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Hyperbolic Coxeter groups of minimal growth rates in higher dimensions

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Abstract. The cusped hyperbolic n-orbifolds of minimal volume are well known for $n \leq 9$. Their fundamental groups are related to the Coxeter n-simplex groups Γ_n. In this work, we prove that Γ_n has minimal growth rate among all non-cocompact Coxeter groups of finite covolume in $\mathrm{Isom}\mathbb{H}^n$. In this way, we extend previous results of Floyd for $n = 2$ and of Kellerhals for $n = 3$, respectively. Our proof is a generalization of the methods developed together with Kellerhals for the cocompact case.

1 Introduction

Let $\mathbb{H}^n$ denote the real hyperbolic n-space with its isometry group $\mathrm{Isom}\mathbb{H}^n$.

A *hyperbolic Coxeter polyhedron* $P \subset \mathbb{H}^n$ is a convex polyhedron of finite volume all of whose dihedral angles are integral submultiples of π. Associated to P is the *hyperbolic Coxeter group* $\Gamma \subset \mathrm{Isom}\mathbb{H}^n$ generated by the reflections in the bounding hyperplanes of P. By construction, Γ is a discrete group with associated orbifold $O^n = \mathbb{H}^n/\Gamma$ of finite volume.

We focus on *non-compact* hyperbolic Coxeter polyhedra, having at least one ideal vertex $v_\infty \in \partial\mathbb{H}^n$. Notice that the stabilizer of the vertex v_∞ is isomorphic to an affine Coxeter group. The group Γ is called *non-cocompact*, and its quotient space O^n has at least one cusp.

The hyperbolic Coxeter group Γ is the geometric realization of an abstract Coxeter system (W, S) consisting of a group W with a finite generating set S together with the relations $s^2 = 1$ and $(ss')^{m_{ss'}} = 1$, where $m_{ss'} = m_{s's} \in \{2, 3, \dots, \infty\}$ for all $s, s' \in S$ with $s \neq s'$. The *growth series* $f_S(t)$ of $W = (W, S)$ is given by

$$f_S(t) = 1 + \sum_{k \geq 1} a_k t^k ,$$

where $a_k \in \mathbb{Z}$ is the number of elements in W with S-length k. The *growth rate* τ_W of $W = (W, S)$ is defined as the inverse of the radius of convergence of $f_S(t)$.

We are interested in small growth rates of non-cocompact hyperbolic Coxeter groups in $\mathrm{Isom}\mathbb{H}^n$ for $n \geq 2$. For $n = 2$, Floyd [6] showed that the Coxeter group $\Gamma_2 = [3, \infty]$ generated by the reflections in the triangle with angles $\pi/2, \pi/3$, and 0 is the (unique) group of minimal growth rate. For $n = 3$, Kellerhals [13] proved that

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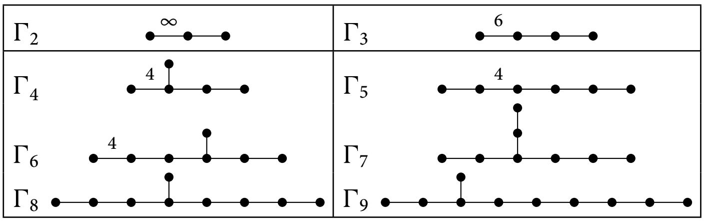
Table 1: The hyperbolic Coxeter n-simplex group Γ_n.



the tetrahedral group Γ_3 generated by the reflections in the Coxeter tetrahedron with symbol $[6, 3, 3]$ realizes minimal growth rate in a unique way.

Consider the hyperbolic Coxeter n-simplices and their reflection groups $\Gamma_n \subset \mathrm{Isom}\mathbb{H}^n$ depicted in Table 1. For their volumes, we refer to [12]. Observe that Γ_n is of minimal covolume among all non-cocompact hyperbolic Coxeter n-simplex groups.

The aim of this work is to prove the following result in the context of growth rates.

Theorem *Let $2 \le n \le 9$. Among all non-cocompact hyperbolic Coxeter groups of finite covolume in* $\mathrm{Isom}\,\mathbb{H}^n$*, the group Γ_n given in Table 1 has minimal growth rate, and as such the group is unique.*

Our Theorem should be compared with the volume minimality results for cusped hyperbolic n-orbifolds O^n for $2 \le n \le 9$. These results are due to Siegel [16] for $n = 2$, Meyerhoff [15] for $n = 3$, Hild and Kellerhals [10] for $n = 4$, and Hild [9] for $n \le 9$. Indeed, the fundamental group of O^n is related to Γ_n in all these cases.

