BULL. AUSTRAL. MATH. SOC. 47B20, 47B47
VOL. 31 (1985), 235-243.

ON SUBNORMALITY OF GENERALIZED DERIVATIONS AND TENSOR PRODUCTS

BOJAN MAGAJNA

Subnormal and quasinormal tensor product operators and generalized derivations on the Hilbert-Schmidt class will be characterized.

Introduction

Let H be a complex Hilbert space, $B(H)$ the algebra of all bounded linear operators on H. For $1 \le p < \infty$ the von Neumann-Schatten class, $C_p(H)$, is defined to be the set of all elements T in $B(H)$ such that $\sum_{k\in K} |\langle T\psi_k, \psi_k\rangle|^p < \infty$ for each orthonormal system $\{\psi_k : k \in K\}$ in H (see [9]). For fixed $A, B \in B(H)$ let $\delta_{A,B}$ and $\tau_{A,B}$ be the operators on $B(H)$ defined by

(1) $$\delta_{A,B}(X) = AX - XB ,$$

(2) $$\tau_{A,B}(X) = AXB .$$

Operators of the form (1) are called generalized derivations and they (as well as their restrictions $\delta_{A,B}|C_p$) have been extensively studied in the past, especially their spectral properties (see, for example, [8], p. 79

Received 28 September 1984.

for some historical notes). In [1] Anderson and Foias obtained the characterization of spectral generalized derivations and Shaw characterized in [10] Hermitian and normal operators of the form $\delta_{A,B}|X$ where X is a subspace of $B(H)$ which satisfies suitable conditions (in particular X can be $C_p(H)$). Now $C_2(H)$ is a Hilbert space with respect to the inner product

$$(3) \qquad \langle X, Y\rangle = \operatorname{tr}(Y^*X) , \quad X, Y \in C_2(H)$$

(where tr denotes the trace) and so the concepts of subnormality and quasinormality make sense. It is a purpose of this note to characterize subnormal and quasinormal operators of type $\delta_{A,B}|C_2(H)$ and $\tau_{A,B}|C_2(H)$. Note that $\tau_{A,B}|C_2(H)$ can be identified with $A \otimes B^*$ (see [7]) and thus we will obtain in this way a characterisation of subnormal and quasinormal tensor products.

Since the Hilbert space H and the operators A, B will be fixed in what follows, we shall denote simply $C_2 = C_2(H)$, $\delta = \delta_{A,B}|C_2$, $\tau = \tau_{A,B}|C_2$.

1. Subnormality

By (a special case of) Theorem 2.2 in [10], δ is normal if and only if A and B are normal operators. The following theorem characterizes subnormal operators δ and τ . Recall that an operator $S \in B(H)$ is subnormal if and only if there exists a bounded normal operator N on some larger Hilbert space $K \supset H$ such that the restriction of N to H is S . N is then called the normal extension of S .

THEOREM 1. *Let δ and τ be defined on C_2 by (1) and (2). Then δ is subnormal if and only if A and B^* are subnormal operators. Moreover, if $A \neq 0$ and $B \neq 0$ the same statement holds for τ .*

Proof. Suppose first that A and B^* are subnormal and denote by M and N^* their (not necessarily minimal) normal extensions. Clearly we may assume that M and N act on the same Hilbert space $K \supset H$. Relative to the decomposition $K = H \oplus H^{\perp}$ the operators M and N^* can be represented by the matrices

https://doi.org/10.1017/S0004972700004718 Published online by Cambridge University Press

$$(4) \qquad M = \begin{pmatrix} A & A_1 \\ 0 & A_2 \end{pmatrix}, \quad N^* = \begin{pmatrix} B^* & B_1 \\ 0 & B_2 \end{pmatrix},$$

where A_1, A_2, B_1, B_2 are certain bounded operators. Now we can regard $C_2 = C_2(H)$ as a subspace of $C_2(K)$ via the embedding

$$X \mapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, \quad X \in C_2 .$$

A straightforward computation with matrices (4) shows that C_2 is an invariant subspace for the operator $\delta_{M,N}$ defined on $C_2(K)$ by $\delta_{M,N}(X) = MX - XN$ and that $\delta_{M,N}|C_2 = \delta$. By Theorem 2.2 of [10], $\delta_{M,N}$ is a normal operator on $C_2(K)$ (this can be also verified directly using (3)). Thus δ is subnormal. The proof that τ is subnormal is the same since an easy computation gives that the operator $\tau_{M,N}$ is normal on $C_2(K)$.

