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THE INFLUENCE ON A FINITE GROUP OF ITS PERMUTABLE SUBGROUPS

BY

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Huppert, Janko and Mann have proved the following theorems for a finite group G.

(Huppert [4]). If each second maximal subgroup of G is normal in G, then G is supersolvable. If the order of G is divisible by at least three different primes, then G is nilpotent.

(Huppert [4]). Let each third maximal subgroup of G be normal in G. Then: (i) G' is nilpotent; (ii) the rank of $G=r(G)\leq 2$; (iii) if $|G|$ is divisible by at least three different primes, then G is supersolvable.

(Janko [5]). Let G be solvable. If each fourth maximal subgroup of G is normal in G, then: (i) $r(G)\leq 3$; (ii) if $|G|$ is divisible by at least four distinct primes, then G is supersolvable.

(Mann [7]). Let G be solvable, and each n-th maximal subgroup of G be quasinormal in G. Then: (i) $r(G)\leq n-1$; (ii) if $|G|$ is divisible by at least $n-k+1$ distinct primes, then $r(G)\leq k$, where $k\geq 1$.

The aim here is to improve these results. In §2, we prove them under the weaker assumption that each i-th maximal subgroup ($i=2, 3, 4$) be π-quasinormal instead of normal or quasinormal (see the definitions below). Incidentally the concept of π-quasinormality as a generalization of quasinormality was introduced by Kegel in [6]. Throughout, the groups are *finite*.

1. Definitions and assumed results.

DEFINITIONS. Subgroups H and K of the group G permute if $\langle H, K\rangle=HK=KH$. A subgroup of G is π-quasinormal (quasinormal) in G if it permutes with every Sylow subgroup (every subgroup) of G. H_G, the core of H in G, is the largest normal subgroup of G contained in H. H_{SG}, the subnormal core of H in G, is the largest subnormal subgroup of G contained in H. If G is solvable, then the rank of G, denoted $r(G)$, is the maximal integer n such that G has a chief factor of order p^n, for some prime p.

We now list for an easy reference some known results which are frequently used later:

(1.1) [6]. A π-quasinormal subgroup of G is subnormal in G.

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(1.2) [6]. A maximal π-quasinormal subgroup of G is normal in G.

(1.3) [6]. If $H \leq K \leq G$ and H is π-quasinormal in G, then H is π-quasinormal in K.

(1.4). If $N \leq H \leq G$ and N is normal in G, then H is π-quasinormal in G if and only if H/N is π-quasinormal in G/N (its proof is straightforward and therefore omitted).

(1.5) [7]. If M is a maximal subgroup of G, then $M_{SG} = M_G$.

(1.6) [3]. If all proper subgroups of the non-nilpotent group G are nilpotent, then G is solvable; $|G| = p^a q^b$ for distinct primes p and q; the Sylow p-subgroup G_p is normal and each Sylow q-subgroup G_q is cyclic.

(1.7) [2]. If each maximal subgroup of G is supersolvable, then: (i) G is solvable; (ii) G has a Sylow tower for the natural (descending) ordering of prime divisors of $|G|$, or G satisfies the hypotheses of (1.6); (iii) if G itself is not supersolvable, then G has exactly one normal Sylow subgroup.

2. **Generalized results.** For a group G, we prove the following theorems:

THEOREM 2.1. *If every second maximal subgroup of G is π-quasinormal in G, then G is supersolvable. Furthermore, if $|G|$ is divisible by at least three different primes, then G is nilpotent.*

THEOREM 2.2. *If every third maximal subgroup of G is π-quasinormal in G, then:*

(i) *if $|G|$ is divisible by three or more different primes, then G is supersolvable;*

(ii) *the commutator subgroup G' of G is nilpotent;*

(iii) *the rank of $G = r(G) \leq 2$.*

THEOREM 2.3. *Let G be solvable. If every fourth maximal subgroup of G is π-quasinormal in G, then:*

(i) *if $|G|$ is divisible by four or more different primes, then G is supersolvable;*

(ii) *$r(G) \leq 3$.*

Proof of Theorem 2.1. Let M be a maximal subgroup of G. Then every maximal subgroup of M is π-quasinormal in G. This means that all maximal subgroups of M are π-quasinormal in M by (1.3) and, therefore, they are normal in M by (1.2). Hence M is nilpotent and so all proper subgroups of G are nilpotent. Now by (1.6), G is solvable. In addition, if $|G|$ is divisible by three or more different primes, then G is nilpotent and we have disposed of this case.