The work is organized as follows. In Section 2.1, we set the background about hyperbolic Coxeter polyhedra and their associated reflection groups. Furthermore, we present a result of Felikson and Tumarkin about their combinatorics as given by [5, Theorem B], which will play a crucial role in our proof. In fact, we will exploit the (non-)simplicity of the Coxeter polyhedra in a most useful way. In Section 2.2, we discuss growth series and growth rates of Coxeter groups and introduce the notion of extension of a Coxeter graph. We also provide some illustrating examples. The monotonicity result of Terragni [18] for growth rates, presented in Theorem 2.2, will be another major ingredient in our proof. Finally, Section 3 is devoted to the proof of our result. We perform it in two steps by assuming that the Coxeter graph under consideration has an affine component of type $\widetilde{A}_1$ or not.

2 Hyperbolic Coxeter groups and growth rates

2.1 Coxeter polyhedra and their reflection groups

Let $\mathbb{X}^n$ denote one of the geometric n-spaces of constant curvature, the unit n-sphere $\mathbb{S}^n$, the Euclidean n-space $\mathbb{E}^n$, or the real hyperbolic n-space $\mathbb{H}^n$. As usual, we embed $\mathbb{X}^n$ in a suitable quadratic space $\mathbb{Y}^{n+1}$. In the Euclidean case, we take the affine model and write $\mathbb{E}^n = \{x \in \mathbb{E}^{n+1} \mid x_{n+1} = 0\}$. In the hyperbolic case, we interpret $\mathbb{H}^n$ as the

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upper sheet of the hyperboloid in $\mathbb{R}^{n+1}$, that is,

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_{n,1} = -1\,,\, x_{n+1} > 0\},$$

where $\langle x, x\rangle_{n,1} = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$ is the standard Lorentzian form. Its boundary $\partial\mathbb{H}^n$ can be identified with the set

$$\partial\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_{n,1} = 0\,,\, \sum_{k=1}^{n+1} x_k^2 = 1\,,\, x_{n+1} > 0\}.$$

In this picture, the isometry group of $\mathbb{H}^n$ is isomorphic to the group $O^+(n,1)$ of positive Lorentzian matrices leaving the bilinear form $\langle\,,\rangle_{n,1}$ and the upper sheet invariant.

It is well known that each isometry of $\mathbb{X}^n$ is a finite composition of reflections in hyperplanes, where a hyperplane $H = H_v$ in $\mathbb{X}^n$ is characterized by a normal unit vector $v \in \mathbb{Y}^{n+1}$. Associated to H_v are two closed half-spaces. We denote by H_v^- the half-space in $\mathbb{X}^n$ with outer normal vector v.

A (convex) *n-polyhedron* $P = \cap_{i\in I} H_i^- \subset \mathbb{X}^n$ is the non-empty intersection of a finite number of half-spaces H_i^- bounded by the hyperplanes $H_i = H_{v_i}$ for $i \in I$. A *facet* of P is of the form $F_i = P \cap H_i$ for some $i \in I$. In the sequel, for $\mathbb{X}^n \neq \mathbb{S}^n$, we always assume that P is of finite volume. In the Euclidean case, this implies that P is compact, and in the hyperbolic case, P is the convex hull of finitely many points $v_1, \ldots, v_k \in \mathbb{H}^n \cup \partial\mathbb{H}^n$. If $v_i \in \mathbb{H}^n$, then v_i is an *ordinary vertex*, and if $v_i \in \partial\mathbb{H}^n$, then v_i is an *ideal vertex* of P, respectively.

If all dihedral angles $\alpha_{ij} = \measuredangle(H_i, H_j)$ formed by intersecting hyperplanes H_i, H_j in the boundary of P are either zero or of the form $\frac{\pi}{m_{ij}}$ for an integer $m_{ij} \geq 2$, then P is called a *Coxeter polyhedron* in $\mathbb{X}^n$. Observe that the Gram matrix $\mathrm{Gr}(P) = (\langle v_i, v_j\rangle_{\mathbb{Y}^{n+1}})_{i,j\in I}$ is a real symmetric matrix with 1's on the diagonal and non-positive coefficients off the diagonal. In this way, the theory of Perron–Frobenius applies. For further details and references about Coxeter polyhedra in $\mathbb{X}^n$, we refer to [4, 19, 20].

