this note we discuss the stability at the origin of the solutions of the differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha\beta & -(\alpha+\beta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

where a dot indicates a differentiation with respect to time, and α, β are real-valued functions of any arguments. We tacitly assume that α, β are such that solutions to (1) do in fact exist. Under the transformation

$$y = \dot{x}; \quad f = \alpha+\beta; \quad g = \alpha\beta,$$

equation (1) takes the equivalent familiar form

$$\ddot{x}+f\dot{x}+gx = 0.$$

As the basis of our work we suppose that the values of the functions α, β lie uniformly between positive real bounds, namely

$$0 < p \leqq \alpha \leqq q \quad \text{and} \quad 0 < r \leqq \beta \leqq s. \tag{2}$$

Then in order to state our results we need the functions P, Q, R and the real number $S = 11.0160938\cdots$ defined by

$$P = P(p, q, r, s) = \left\{ \left(\frac{s-p}{q-r}\right)^{rs-pq} \left(\frac{q-p}{s-r}\right)^{ps-qr} \left(\frac{r^r}{p^p}\right)^{s-q} \left(\frac{q^q}{s^s}\right)^{r-p} \right\}^{1/((r-p)(s-q))},$$

$$Q = Q(p, q, s) = p \left\{ \left(\frac{s-p}{q-p}\right)^{s+q} \frac{q^q}{s^s} \right\}^{1/(s-q)} \exp\left\{ \frac{qs-p^2}{(q-p)(s-p)} \right\},$$

$$\begin{aligned} R = R(p, q, r) &= \{Q(q, p, r)\}^{-1} \\ &= \frac{1}{q} \left\{ \left(\frac{q-p}{q-r}\right)^{r+p} \left(\frac{r^r}{p^p}\right) \right\}^{1/(r-p)} \exp\left\{ \frac{q^2-pr}{(q-p)(q-r)} \right\}, \end{aligned}$$

and

$$S = \exp\left\{ \frac{2(S+1)}{S-1} \right\}.$$

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The functions P, Q, R will only be used for values of p, q, r, s for which they are well defined and positive real valued. Our principal result is

THEOREM 1. *If, for all values of their arguments, the functions* α *and* β *satisfy the bounds* (2), *then for equation* (1) *the origin* $x = y = 0$ *is uniformly asymptotically stable in the large, except possibly if there is a real number* τ *such that* $p < \tau < q$, $r < \tau < s$ *and either*

(i) $p \neq r$; $q \neq s$ *and* $P \leqq 1$,
(ii) $p = r$; $q \neq s$ *and* $Q \leqq 1$,
(iii) $p \neq r$; $q = s$ *and* $R \leqq 1$,
or (iv) $p = r$; $q = s$ *and* $pS/q \leqq 1$.

This theorem is the best obtainable from uniform bounds of the form (2). For each of the exceptional cases (i)—(iv) of the theorem, in section 6 we give examples of functions α, β for which the solution to (1) is unstable. However if one of the cases (i)—(iv) hold with equality in the inequality involving P, Q, R or S, as the case may be, then the solutions to (1) converge either to the origin or to limit cycles. There is no point in considering stability of (1) when the bound p or r is non-positive or when the bounds q and s are both non-finite, because it is easy to choose functions α, β which take arbitrarily small or arbitrarily large values for which solutions to (1) are unstable. It is also worth remarking that, if w is a positive real number, then the values of the functions P, Q, R and pS/q are unchanged by the transformation

$$p \to pw; \quad q \to qw; \; r \to rw; \quad s \to sw.$$

This fact is not surprising because such a transformation corresponds to the change $t \to tw$ of the time variable t.

It is well known (cf. [2], p. 48) that when α and β are real constants then the solution to (1) is stable at the origin if (and only if) $\alpha > 0$ and $\beta > 0$. Also H. H. Rosenbrock recently proved [1] that we get stability of (1) if (2) hold and $q < r$ or $s < p$. Both these results are generalised by theorem 1.