To prove the converse we shall use the following theorem of Halmos and Bram (see [2] or [4]).

An operator $T \in B(H)$ is subnormal if and only if

$$(5) \qquad \sum_{j,k=0}^{n} \left\langle T^j f_k, T^k f_j \right\rangle \geq 0$$

for every finite subset $f_0, \ldots, f_n$ of H.

Suppose that δ is subnormal. In order to apply (5) with δ instead of T express the powers δ^j by

$$\delta^j X = \sum_{s=0}^{j} (-1)^s \binom{j}{s} A^{j-s} X B^s , \quad X \in C_2 .$$

Taking into account also the definition (3) of inner product in C_2 we see that (5) assumes the form

$$(6) \qquad \sum_{j,k=0}^{n} \sum_{r=0}^{j} \sum_{s=0}^{k} (-1)^{r+s} \binom{j}{r} \binom{k}{s} \operatorname{tr}\left(B^{*s} X_j^* A^{*k-s} A^{j-r} X_k B^r\right) \geq 0$$

https://doi.org/10.1017/S0004972700004718 Published online by Cambridge University Press

where $X_1, \ldots, X_n$ are arbitrary elements of C_2. Now let f_j, g_j, $j = 1, \ldots, n$ be any vectors in H and put $X_j = f_j \otimes g_j$ (that is, $X_j(h) = \langle h, g_j\rangle f_j$, $h \in H$). Then, after a simple computation, we get, from (6),

$$(7) \qquad \sum_{j,k=0}^{n} \sum_{r=0}^{j} \sum_{s=0}^{k} (-1)^{r+s} \binom{j}{r}\binom{k}{s} \left\langle A^{j-r} f_k, A^{k-s} f_j \right\rangle \left\langle B^{*s} g_j, B^{*r} g_k \right\rangle \geq 0 .$$

We will show how (7) implies that A is subnormal. The proof that B^* is subnormal is similar and will be omitted. Without loss of generality we may assume that 0 is an approximate eigenvalue of B^*. (Otherwise we can replace A and B with $A - \alpha$ and $B - \alpha$ respectively, where $\bar{\alpha}$ is an approximate eigenvalue for B^*; this is possible since $\delta_{A,B} = \delta_{A-\alpha,B-\alpha}$.) Let (h_m) be the corresponding sequence of approximate eigenvectors (that is, $\|h_m\| = 1$ and $\lim\|B^* h_m\| = 0$). For fixed m put $g_1 = g_2 = \ldots = g_n = h_m$ in (7), then let m tend to infinity. It follows that

$$\sum_{j,k=0}^{n} \left\langle A^j f_k, A^k f_j \right\rangle \geq 0$$

and this implies that A is subnormal by the Bram-Halmos theorem.

The proof that subnormality of τ implies subnormality of A and B^* is similar. Instead of (7) we have here an analogous condition (derived in the same way as (7))

$$(8) \qquad \sum_{j,k=0}^{n} \left\langle A^j f_k, A^k f_j \right\rangle \left\langle B^{*k} g_j, B^{*j} g_k \right\rangle \geq 0 .$$

Since $A \neq 0$, $B \neq 0$ by assumption it follows that $\tau \neq 0$ and hence $\sigma(\tau) \neq \{0\}$ by subnormality. Now the theorem of Brown and Pearcy in [3] tells that $\sigma(\tau) = \sigma(A) \cdot \sigma(B)$, hence there is a $\beta \neq 0$ in the boundary of $\sigma(B^*)$. Then β is an approximate eigenvalue of B^*; let (h_m) be the corresponding sequence of eigenvectors. Replace now in (8) all g_j, $j = 1, \ldots, n$, with the same vector h_m and then take the limit as m tends to infinity. It follows

$$\sum_{j,k=0}^{n} \beta^k \bar{\beta}^j \left\langle A^j f_k, A^k f_j \right\rangle \geq 0$$

and this implies that A is subnormal since f_j are arbitrary and $\beta \neq 0$. The proof that B^* is subnormal is similar. //

2. Quasinormality

An operator $T \in \mathcal{B}(H)$ is called quasinormal if and only if it commutes with T^*T ([4], [6]).

THEOREM 2. *Let* δ *and* τ *be defined on* C_2 *by* (1) *and* (2).

(i) δ *is quasinormal if and only if one of the following holds:*

(a) A *and* B *are both normal;*

(b) there exists $\lambda \in \mathbb{C}$ *such that* $A = \lambda I$ *and* $(B-\lambda I)^*$ *is quasinormal;*

(c) there exists $\lambda \in \mathbb{C}$ *such that* $B = \lambda I$ *and* $A - \lambda I$ *is quasinormal.*

Here of course I *is the identity operator on* H.