Let $P = \cap_{i=1}^N H_i^- \subset \mathbb{X}^n$ be a Coxeter n-polyhedron. Denote by $r_i = r_{H_i}$ the reflection in the bounding hyperplane H_i of P, and let $G = G_P$ be the group generated by $r_1, \ldots, r_N$. It follows that G is a discrete subgroup of finite covolume in $\mathrm{Isom}\mathbb{X}^n$, called a *geometric Coxeter group*.

A geometric Coxeter group $G \subset \mathrm{Isom}\mathbb{X}^n$ with generating system $S = \{r_1, \ldots, r_N\}$ is the geometric realization of an abstract Coxeter system (W, S). In fact, we have $r_i^2 = 1$ and $(r_i r_j)^{m_{ij}} = 1$ with $m_{ij} = m_{ji} \in \{2, 3, \ldots, \infty\}$ as above. Here, $m_{ij} = \infty$ indicates that $r_i r_j$ is of infinite order.

For $\mathbb{X}^n = \mathbb{S}^n$, G is a *spherical* Coxeter group and as such finite. For $\mathbb{X}^n = \mathbb{E}^n$, G is a *Euclidean* or *affine* Coxeter group and of infinite order. By a result of Coxeter [3], the irreducible spherical and Euclidean Coxeter groups are entirely classified. In contrast to this fact, *hyperbolic* Coxeter groups are far from being classified. For a survey about partial classification results, we refer to [4].

For the description of abstract and geometric Coxeter groups, one commonly uses the language of weighted graphs and Coxeter symbols. Let (W, S) be an abstract Coxeter system with generating system $S = \{s_1, \ldots, s_N\}$ and relations of the form $s_i^2 = 1$ and $s_i s_j^{m_{ij}} = 1$ with $m_{ij} = m_{ji} \in \{2, 3, \ldots, \infty\}$. The *Coxeter graph* of the Coxeter

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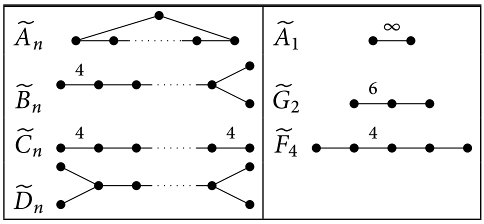
Table 2: Connected affine Coxeter graphs of order $n+1$.



system (W,S) is the non-oriented graph Σ whose nodes correspond to the generators $s_1,\dots,s_N$. If s_i and s_j do not commute, their nodes n_i,n_j are connected by an edge with weight $m_{ij}\geq 3$. We omit the weight $m_{ij}=3$ since it occurs frequently. The number N of nodes is the *order* of Σ. A subgraph $\sigma\subset\Sigma$ corresponds to a *special* subgroup of (W,S), that is, a subgroup of the form (W_T,T) for a subset $T\subset S$. Observe that the Coxeter graph Σ is connected if (W,S) is irreducible.

In the case of a geometric Coxeter group $G=(W,S)\subset\mathrm{Isom}\mathbb{X}^n$, we call its Coxeter graph Σ *spherical*, *affine*, or *hyperbolic*, if $\mathbb{X}^n=\mathbb{S}^n,\mathbb{E}^n$, or $\mathbb{H}^n$, respectively. In Table 2, we reproduce all the connected affine Coxeter graphs, using the classical notation, with the exception of the three groups $\widetilde{E}_6,\widetilde{E}_7,\widetilde{E}_8$ (they will not appear in the following).

An abstract Coxeter group with a simple presentation can conveniently be described by its *Coxeter symbol*. For example, the linear Coxeter graph with edges of successive weights $k_1,\dots,k_N\geq 3$ is abbreviated by the Coxeter symbol $[k_1,\dots,k_N]$. The Y-shaped graph made of one edge with weight p and of two strings of k and l edges emanating from a central vertex of valency 3 is denoted by $[p,3^{k,l}]$ (see [12]).