Next we consider the case where $|G|$ is divisible by, at most, two distinct primes. To prove that G is supersolvable, we must show that $[G:M]$, the index of M in G, is a prime for an arbitrary but fixed maximal subgroup M of G since a theorem of Huppert states that a group is supersolvable if and only if its maximal subgroups have prime index. If $M_G \neq 1$, then, since by (1.4) G/M_G satisfies the hypothesis of the theorem, G/M_G is supersolvable by induction. From this it follows that $[G/M_G:M/M_G]=[G:M]$ is a prime. Therefore, we may assume that $M_G=1$, and form the maximal chain: $M_1<M<G$, where M_1 is maximal in M. Since M_1 is π-quasinormal in G, by (1.1) M_1 is subnormal in G. Hence $M_1 \leq M_{SG}=M_G=1$ which implies that $M_1=1$. But M is nilpotent and, therefore, $|M|=p$, a prime. Now consider $[G:M]$ which is a power of a prime since G is solvable. If $[G:M]$ is a power of p, then G is a p-group and we are finished. On the other hand, if $[G:M]=q^m$, $q \neq p$, then $|G|=pq^m$. Let G_q be a Sylow q-subgroup of G and L be a maximal subgroup of G_q. Then G_q is maximal in G, and L is π-quasinormal in G. Since M is a Sylow p-subgroup of G, $LM=ML$ is a subgroup of G. But M is maximal in G and $LM \neq G$. Therefore, $LM=M$. This implies that $L \leq M$ and so $L=1$. Hence $|G_q|=q$ showing that $[G:M]=q$, a prime. This completes the proof.

Remark. If we simply require that every second maximal subgroup of G be subnormal in G, then G is not necessarily supersolvable, as confirmed by A_4, the alternating group of degree 4.

Proof of Theorem 2.2. (i) From (1.3) and the Theorem 2.1 it follows that every maximal subgroup of G is supersolvable. Hence by (1.7), G is solvable. Moreover, if the order of G is divisible by at least four different primes, then G is supersolvable. Thus we need only consider the case in which $|G|$ is divisible by three different primes. Before proceeding, it should be noted that every second maximal subgroup of G is nilpotent by (1.2) and (1.3) and therefore every third maximal subgroup of G is also nilpotent.

Let $|G|=p^\alpha q^\beta r^\gamma$ where $p>q>r$ and $\alpha, \beta, \gamma>0$. Suppose that G is not supersolvable. Then, since (1.6) does not hold, it follows from (1.7) that the Sylow p-subgroup G_p is normal in G and no other Sylow subgroup of G is normal in G. Since G is solvable, there exist Sylow subgroups G_q and G_r such that G_qG_r is a subgroup. Let $H=G_qG_r$. If H is not maximal in G, then G_q is contained in a third maximal subgroup of G. Since each third maximal subgroup is nilpotent and subnormal (being π-quasinormal; see (1.1)), it follows that G_q is subnormal in G. But a subnormal Sylow subgroup is always normal and so G_q is normal in G, a contradiction. Hence H is maximal in G.

Now suppose that $\beta \geq 2$. Since every maximal subgroup of G is supersolvable, H is supersolvable, too. Therefore G_r is properly contained in a maximal subgroup of H. This means that G_r is contained in a third maximal subgroup of G. Hence, as before, $G_r \lhd G$, again a contradiction. Thus $\beta=1$. By a similar argument, $\gamma=1$ and so $|G|=p^\alpha qr$. Next suppose L is a maximal subgroup of G_p and consider the following maximal chain:

$$L<G_p<G_pG_q<G.$$

From this we see that L is π-quasinormal in G. Hence L permutes with H and therefore LH is a subgroup. Since H is maximal in G and $LH \neq G$, $LH=H$. Thus

$L \leq H$ and so $L=1$, which means that $\alpha=1$. Hence G is supersolvable, a contradiction to our assumption that G is not supersolvable. Therefore, we have the desired result.

(ii) In view of part (i) and the fact that the commutator subgroup of a supersolvable group is always nilpotent, we may assume that G is not supersolvable and $|G|$ is divisible by two different primes p and q. We may further assume without loss of generality (see (1.7)) that $G_p \lhd G$. Then G_q is not normal in G and we will show that G_q is either abelian or cyclic.