(ii) If $A \neq 0$ *and* $B \neq 0$ *then* τ *is quasinormal if and only if* A *and* B^* *are quasinormal.*

In the proof of this theorem the following result of Fong and Sourour will be used (see [5]).

(FS) Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ be finite subsets of $\mathcal{B}(H)$. Suppose that

$$A_1XB_1 + \ldots + A_nXB_n = 0$$

for all $X \in C_2$ and that $A_1, \ldots, A_k$ are linearly independent. Then $B_1, \ldots, B_k$ can be expressed as linear combinations of $B_{k+1}, \ldots, B_n$. (In particular for $k = n$ this means $B_1 = \ldots = B_n = 0$. Also the role of $\mathcal{A}$ and $\mathcal{B}$ can be interchanged.)

Actually in [5] this result is stated for $\mathcal{B}(X)$ (where X is any Banach space) instead of C_2 but (FS) follows at once since C_2 is strongly dense in $B(H)$.

Proof of Theorem 2. *(i)* Since δ^* , the adjoint of δ , is given by $\delta^*(X) = A^*X - XB^*$ (as a direct verification would show) the quasinormality condition $\delta^*\delta^2 - \delta\delta^*\delta = 0$ can be written as

$$(9)\quad \left(A^*A^2-AA^*A\right)XI + (AA^*-A^*A)XB - AX(B^*B-BB^*) - X\left(B^2B^*-BB^*B\right) = 0 , \quad \text{for all } X \in C_2 ,$$

where I denotes the identity operator.

If B is not normal then I and $B^*B - BB^*$ are linearly independent since 0 is the only scalar commutator ([6], Problem 230). Hence it follows from (9) by (FS) that A can be expressed as a linear combination of commuting self-adjoint operators I and $AA^* - A^*A$. Thus A is normal and in fact a scalar multiple of I . If we put $A = \lambda I$ in (9) we get

$$B^2B^* - BB^*B + \lambda(B^*B-BB^*) = 0 .$$

This equation can be written also as

$$(B-\lambda I)^2(B-\lambda I)^* - (B-\lambda I)(B-\lambda I)^*(B-\lambda I) = 0$$

which is obviously equivalent to the quasinormality of $(B-\lambda I)^*$.

The case when A is not normal is treated in the same way. Now only the case when A and B are both normal remains, but then δ is normal.

(ii) Since $\tau^*(X) = A^*XB^*$, $X \in C_2$, the quasinormality condition $\tau^*\tau^2 - \tau\tau^*\tau = 0$ is equivalent to

$$(10)\qquad AA^*AXBB^*B - A^*A^2XB^2B^* = 0 , \quad X \in C_2 .$$

If A and B^* are quasinormal then obviously (10) is satisfied. Conversely, if (10) is satisfied then AA^*A and A^*A^2 are linearly dependent. (Otherwise it would follow that $BB^*B = 0$ by (FS) and hence $\|B\|^4 = \|B^*BB^*B\| = 0$, but $B \neq 0$ by assumption.) Thus we have

$$(11)\qquad A^*A^2 = \lambda AA^*A$$

for some $\lambda \in \mathbf{C}$. If we prove that $\lambda = 1$ then A will be quasinormal and since the quasinormality of B^* can be proved similarly this will complete the proof of the theorem. Now (11) implies that

$$A^{*2}A^2 = \lambda A^*AA^*A \ . \tag{12}$$

Since $A^{*2}A^2$ and A^*AA^*A are non-negative operators different from 0, (12) implies that $\lambda \geq 0$. From Theorem 1 and the fact that every quasinormal operator is subnormal ([6], Problem 195) we see that A is subnormal, hence $\|A^2\| = \|A\|^2$. From comparing the norms of the left and the right side of (12) it follows that $\lambda = 1$. //

3. Hyponormality

An operator $T \in B(H)$ is hyponormal (by definition) if and only if $T^*T - TT^* \geq 0$.

If A and B^* are hyponormal operators then δ is also hyponormal by [10], p. 141. Actually the argument of [10] together with the fact that 0 is always in the closure of the numerical range of $A^*A - AA^*$ (where $A \in B(H)$) imply that the converse is also true. A similar statement can be proved for τ.