Let us specify the context and consider a Coxeter polyhedron $P=\cap_{i=1}^N H_i^-\subset\mathbb{H}^n$. Denote by $\Gamma=G_P\subset\mathrm{Isom}\mathbb{H}^n$ its associated Coxeter group and by Σ its Coxeter graph. Since P is of finite volume, the graph Σ is connected. Furthermore, if P is not compact, then P has at least one ideal vertex.

Let $v\in\mathbb{H}^n$ be an ordinary vertex of P. Then, its *link* L_v is the intersection of P with a small sphere of center v that does not intersect any facet of P not incident to v. It corresponds to a spherical Coxeter polyhedron of $\mathbb{S}^{n-1}$ and therefore to a spherical Coxeter subgraph σ of order n in Σ.

Let $v_\infty\in\partial\mathbb{H}^n$ be an ideal vertex of P. Then, its link, denoted by L_∞, is given by the intersection of P with a sufficiently small horosphere centered at v_∞ as above. The link L_∞ corresponds to a Euclidean Coxeter polyhedron in $\mathbb{E}^{n-1}$ and is related to an affine Coxeter subgraph σ_∞ of order $\geq n$ in Σ.

More precisely, if v_∞ is a *simple* ideal vertex, that is, v_∞ is the intersection of exactly n among the N bounding hyperplanes of P, the Coxeter graph σ_∞ is connected and of order n. Otherwise, σ_∞ has $n_c(\sigma_\infty)\geq 2$ affine components, and we have the following formula:

$$n-1=\mathrm{order}(\sigma_\infty)-n_c(\sigma_\infty). \tag{1}$$

Recall that a polyhedron is *simple* if all of its vertices are simple.

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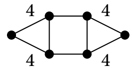


Figure 1: The Coxeter polyhedron $P_0 \subset \mathbb{H}^4$.

As in the spherical and Euclidean cases, hyperbolic Coxeter simplices in $\mathbb{H}^n$ are all known, and they exist for $n \leq 9$ (see [1] or [20]). A list of their Coxeter graphs, Coxeter symbols, and volumes can be found in [12]. Among the related Coxeter n-simplex groups, the group Γ_n, as given in Table 1, is of minimal covolume.

The following structural result for *simple* hyperbolic Coxeter polyhedra due to Felikson and Tumarkin [5, Theorem B] will be a corner stone for the proof of our Theorem.

Theorem 2.1 *Let $n \leq 9$, and let $P \subset \mathbb{H}^n$ be a non-compact simple Coxeter polyhedron. If all facets of P are mutually intersecting, then P is either a simplex or isometric to the polyhedron P_0 whose Coxeter graph is depicted in Figure 1.*

2.2 Growth rates and their monotonicity

Let (W, S) be a Coxeter system and denote by $a_k \in \mathbb{Z}$ the number of elements $w \in W$ with S-length k. The *growth series* $f_S(t)$ of (W, S) is defined by

$$f_S(t) = 1 + \sum_{k \geq 1} a_k t^k .$$

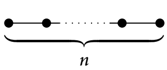
In the following, we list some properties of $f_S(t)$. For references, we refer to [11].

There is a formula due to Steinberg expressing the growth series $f_S(t)$ of a Coxeter system (W, S) in terms of its finite special subgroups W_T for $T \subseteq S$,

$$\frac{1}{f_S(t^{-1})} = \sum_{\substack{W_T < W \\ |W_T| < \infty}} \frac{(-1)^{|T|}}{f_T(t)}, \tag{2}$$

where $W_\varnothing = \{1\}$. By a result of Solomon, the growth polynomial of each term $f_T(t)$ in (2) can be expressed by means of its exponents $\{m_1, m_2, \dots, m_p\}$ according to the formula

$$f_T(t) = \prod_{i=1}^{p} [m_i + 1], \tag{3}$$

where $[k] = 1 + t + \cdots + t^{k-1}$ and, more generally, $[k_1, \dots, k_r] := [k_1] \cdots [k_r]$. A complete list of the irreducible spherical Coxeter groups together with their exponents can be found in [14]. For example, the exponents of the Coxeter group A_n with Coxeter graph  are $\{1, 2, \dots, n\}$ so that

$$f_{A_n}(t) = [2, \dots, n+1]. \tag{4}$$

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Furthermore, the growth series of a reducible Coxeter system (W,S) with factor groups (W_1,S_1) and (W_2,S_2) such that $S=(S_1\times\{1_{W_2}\})\cup(\{1_{W_1}\}\times S_2)$ satisfies the product formula