If $(G_q)_G \neq 1$, then, since $|G/(G_q)_G|$ is divisible by both primes p and q, $(G/(G_q)_G)'$ is nilpotent by induction. Clearly $(G/G_p)'$ is nilpotent. Since $(G/G_p)' \cong G'/G' \cap G_p$ and $(G/(G_q)_G)' \cong G'/G' \cap (G_q)_G$, it follows that $G'/(G' \cap G_p) \cap (G' \cap (G_q)_G) \cong G'$ is nilpotent. So suppose that $(G_q)_G=1$. If G_q is maximal in G, then every second maximal subgroup of G_q is π-quasinormal (hence subnormal) in G. Since $(G_q)_G = (G_q)_{SG}=1$, all second maximal subgroups of G_q are 1. Therefore $|G_q| \leq q^2$ which implies that G_q is abelian. On the other hand, if G_q is not maximal in G, then there exists a maximal subgroup M of G such that $G_q < M < G$. Now if G_q is not maximal in M, then, as in part (i), $G_q \lhd G$, a contradiction. Therefore $G_q < M < G$ is a maximal chain. Hence every maximal subgroup of G_q is subnormal (being π-quasinormal) in G. Since G_q is not subnormal in G, G_q must have a unique maximal subgroup and so G_q is cyclic. Now to show that G' is nilpotent we need only note that $G' \leq G_p$ since $G/G_p (\cong G_q)$ is abelian. This proves part (ii).

(iii) Again, the only case that requires a proof is the one in which $|G|$ is divisible by two distinct primes, p and q. We further assume that G is not supersolvable, otherwise $r(G)=1$. As in part (ii), we suppose that G_p is the only Sylow subgroup of G which is normal in G.

Let $N_i \lhd G$ and $N_i \neq 1$ for $i=1$ and 2. If p and q both divide $|G/N_i|$, then by induction $r(G/N_i) \leq 2$, and if G/N_i is a p or q-group, then $r(G/N_i)=1 \leq 2$. Hence if $N_1 \cap N_2=1$, then $r(G/N_1 \cap N_2)=r(G)=\max \{r(G/N_i)\} \leq 2$ and we are done. Thus we may assume that G has a unique minimal normal subgroup N. Since $G_p \lhd G$, N is a p-subgroup. It now suffices to show that $|N| \leq p^2$ because we already have $r(G/N) \leq 2$.

Let G_q be a Sylow q-subgroup of G. If $N \neq G_p$, then $NG_q \neq G$. Hence NG_q is supersolvable. Since $G_p \lhd G$, it follows that its center $Z(G_p) \lhd G$. Thus $N \leq Z(G_p)$ and so every subgroup of N is normal in G_p. From this it easily follows that N is also a minimal normal subgroup of NG_q, which implies that $|N|=p<p^2$. On the other hand, if $N=G_p$, then G_p is abelian and G_q is maximal in G. Since G has a unique minimal normal subgroup, it follows that $(G_q)_G=(G_q)_{SG}=1$. Hence every second maximal subgroup of G_q is 1 and so $|G_q| \leq q^2$. First, suppose that $|G_q|=q^2$ and consider the following maximal chain:

$$L < G_p < G_pK < G,$$

where L is maximal in G_p and K is maximal in G_q. Now L is π-quasinormal in G, and so LG_q is a subgroup of G. But $LG_q \neq G$, therefore $LG_q = G_q$. From this we conclude that $L=1$ which shows that $|N|=|G_p|=p \leq p^2$. Next, assume that $|G_q|=q$. Then, in the same manner, it follows that every second maximal subgroup of G_p is 1 which, in turn, proves that $|N|=|G_p| \leq p^2$. This completes the proof of part (iii) and of the thoerem.

REMARK. The group A_4 shows that if the order of a group is divisible by two different primes and if its third maximal subgroups are π-quasinormal, then the group need not be supersolvable in general.

Proof of Theorem 2.3. (i) Let M be an arbitrary but fixed maximal subgroup of G. By (1.3), third maximal subgroups of M are π-quasinormal in M. Since $|G|$ is divisible by at least four different primes and G is solvable, $|M|$ is divisible by at least three different primes. Hence by part (i) of Theorem 2.2, M is supersolvable. Now G is supersolvable by a theorem of Huppert [4].

(ii) We use induction on $|G|$. In view of part (i), the only cases that need proof are the ones in which $|G|$ is divisible by three and two different primes, respectively. We treat these cases separately. Before proceeding, we should observe that each second maximal subgroup of G is supersolvable by (1.3) and Theorem 2.1.