After seeing a typescript of this paper, Rosenbrock kindly drew the authors' attention to the case in which one of q and s is finite whilst the other is infinite. By symmetry in α and β we may as well assume that $q < s = \infty$. He had proved that we then get stability of (1) if (2) hold and $q \leqq r$. Taking the limit $s \to \infty$ of the functions P and Q gives, as a corollary to theorem 1, the most general result, namely

THEOREM 2. *If, for all values of their arguments, the functions* α *and* β *satisfy the bounds*

$$0 < p \leqq \alpha \leqq q \quad \textit{and} \quad 0 < r \leqq \beta,$$

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then for equation (1) *the origin* $x = y = 0$ *is uniformly asymptotically stable in the large, except possibly if* $r < q$ *and either*

$$\text{(i)} \qquad p \neq r \quad \textit{and} \quad \left\{\left(\frac{q-p}{p}\right)^p \left(\frac{r}{q-r}\right)^r\right\}^{1/r-p} \leqq 1,$$

or

$$\text{(ii)} \qquad p = r \quad \textit{and} \quad \frac{p}{(q-p)} \exp\left\{\frac{q}{q-p}\right\} \leqq 1.$$

2. An elementary differential equation

If we consider the vector matrix equation (1) as a pair of simultaneous equations, then by eliminating the time variable t, we have

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-\alpha\beta x - (\alpha+\beta)y}{y}. \tag{3}$$

For arbitrary functions α, β we cannot write down the solution to this equation. However if u, v are positive real numbers, then the general solution of

$$\frac{dy}{dx} = \frac{-uvx - (u+v)y}{y}, \tag{4}$$

is given by

$$|ux+y|^u = c|vx+y|^v \quad \text{if} \quad u \neq v, \tag{5}$$

but by

$$|ux+y| = c \exp\{-ux/(ux+y)\} \quad \text{if} \quad u = v, \tag{6}$$

where c denotes an arbitrary constant. These facts can be verified either by differentiating (5) and (6) or by making the transformation $y = zx$ in the homogeneous equation (4) and integrating.

Consider now the solution curve of (4) which passes through the point $(\lambda, 0)$, where λ is arbitrary. This curve is given by (5) or (6) with

$$c = \begin{cases} |\lambda|^{u-v} u^u v^{-v} & \text{if} \quad u \neq v, \\ |\lambda|\, ue & \text{if} \quad u = v. \end{cases} \tag{7}$$

Further the curve meets the fixed line $mx+y = 0$ at a point $(x(\lambda), y(\lambda))$ depending on λ. Putting $x = -y/m$ in (5) and (6) we find that

$$y(\lambda) = \pm\lambda a(u, v), \tag{8}$$

where $a(u, v)$ denotes the constant

(9) $$a(u, v) = \begin{cases} (|u^{-1}-m^{-1}|^{u}|v^{-1}-m^{-1}|^{-v})^{1/(v-u)} & \text{if} \quad u \neq v, \\ |u|\,|m/(m-u)| \exp\{m/(m-u)\} & \text{if} \quad u = v. \end{cases}$$

The particular value of m which will interest us is given by

(10) $$m = \nu/\mu \text{ where } \nu = qs-pr \text{ and } \mu = q-p+s-r.$$

It is easy to show that if there is a τ such that $p < \tau < q$ and $r < \tau < s$, then

(11) $$p, r < m < q, s.$$

3. The fundamental case

In this section we suppose that the functions α, β satisfy (2) and that

(12) $$p < r < q; \quad r < s; \quad s \neq q.$$

This case is the fundamental case of theorem 1 and the other cases of the theorem will be deduced from lemma 2 below. For the moment we wish to find an upper bound for the value of $\dot{y}$ in (1). We let Γ, Π denote the regions of the (x, y) plane

$$\Gamma = \{(x, y) \mid y \geqq 0, mx+y \geqq 0\} \text{ and } \Pi = \{(x, y) \mid y \geqq 0, mx+y \leqq 0\},$$

where m is given by (10). Then we have

LEMMA 1.

$$\dot{y} \leqq \begin{cases} -prx-(p+r)y & \textit{at the point } (x, y) \textit{ of } \Gamma, \\ -qsx-(q+s)y & \textit{at the point } (x, y) \textit{ of } \Pi. \end{cases}$$

PROOF. Equation (1) shows that

$$\dot{y} = \alpha(-\beta x-y)-\beta y.$$

Since $p \leqq \alpha \leqq q$, for any values of β, x, y the value of $\dot{y}$ must therefore lie between $p(-\beta x-y)-\beta y$ and $q(-\beta x-y)-\beta y$. In turn, since $r \leqq \beta \leqq s$, for any values of x, y the value of $\dot{y}$ must lie between the maximum and the minimum of the numbers

$$-prx-(p+r)y; \quad -psx-(p+s)y; \quad -qrx-(q+r)y; \quad -qsx-(q+s)y.$$

The lemma follows by finding which of these numbers is the largest, subject to (12), and subject to (11) which holds by virtue of (12).