$$f_S(t)=f_{S_1}(t)\cdot f_{S_1}(t).$$

In its disk of convergence, the growth series $f_S(t)$ is a rational function, which can be expressed as the quotient of coprime monic polynomials $p(t),q(t)\in\mathbb{Z}[t]$ of the same degree. The *growth rate* $\tau_W=\tau_{(W,S)}$ is defined by the inverse of the radius of convergence of $f_S(t)$ and can be expressed by

$$\tau_W=\limsup_{k\to\infty} {a_k}^{1/k}.$$

It is the inverse of the smallest positive real pole of $f_S(t)$ and hence an algebraic integer.

Important for the proof of our Theorem is the following result of Terragni [17] about the growth monotonicity.

Theorem 2.2 *Let (W,S) and (W',S') be two Coxeter systems such that there is an injective map $\iota:S\to S'$ with $m_{st}\le m'_{\iota(s)\iota(t)}$ for all $s,t\in S$. Then, $\tau_{(W,S)}\le\tau_{(W',S')}$.*

For $n\ge 2$, consider a Coxeter group $\Gamma\subset\mathrm{Isom}\mathbb{H}^n$ of finite covolume. By results of Milnor and de la Harpe, we know that $\tau_\Gamma>1$. More precisely, and as shown by Terragni [17], $\tau_\Gamma\ge\tau_{\Gamma_9}\approx 1.1380$, where Γ_9 is the Coxeter simplex group given in Table 1.

Next, we introduce another tool in the proof of our result, the *(simple) extension* of a Coxeter graph.

Definition 2.1 Let Σ be an abstract Coxeter graph. A *(simple) extension* of Σ is a Coxeter graph Σ' obtained by adding one node linked with a (simple) edge to the Coxeter graph Σ.

As a direct consequence of Theorem 2.2, if W is a Coxeter group with Coxeter graph Σ, any extension Σ' of Σ encodes a Coxeter group W' such that $\tau_W\le\tau_{W'}$.

Example 2.3 Consider an irreducible affine Coxeter graph of order 3 as given in Table 2. Up to symmetry, the graph $\widetilde{A}_2$ has a unique extension given by the Coxeter graph at the top left in Figure 2. This graph describes the Coxeter tetrahedron $[3,3^{[3]}]$ of finite volume. The Coxeter graphs $\widetilde{C}_2$ and $\widetilde{G}_2$ give rise to the remaining five extensions depicted in Figure 2. By a result of Kellerhals [13], these six Coxeter graphs describe Coxeter tetrahedral groups Λ of finite covolume in $\mathrm{Isom}\mathbb{H}^3$ whose growth rates satisfy $\tau_\Lambda\ge\tau_{\Gamma_3}$.

Example 2.4 In a similar way, any extension of an irreducible affine Coxeter graph of order 4 yields a Coxeter simplex group of finite covolume in $\mathrm{Isom}\mathbb{H}^4$. They are given in Figure 3. Notice that $\Gamma_4=[4,3^{2,1}]$ is part of them.

Remark 2.5 When considering irreducible affine Coxeter graphs of order greater than or equal to 5, the resulting extensions do *not* always relate to hyperbolic Coxeter

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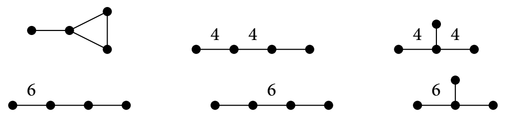

Figure 2: Extensions of $\widetilde{A}_2$, $\widetilde{C}_2$, and $\widetilde{G}_2$.

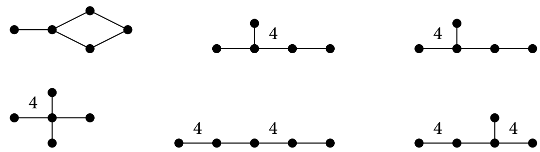

Figure 3: Extensions of $\widetilde{A}_3$, $\widetilde{B}_3$, and $\widetilde{C}_3$.

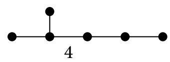

Figure 4: An infinite volume 5-simplex.

n-simplex groups of finite covolume. For example, among the extensions of $\widetilde{F}_4$, the graph depicted in Figure 4 describes an infinite volume simplex in $\mathbb{H}^5$.