PROPOSITION. *Suppose* $A \neq 0$, $B \neq 0$. *Then* τ *is hyponormal if and only if* A *and* B^* *are hyponormal.*

Proof. Note first that the hyponormality condition for τ,

$$0 \leq \langle (\tau^*\tau-\tau\tau^*)X, X\rangle = \mathrm{tr}\big(X^*(A^*AXBB^*-AA^*XB^*B)\big) \ , \quad X \in C_2 \ , \tag{13}$$

can be written in the form

$$\mathrm{tr}\big(B^*X^*(A^*A-AA^*)XB\big) + \mathrm{tr}\big(A^*X(BB^*-B^*B)X^*A\big) \geq 0 \ , \quad X \in C_2 \ . \tag{14}$$

(Here we have used the identity $\mathrm{tr}(YZ) = \mathrm{tr}(ZY)$ for $Y \in B(H)$, $Z \in C_1(H)$ and for Y, $Z \in C_2(H)$ ([9], p. 100).) If A and B^* are hyponormal then $(XB)^*(A^*A-AA^*)XB \geq 0$ and $(X^*A)^*(BB^*-B^*B)X^*A \geq 0$ for all $X \in C_2$ and so (14) holds.

Conversely, if τ is hyponormal then put $X = f \otimes g$ in (13) where $f, g \in H$. It follows, after a short computation,

$$\|Af\|\|B^*g\| - \|A^*f\|\|Bg\| \geq 0 \ . \tag{15}$$

We shall prove that B^* is hyponormal, the proof that A is hyponormal is similar. Assume for a moment that there exists a sequence (f_m) of unit vectors in H such that $\lim\|A^*f_m\| = \lim\|Af_m\| > 0$. Then the hyponormality of B^* follows at once from (15) if we put $f = f_m$ and take the limit when m tends to infinity. To prove that the sequence (f_m) exists put $C = A^*A - AA^*$. If $C \geq 0$ (respectively $C \leq 0$) let α be any non-zero approximate eigenvalue for A (respectively A^*); then the corresponding sequence of unit approximate eigenvectors satisfies the requirement. (Proof. The relations $\lim(A-\alpha)f_n = 0$ and $(A-\alpha)^*(A-\alpha) - (A-\alpha)(A-\alpha)^* = C \geq 0$ imply $\lim(A-\alpha)^*f_n = 0$, thus $\lim\|A^*f_m\| = |\alpha| = \lim\|Af_m\|$.) If neither $C \geq 0$ nor $C \leq 0$ then there exists $f \in H$, $\|f\| = 1$, such that $\langle Cf, f\rangle = 0$ and $\|Cf\| \neq 0$. (This can be seen from the spectral theorem when C is represented as a multiplication with a bounded measurable real function on a suitable $L^2(\mu)$.) Now the constant sequence, $f_m = f$, satisfies the requirement. //

Let us finally remark that the same kind of characterization can not hold for general elementary operators. For example the operator $X \mapsto AXB + A^*XB^*$ is self-adjoint on C_2 for arbitrary $A, B \in B(H)$.

References

[1] J. Anderson and C. Foias, "Properties which normal operators share with normal derivations and related operators", *Pacific J. Math.* 61 (1975), 313-325.

[2] J. Bram, "Subnormal operators", *Duke Math. J.* 22 (1955), 75-94.

[3] A. Brown and C.M. Pearcy, "Spectra of tensor product of operators", *Proc. Amer. Math. Soc.* 17 (1966), 162-169.

[4] J.B. Conway, *Subnormal operators* (Pitman, Boston, 1981).

[5] C.K. Fong and A.R. Sourour, "On the operator identity $\sum A_kXB_k = 0$", *Canad. J. Math.* 31 (1979), 845-857.

[6] Paul R. Halmos, *A Hilbert space problem book* (Springer-Verlag, New York, Heidelberg, Berlin, 1982).

[7] Paul R. Halmos and V.S. Sunder, *Bounded integral operators on L^2 spaces* (Springer-Verlag, Berlin, Heidelberg, New York, 1978).

[8] D.A. Herrero, *Approximation of Hilbert space operators*, Volume I (Pitman, Boston, 1982).

[9] J.R. Ringrose, *Compact non-self-adjoint operators* (Van Nostrand, London, 1971).

[10] S.Y. Shaw, "On numerical ranges of generalized derivations and related properties", *J. Austral. Math. Soc. Ser. A* **36** (1984), 134-142.

Department of Mathematics,
University of Ljubljana,
Jadranska 19,
Ljubljana 61000,
Yugoslavia.