Let us assume for the present that

(13) $$a(p, r)/a(q, s) = P(p, q, r, s), \text{ whenever (12) holds.}$$

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We will prove this result in section 4 and by means of it we can prove our fundamental result, which is

LEMMA 2. *If* (2) *and* (12) *hold and if* $P(p, q, r, s) > 1$, *then for equation* (1) *the origin* $x = y = 0$ *is uniformly asymptotically stable in the large.*

PROOF. We use a Liapunov function in the standard way. In other words we simply define a closed contour V containing the origin with the properties that, (i) each ray from the origin cuts V once, and (ii) there is a $\delta > 0$ such that

$$(\dot{x}, \dot{y}) \cdot \boldsymbol{n} > \delta, \tag{14}$$

at each point η of V, where the vector $(\dot{x}, \dot{y})$ in the scalar product is evaluated from (1) with (x, y) at η, and $\boldsymbol{n}$ denotes the unit inward normal to V at η. Our contour in fact has corners at which $\boldsymbol{n}$ is not defined. For such corners η we need only assume that η belongs to both arcs adjacent to η. If for each positive real number ρ we were to construct a contour $V(\rho)$ by multiplying every coordinate of V by ρ, then from the form of (1) we see that $V(\rho)$ would have the same properties as V. Therefore the lemma will follow from Liapunov's stability theorem (cf. [2], p. 59) if V does in fact have the properties (i) and (ii).

Now the functions P and m are continuous in each of their variables p, q, r, s. We are given that $P(p, q, r, s) > 1$, and (11) holds by virtue of (12). Hence we can choose p', q', r', s' so that

$$\begin{aligned} &p' < p;\ q < q';\ r' < r;\ s < s';\ p' \neq r';\ q' \neq s',\\ &P(p', q', r', s') > 1,\\ &p, r < m' < q, s. \end{aligned} \tag{15}$$

For convenience we will use dashes to denote the effect of replacing the bounds p, q, r, s by the new set p', q', r', s'. For example

$$P' = P(p', q', r', s')$$

whilst

$$\Gamma' = \{(x, y) \mid y \geqq 0,\ m'x + y \geqq 0\},$$

and so on.

Let ζ denote the portion of the solution curve of (4) with $u = q'$, $v = s'$ which starts at $(-\sqrt{P'}, 0)$ and lies in Π'. Similarly let ξ denote that portion of the solution curve of (4) with $u = p'$, $v = r'$ which starts at $(1/\sqrt{P'}, 0)$ and lies in Γ'. We can obtain ζ and ξ from (5) and (7). By virtue of (8) and our assumption (13), we have

$$y'(-\sqrt{P'})/y'(1/\sqrt{P'}) = P'a(q', s')/a(p', r') = 1,$$

and hence ζ and ξ meet the line $m'x+y=0$ at a common point $C=(x'(-\sqrt{P'}), y'(-\sqrt{P'}))$. Next we take a small positive number ε and write B and D for the points of intersection of ζ and ξ respectively with the line $y=\varepsilon$. Further we denote the portions of ζ and ξ between B and C and between C and D by ϕ_2 and ϕ_3 respectively. The lines joining the points $A=(-1, 0)$ and B and the points D and $E=(1, 0)$ we denote by ϕ_1 and ϕ_4 respectively. Finally our contour V consists of the curves ϕ_i and the curves ψ_i obtained from the ϕ_i by symmetry through the origin.

Clearly a ray from the origin meets ϕ_1 or ϕ_4 in at most one point. Moreover a ray will meet ϕ_2 or ϕ_3 in at most one point, as can be seen from (5), (7), (8) and (9). Thus V does have the property that each ray from the origin cuts it just once.

Now the lines ϕ_1, ϕ_4 are nearly horizontal. By inspection of (1) and (2) we see that the vector $(\dot{x}, \dot{y})$ is of bounded magnitude and is nearly vertically upwards on ϕ_1, but nearly vertically downwards on ϕ_4. Hence there is a $\delta > 0$ such that (14) holds on ϕ_1 and ϕ_4.