3 Proof of the Theorem

Let $2 \le n \le 9$, and consider the Coxeter simplex group $\Gamma_n \subset \mathrm{Isom}\mathbb{H}^n$ whose Coxeter graph is depicted in Table 1. In this section, we provide the proof of our main result stated as follows.

Theorem *For any $2 \le n \le 9$, the group Γ_n has minimal growth rate among all non-cocompact hyperbolic Coxeter groups of finite covolume in* $\mathrm{Isom}\mathbb{H}^n$*, and as such the group is unique.*

For $n = 2$ and for $n = 3$, the result has been established by Floyd [6] and Kellerhals [13]. Therefore, it suffices to prove the Theorem for $4 \le n \le 9$.

Observe that the growth rates of all Coxeter simplex groups in $\mathrm{Isom}\mathbb{H}^n$ are known. Their list can be found in [17]. In particular, one deduces the following strict inequalities:

$$\tau_{\Gamma_9} \approx 1.1380 < \cdots < \tau_{\Gamma_5} \approx 1.2481 < \tau_{\Gamma_4} \approx 1.3717, \tag{5}$$

$$\tau_{\Gamma_5} < \tau_{\Gamma_3} \approx 1.2964. \tag{6}$$

For a fixed dimension n, one also checks that Γ_n has minimal growth rate among (all the finitely many) non-cocompact Coxeter simplex groups $\Lambda \subset \mathrm{Isom}\mathbb{H}^n$.

As a consequence, we focus on hyperbolic Coxeter groups $\Gamma \subset \mathrm{Isom}\mathbb{H}^n$ generated by at least $N \ge n + 2$ reflections in the facets of a non-compact finite volume Coxeter

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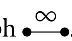
Figure 5: The Coxeter group $W_0 = [\infty, 3, 3]$.

Figure 6: The Coxeter groups $W_1 = [3, \infty, 3]$ and $W_2 = [\infty, 3^{1,1}]$.

polyhedron $P \subset \mathbb{H}^n$. We have to show that $\tau_{\Gamma_n} < \tau_\Gamma$, which yields unicity of the group Γ_n with this property.

Suppose that the Coxeter polyhedron P is simple. By Theorem 2.1, P is either isometric to the polyhedron $P_0 \subset \mathrm{Isom}\mathbb{H}^4$ depicted in Figure 1, or P has a pair of disjoint facets. For the growth rate τ of the Coxeter group associated to P_0, one easily checks with help of the software CoxIter [7, 8] that $\tau_{\Gamma_4} < \tau \approx 2.8383$. Hence, we can assume that P is not isometric to P_0. If P has a pair of disjoint facets, then the Coxeter graph Σ of P and its associated group Γ contains a subgraph .

The property that the Coxeter graph Σ contains such a subgraph of type $\widetilde{A}_1 = [\infty]$ allows us to conclude the proof, whether the polyhedron P is simple or not. In the following, we first look at this property and analyze it more closely.

3.1 In the presence of $\widetilde{A}_1$

We start by considering particular Coxeter graphs of order 4 containing $\widetilde{A}_1$. Their related growth rates will be useful when comparing with the one of Γ. This approach is similar to the one developed in [2].

Let $W_0 = [\infty, 3, 3]$ be the abstract Coxeter group depicted in Figure 5. By means of the software CoxIter, one checks that

(7) $$\tau_{\Gamma_4} < \tau_{W_0} \approx 1.4655.$$

Furthermore, consider the two abstract Coxeter groups $W_1 = [3, \infty, 3]$ and $W_2 = [\infty, 3^{1,1}]$ given in Figure 6.

For their growth rates, we prove the following auxiliary result.

Lemma 3.1 *$\tau_{W_0} < \tau_{W_1}$ and $\tau_{W_0} < \tau_{W_2}$.*

Proof For $0 \le i \le 2$, denote by $f_i := f_{W_i}$ the growth series of W_i and by R_i its radius of convergence. Recall that R_i is the smallest positive pole of f_i, and that $\tau_{W_i} = \frac{1}{R_i}$.