Case 1. $|G|$ is divisible by two primes, p and q. Then, as in part (iii) of Theorem 2.2, we can assume that G has a unique minimal normal subgroup N and $r(G/N) \leq 3$. Without loss of generality, let $|N|=p^n$. Now it is enough to show that $n \leq 3$. For this, let G_p be a Sylow p-subgroup and G_q be a Sylow q-subgroup of G. First, suppose that $N \neq G_p$, and consider NG_q. If NG_q is not maximal in G, then NG_q is supersolvable. From this, it easily follows that G_q is maximal in NG_q and so $|N|=p<p^3$. On the other hand, if NG_q is maximal in G, then we claim that $|G_q| \leq q^2$. To show this, let $|G_q| \geq q^3$, and consider the chain:

$$L_2 < NL_2 < NL_1 < NG_q < G,$$

where L_2 is maximal in L_1, L_1 is maximal in G_q and $L_2 \neq 1$. This implies that L_2 is contained in a fourth maximal subgroup of G. But fourth maximal subgroups are nilpotent and subnormal, and so L_2 is subnormal in G. Thus L_2 is contained in every Sylow q-subgroup of G, which means that there is a nontrivial normal q-subgroup of G, a contradiction. Hence $|G_q| \leq q^2$. We now have two possibilities: (a) Suppose $|G_q|=q^2$. Let L be a maximal subgroup of G_q and H be a maximal subgroup of N. If $H=1$, then $|N|=p$ and if $H \neq 1$, then we form the following maximal chain:

$$H < N < NL < NG_q < G.$$

From this, we see that H is π-quasinormal in G. Therefore HG_q is a subgroup and is clearly maximal in NG_q. Now consider the chain:

$$G_q < HG_q < NG_q < G.$$

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If G_q is not maximal in HG_q, then G_q is subnormal in G, a contradiction. Hence G_q must be maximal in HG_q. Since HG_q is supersolvable, we have $|H|=p$, which shows that $|N|=p^2<p^3$, the desired conclusion. (b) Next, suppose $|G_q|=q$ and form the maximal chain:

$$A < B < N < NG_q < G.$$

If $A=1$ or $B=1$, then $|N|\leq p^2$ and if $A\neq 1$, then, as before, the chain:

$$G_q < AG_q < NG_q < G$$

implies that $|A|=p$, which means that $|N|=p^3$ and we are finished.

Now suppose $N=G_p$. Then G_q is maximal in G and, since N is the unique minimal normal subgroup of G, $(G_q)_G=(G_q)_{SG}=1$ and so every third maximal subgroup of G_q is 1. Hence $|G_q|\leq q^3$. First, suppose that $|G_q|=q^3$ and let L be a maximal subgroup of G_q and K be a maximal subgroup of L. Then

$$N < NK < NL < G = NG_q$$

is a maximal chain. Let M be a maximal subgroup of N. Since M is π-quasinormal, MG_q is a subgroup. But $MG_q\neq G$, hence $G_q=MG_q$ by the maximality of G_q. Thus $M=1$ and so $|N|=|G_p|=p$. Likewise, it can be shown that if $|G_q|=q^2$, then $|N|\leq p^2$ and if $|G_q|=q$, then $|N|\leq p^3$. This completes the proof of Case 1.

Case 2. $|G|$ is divisible by three distinct primes p, q, and r. Let G/K be any proper factor group of G. If $|G/K|$ is a prime-power, then $r(G/K)=1<3$; if $|G/K|$ is divisible by two distinct primes, then by Case 1, $r(G/K)\leq 3$; and if $|G/K|$ is divisible by all primes p, q and r, then by induction, $r(G/K)\leq 3$. So, as before, we may assume that N is the unique minimal normal subgroup of G. Without loss of generality, let $|N|=p^n$. We must show $n\leq 3$. Since G is solvable, there exists a subgroup M such that $M=G_qG_r$ for some G_q and G_r. Now if $N\neq G_p$, then $NM\neq G$. Hence from the chain:

$$G_q < NG_q < NM < G,$$

it follows that G_q is maximal in NG_q, for otherwise G_q would be subnormal in G, which is impossible. But NG_q is supersolvable, hence $[NG_q:G_q]=|N|=p$. On the other hand, if $N=G_p$, then M is maximal in G. Since $N \cap M_G=1$, it follows from the uniqueness of N that $M_{SG}=M_G=1$. Thus every third maximal subgroup of M is 1. This implies that N is either a third or a second maximal subgroup of G. Note that N cannot be a first maximal subgroup of G. First, suppose that N is a third maximal subgroup of G and let H be a maximal subgroup of N. Then H is π-quasinormal in G and, since $M=G_qG_r$, $HM=MH$ is a subgroup. But M is maximal and $HM\neq G$. Hence $HM=M$, which means that $H=1$. Thus $|N|=p$ and we are done. Similarly, one can verify that $|N|\leq p^2$ when N is second maximal in G. This proves Case 2 and the theorem.

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