Next for $i=2, 3$ let η be a point of ϕ_i lying in any line $m_i x+y=0$. Then at η we have $\dot{x}=y \geqq \varepsilon > 0$. On the other hand $\dot{y}$ is bounded above as shown in lemma 1. Hence if θ, $-\pi \leqq \theta \leqq \pi$, is the argument of the vector $(\dot{x}, \dot{y})$ at η then

$$-\frac{\pi}{2} < \theta = \tan^{-1}(\dot{y}/\dot{x}) \leqq \begin{cases} \tan^{-1}\{[-qsx-(q+s)y]/y\} & \text{on } \phi_2, \\ \tan^{-1}\{[-prx-(p+r)y]/y\} & \text{on } \phi_3. \end{cases}$$

By definition of ζ, ξ the slope $\sigma = dy/dx$ of ϕ_i is given by (4) with $u=q'$, $v=s'$ if $i=2$, but with $u=p'$, $v=r'$ if $i=3$. Hence at η we have

$$\sigma - \tan\theta \geqq \begin{cases} [qsx+(q+s)y-q's'x-(q'+s')y]/y & \text{on } \phi_2, \\ [prx+(p+r)y-p'r'x-(p'+r')y]/y & \text{on } \phi_3, \end{cases}$$
$$= \begin{cases} [(s'-m_2)(q'-q)+(q-m_2)(s'-s)]/m_2 & \text{if } i=2, \\ [(m_3-r')(p-p')+(m_3-p)(r-r')]/m_3 & \text{if } i=3. \end{cases}$$

We now point out that there is a $\delta > 0$ such that (14) holds on ϕ_2 and ϕ_3 by showing that $\sigma-\tan\theta$ is bounded below on ϕ_2 and ϕ_3. This result follows from (15) when we make the following observations. For η on ϕ_2 we note that m_2 has a positive lower bound and $m_2 \leqq m'$. For η on that part of ϕ_3 in the second quadrant we have $m' \leqq m_3 < \infty$. For η on that part of ϕ_3 in the first quadrant we have $x \geqq 0$ and $y > \varepsilon$.

We have therefore shown that there is a $\delta > 0$ such that (14) holds on the ϕ_i's. By symmetry through the origin, it also holds on the ψ_i's and the lemma is proved.

Suppose now that (2) and (12) hold and that $P(p, q, r, s) = 1$. Also let V be as defined above but with $p'=p$; $q'=q$; $r'=r$; $s'=s$ and

$\varepsilon = 0$. Then it is clear that solutions to (1) which start on the contour V either follow the contour of V or move inside V. This fact is the basis for our remark in the introduction concerning convergence to the origin or a limit cycle for the first exceptional case of theorem 1. The other exceptional cases could be discussed in a similar way but we shall not do so.

4. The functions P, Q, R and pS/q

Throughout this section we assume that there is a τ such that $p < \tau < q$ and $r < \tau < s$. From (10) we have

$$\begin{aligned} p^{-1} - m^{-1} &= (s-p)(q-p)/p\mu, \\ r^{-1} - m^{-1} &= (q-r)(s-r)/r\mu, \\ m^{-1} - q^{-1} &= (q-r)(q-p)/q\mu, \\ m^{-1} - s^{-1} &= (s-p)(s-r)/s\mu, \end{aligned}$$

and hence by straightforward manipulations from (9),

$$\text{(16)} \qquad a(p, r)/a(q, s) = \begin{cases} P(p, q, r, s) & \text{if } p \neq r;\ q \neq s, \\ Q(p, q, s) & \text{if } p = r;\ q \neq s, \\ R(p, q, r) = [Q(q, p, r)]^{-1} & \text{if } p \neq r;\ q = s, \\ (p/q) \exp\{2(p+q)/(q-p)\} & \text{if } p = r;\ q = s. \end{cases}$$

In view of the statement of theorem 1 the last result may seem out of place so we hasten to point out that

$$\text{(17)} \qquad pS/q \gtreqless 1 \text{ according as } (p/q) \exp\{2(q+p)/(q-p)\} \gtreqless 1.$$

The results (16) and (17) show the origin of the functions P, Q, R, pS/q. Moreover the functions of (16) are related in the way shown by the following elementary limits,

$$\text{(18)} \qquad \lim_{h \to 1-} P(hp, q, p, s) = Q(p, q, s) \text{ if } p < q;\ p < s;\ q \neq s,$$

$$\text{(19)} \qquad \lim_{h \to 1} P(p, q, r, hq) = R(p, q, r) \text{ if } p < q;\ r < q;\ p \neq r,$$

$$\text{(20)} \qquad \lim_{h \to 1-} P(hp, q, p, q/h) = (p/q) \exp\{2(p+q)/(q-p)\} \text{ if } p < q.$$

We will make use of these limits, and the result of

LEMMA 3. *If $\omega > 0$ and T is any one of the functions in (16) then there exist values of p, q, r, s for which T takes the value ω, and there is a τ such that $p < \tau < q$, $r < \tau < s$.*