We establish the growth functions f_i according to Steinberg's formula (2). They are given as follows:

$$\frac{1}{f_0(t^{-1})} = 1 - \frac{4}{[2]} + \frac{3}{[2,2]} + \frac{2}{[2,3]} - \frac{1}{[2,2,3]} - \frac{1}{[2,3,4]},$$

$$\frac{1}{f_1(t^{-1})} = 1 - \frac{4}{[2]} + \frac{3}{[2,2]} + \frac{2}{[2,3]} - \frac{2}{[2,2,3]},$$

$$\frac{1}{f_2(t^{-1})} = 1 - \frac{4}{[2]} + \frac{3}{[2,2]} + \frac{2}{[2,3]} - \frac{1}{[2,2,2]} - \frac{1}{[2,3,4]}.$$

Hence, for any $t > 0$, one has the positive difference functions given by

$$\frac{1}{f_0(t^{-1})} - \frac{1}{f_1(t^{-1})} = \frac{1}{[2,2,3]} - \frac{1}{[2,3,4]} = \frac{t^2+t^3}{[2,2,3,4]} > 0,$$

$$\frac{1}{f_0(t^{-1})} - \frac{1}{f_2(t^{-1})} = \frac{1}{[2,2,2]} - \frac{1}{[2,2,3]} = \frac{t^2}{[2,2,2,3]} > 0.$$

Therefore, for $i = 1, 2$, and for $u = t^{-1} \in (0,1)$, the smallest positive root R_0 of $\frac{1}{f_0(u)}$ is strictly bigger than the one of $\frac{1}{f_i(u)}$. This finishes the proof.

As a first consequence, combining (5), (7), and Lemma 3.1, one obtains that

$$\tau_{\Gamma_n} < \tau_{W_i}, \tag{8}$$

for all $4 \le n \le 9$ and $0 \le i \le 2$.

Next, suppose that the Coxeter graph Σ of Γ contains a subgraph $\widetilde{A}_1$. Since Σ is connected of order $N \ge n+2 \ge 6$, the subgraph $\widetilde{A}_1$ is contained in a connected subgraph σ of order 4 in Σ, which is related to a special subgroup W of Γ. By Theorem 2.2, one has that $\tau_{W_i} \le \tau_W$ for some $0 \le i \le 2$. By combining (8) with these findings, and by Theorem 2.2 and Lemma 3.1, one deduces that

$$\tau_{\Gamma_n} < \tau_{W_0} \le \tau_W \le \tau_\Gamma. \tag{9}$$

This finishes the proof of the Theorem in the presence of a subgraph $\widetilde{A}_1$ in Σ.

3.2 In the absence of $\widetilde{A}_1$

Suppose that the Coxeter graph Σ with $N \ge n+2$ nodes does *not* contain a subgraph of type $\widetilde{A}_1$. In particular, by Theorem 2.1, the corresponding Coxeter polyhedron $P \subset \mathrm{Isom}\mathbb{H}^n$ is not simple, and it follows that $5 \le n \le 9$.

Consider a non-simple ideal vertex $v_\infty \in P$. Its link $L_\infty \subset \mathbb{E}^{n-1}$ is described by a reducible affine subgraph σ_∞ with $n_c = n_c(\infty) \ge 2$ components which satisfies $n-1 = \mathrm{order}(\sigma_\infty) - n_c$ by (1). In Table 3, we list all possible realizations for σ_∞ by using the following notations.

Let $\widetilde{\sigma}_k$ be a connected affine Coxeter graph of order $k \ge 3$ as listed in Table 2, and denote by $\bigsqcup_k \widetilde{\sigma}_k$ the Coxeter graph consisting of the components of type $\widetilde{\sigma}_k$.

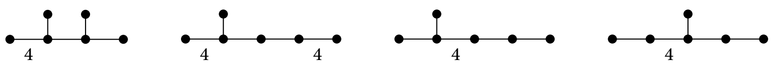
Observe that for any graph $\bigsqcup_k \widetilde{\sigma}_k$ in Table 3, one has $3 \le \min_k k \le 5$, and that the case $\min_k k = 5$ appears only when $n = 9$.

Among the different components of σ_∞, we consider the ones of smallest order ≥ 3 together with their extensions.

- Assume that the graph σ_∞ of the vertex link L_∞ contains an affine component $\widetilde{\sigma}$ of order 3. By Example 2.3, we know that any extension of $\widetilde{\sigma}$ encodes a Coxeter tetrahedral group $\Lambda \subset \mathrm{Isom}\mathbb{H}^3$ of finite covolume. The graph Σ itself contains a subgraph σ of order 4 which in turn comprises $\widetilde{\sigma}$. The Coxeter graph σ corresponds to a special subgroup W of Γ, and by Theorem 2.2, we deduce that $\tau_\Lambda \le \tau_W$.