PROOF. For the last function in (16) the result follows from (17). If T is one of P, Q and R then T is continuous in each of the variables

p, q, r, s. Hence the lemma holds by virtue of the limits below, in which we assume that (12) holds. First we have

$$\text{(21)}\quad \operatorname*{Lim}_{k\to 1+} P(p, kr, r, s) = \operatorname*{Lim}_{p\to \min(q,s)} Q(p, q, s) = \operatorname*{Lim}_{p\to \max(p,r)} R(p, q, r) = \infty.$$

On the other hand

$$P(p, kq, r, ks) = \frac{1}{k}\left\{\left(\frac{ks-p}{kq-r}\right)^{rs-pq}\left(\frac{kq-p}{ks-r}\right)^{ps-qr}\left(\frac{r^r}{p^p}\right)^{s-q}\left(\frac{q^q}{s^s}\right)^{r-p}\right\}^{1/(r-p)(s-q)},$$

so that

$$\operatorname*{Lim}_{k\to\infty} P(p, kq, r, ks) = 0,$$

and hence

$$\operatorname*{Lim}_{k\to\infty} Q(p, kq, ks) = \operatorname*{Lim}_{k\to\infty} R(p, kq, r) = 0.$$

This proves the lemma.

One further limit that we will use is

$$\text{(22)}\qquad \operatorname*{Lim}_{k\to 1+} P(p, q, r, kr) = \infty \quad \text{if} \quad p < r < q.$$

5. Proof of theorem 1

We prove the theorem by cases. In each case we replace the bounds p, q, r, s of (2) by new bounds in such a way that with the new bounds (2) still hold, inequalities (12) hold, and $P > 1$. Thus we satisfy the conditions of lemma 2 and so have uniform asymptotic stability in the large for the solutions of (1). In each of the cases below the values of the arguments in the expression $P > 1$ indicate the values of the new bounds. We adopt this convention to save repeating ourselves. Also by symmetry in α, β in (1), without loss of generality, we assume that $p \leqq r$ in (2).

Case a. There is no τ for which $p < \tau < q$ and $r < \tau < s$.

Case a (i). $q \leqq r$. We choose s' with $s < s'$ and then, in view of (21), we choose $k > 1$ but sufficiently close to 1 that $kr \neq s'$ and $P(p, kr, r, s') > 1$.

Case a (ii). $r < q$. Then we must have $r = s$. We choose p' with $p' < p$ so that $p' < r < q$ and then, in view of (22), we choose $k > 1$ so that $q \neq kr$ and $P(p', q, r, kr) > 1$.

Case b. There is a τ such that $p < \tau < q$ and $r < \tau < s$.

Case b (i). $p \neq r$; $q \neq s$; $P > 1$. This is the case of lemma 2.

Case b (ii). $p = r$; $q \neq s$; $Q > 1$. In view of (18) we choose h, $0 < h < 1$, so that $P(hp, q, p, s) > 1$.

Case b (iii). $p \neq r$; $q = s$; $R > 1$. In view of (19) we choose $h > 1$ so that $q \neq hq$ and $P(p, q, r, hq) > 1$.

Case b (iv). $p = r$; $q = s$; $pS/q > 1$. Then in view of (17) and (20)

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we choose h, $0 < h < 1$, so that $P(hp, q, p, q/h) > 1$, and this completes the proof of theorem 1.

6. Unstable solutions

For each of the exceptional cases of theorem 1, we now give examples of functions α, β for which the solution to (1) is unstable. Suppose therefore that there exists a τ such that $p < \tau < q$ and $r < \tau < s$. Then choose one of the exceptional cases (i)—(iv) of theorem 1 and let T denote the *value* of the corresponding function in (16). We then define functions α, β by the rule, if (x, y) or $(-x, -y)$ is in Γ then $\alpha = p$ and $\beta = r$, otherwise $\alpha = q$ and $\beta = s$. Examination of (1) then shows that the solution to (1) moves clockwise round the origin in the (x, y) plane. Using the results of section 2 and equations (16), we then see that the solution curve to (1) which starts at the point $(\lambda, 0)$ with $\lambda < 0$ passes in turn through the following points of the x-axis and line $mx+y = 0$,

$$(\lambda, 0); \quad \big(\lambda a(q, s)/m, -\lambda a(q, s)\big) = \big(\lambda a(p, r)/mT, -\lambda a(p, r)/T\big);$$
$$(-\lambda/T, 0); \quad \big(-\lambda a(q, s)/mT, \lambda a(q, s)/T\big) = \big(-\lambda a(p, r)/mT^2, \lambda a(p, r)/T^2\big); \ (\lambda/T^2, 0).$$

Hence if $T = 1$ the solution is a limit cycle. If on the other hand $T < 1$ then the solution diverges. That T can take such values was shown in Lemma 3.

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