Table 3: Reducible affine Coxeter graphs σ_∞ with $n_c \geq 2$ components $\widetilde{\sigma}_k$ of order $k \geq 3$ such that $n = \text{order}(\sigma_\infty) - n_c + 1$.

n	5	6	7	8	9
	$\widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_3$	$\widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_4$	$\widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_5$ $\widetilde{\sigma}_4 \sqcup \widetilde{\sigma}_4$ $\widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_3$	$\widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_6$ $\widetilde{\sigma}_4 \sqcup \widetilde{\sigma}_5$ $\widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_4$	$\widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_7$ $\widetilde{\sigma}_4 \sqcup \widetilde{\sigma}_6$ $\widetilde{\sigma}_5 \sqcup \widetilde{\sigma}_5$ $\widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_4 \sqcup \widetilde{\sigma}_4$ $\widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_3 \sqcup \widetilde{\sigma}_3$



$\tau_{\Delta_1} \approx 1.678$ $\tau_{\Delta_2} \approx 1.599$ $\tau_{\Delta_3} \approx 1.668$ $\tau_{\Delta_4} \approx 1.702$

Figure 7: The Coxeter groups Δ_i, $i = 1, \ldots, 4$.

Since $\tau_{\Gamma_3} \leq \tau_\Lambda$, and in view of (5) and (6), Theorem 2.2 yields the desired inequality

$$\tau_{\Gamma_n} < \tau_{\Gamma_3} \leq \tau_\Lambda \leq \tau_W \leq \tau_\Gamma, \tag{10}$$

which finishes the proof in this case, and for $n = 5$ and $n = 6$; see Table 3.

- Assume that the graph σ_∞ contains an affine component $\widetilde{\sigma}$ of order 4. We apply the same reasoning as above. By Example 2.4, any extension of $\widetilde{\sigma}$ corresponds to a Coxeter 4-simplex group Λ of finite covolume, and $\tau_{\Gamma_4} \leq \tau_\Lambda$. Again, Σ contains a subgraph σ comprising $\widetilde{\sigma}$. Hence, there exists a special subgroup W of Γ described by σ so that

$$\tau_{\Gamma_n} < \tau_{\Gamma_4} \leq \tau_\Lambda \leq \tau_W \leq \tau_\Gamma. \tag{11}$$

By (10) and (11), the proof is finished in this case, and for $n = 7$ and $n = 8$; see Table 3.

- Assume that σ_∞ contains an affine component $\widetilde{\sigma}$ of order 5. By Table 3, one has $7 \leq n \leq 9$. It is not difficult to list all possible extensions of $\widetilde{\sigma}$. There are exactly 15 such extensions. It turns out that there are 11 extensions that encode Coxeter 5-simplex groups of finite covolume, whereas the remaining 4 extensions describe 5-simplex groups Δ_i, $i = 1, \ldots, 4$, of *infinite* covolume. These last four simplices arise by extending $\widetilde{B}_4$, $\widetilde{C}_4$, and $\widetilde{F}_4$. They are given in Figure 7, together with their associated growth rates computed with CoxIter.

In view of (5), it turns out that

$$\tau_{\Gamma_5} < \tau_{\Delta_i}, \quad \text{for } i = 1, \ldots, 4. \tag{12}$$

As above, the component $\widetilde{\sigma}$ lies in a subgraph σ of order 6 in Σ, and the latter corresponds to a special subgroup W of Γ so that

$$\text{either} \quad \tau_\Lambda \leq \tau_W \quad \text{or} \quad \tau_{\Delta_i} \leq \tau_W, \qquad 1 \leq i \leq 4$$

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where Λ is a Coxeter 5-simplex group of finite covolume. Since $\tau_{\Gamma_5} \leq \tau_\Lambda$, and by (5) and (12), one deduces that

$$\tau_{\Gamma_n} < \tau_{\Gamma_5} \leq \tau_W \leq \tau_\Gamma. \tag{13}$$

This finishes the proof of this case.

Finally, all the above considerations allow us to conclude the proof of the Theorem.

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