Congruences on a Distributive Lattice

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Among the many papers on the subject of lattices I have not seen any simple discussion of the congruences on a distributive lattice. It is the purpose of this note to give such a discussion for lattices with a certain finiteness. Any distributive lattice is isomorphic with a ring of sets (G. Birkhoff, *Lattice Theory*, revised edition, 1948, p. 140, corollary to Theorem 6); I take the case where the sets are finite. All finite distributive lattices are covered by this case.

Let $\mathscr{L}$ be a lattice of subsets of S and K be a subset of S (not necessarily an element of $\mathscr{L}$). Let the definition of the relation q_K between elements of the lattice be that $X q_K Y$ if and only if $X \cap K = Y \cap K$. q_K is clearly an equivalence (this would be true for any lattice). In fact, q_K is a congruence. For if $X q_K Y$ and $Z \in L$, then $(X \cap Z) \cap K = (X \cap K) \cap Z = (Y \cap K) \cap Z = (Y \cap Z) \cap K$ and $(X \cup Z) \cap K = (X \cap K) \cup (Z \cap K) = (Y \cap K) \cup (Z \cap K) = (Y \cup Z) \cap K$. Therefore $(X \cap Z)\, q_K\, (Y \cap Z)$ and $(X \cup Z)\, q_K\, (Y \cup Z)$.

The chief theorem is the converse of this: *If $\mathscr{L}$ is a lattice of finite subsets of S, and q is a congruence on $\mathscr{L}$, then there is a subset K of S such that $q = q_K$.*

The set Xq of all elements in the relation q to X is a sub-lattice. For if $Y q X q Z$, then $(Y \cap Z)\, q\, (X \cap X) = X$ and $(Y \cup Z)\, q\, (X \cup X) = X$. Let X_g and X_l be the greatest and least elements of Xq.

Let K be $\bigcup X - \bigcup (X_g - X_l)$. (Unions are over all X of $\mathscr{L}$.) First we see that $q \subseteq q_K$. For $X_g - X_l \subseteq \bigcup X - K$. Therefore $(X_g - X_l) \cap K \subseteq K - K = 0$, and so $X_g \cap K = X_l \cap K$. But $X_g \cap K \supseteq X \cap K \supseteq X_l \cap K$, giving $X \cap K = X_l \cap K$. If $Xq = Yq$ then $X_l = Y_l$. Therefore $Y \cap K = Y_l \cap K = X_l \cap K = X \cap K$, and so $X q_K Y$.

We have now to see that if $X q_K Y$, then $X q Y$. We take first the case $X \supseteq Y$; the proof is by induction on the number of elements in $X - Y$. It is clearly true when $X - Y$ has no elements. Let $X - Y$ have n elements $(n > 0)$ and $a \in X - Y$ and $X \cap K = Y \cap K$. Then a is not in K and so $a \in \bigcup X - K = \bigcup (X_g - X_l)$. Therefore there are elements P and Q of $\mathscr{L}$ such that $P q Q$ and $a \in P - Q$. Then $U q V$ where $U = Y \cup P . \cup X$ and $V = Y \cup Q . \cap X$.

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Then $$Y \subseteq U \cap V \subseteq U \subseteq X. \tag{1}$$

Now a is not in $U \cap V$ and $a \in U$. The number of elements in $X - U$ and the number of elements in $(U \cap V) - Y$ are therefore less than n. But, from (1), $K \cap Y \subseteq K \cap U \cap V \subseteq K \cap U \subseteq K \cap X$. And $X \cap K = Y \cap K$. Therefore all these are equal. Therefore we have $K \cap X = K \cap U$, $U \subseteq X$, and the number of elements in $X - U$ is less than n. Therefore $U \mathfrak{q} X$. In the same way, $Y \mathfrak{q} U \cap V$. But $U \mathfrak{q} U \cap V$. Therefore $X \mathfrak{q} Y$.

Now let (X, Y) be any element of $\mathfrak{q}_K$. Then $X \cap K = X \cap Y \cap K$, and $X \supset X \cap Y$, and so $X \mathfrak{q} X \cap Y$. In the same way, $Y \mathfrak{q} Y \cap X$, and so $X \mathfrak{q} Y$.

Note. A similar theorem for complemented modular lattices is given by Birkhoff, *loc. cit.*, p. 119, Theorem 5.

Definition: If $\mathfrak{p}$ and $\mathfrak{r}$ are any two relations, then $\mathfrak{pr}$ is the relation for which $X \mathfrak{pr} Y$ if and only if there is a Z for which $X \mathfrak{p} Z$ and $Z \mathfrak{r} Y$.

We can now prove that

If $\mathfrak{p}$ and $\mathfrak{r}$ are any two congruences on a lattice of finite subsets, then $\mathfrak{pr} = \mathfrak{rp}$.

Let $\mathfrak{p}$ be $\mathfrak{q}_P$ and $\mathfrak{r}$ be $\mathfrak{q}_R$. If $A \mathfrak{pr} B$, then, for some element C of the lattice, $A \mathfrak{q}_P C \mathfrak{q}_R B$, and so $A \cap P = C \cap P$ and $B \cap R = C \cap R$.

Then $A \cap P \cap R = C \cap P \cap R = C \cap R \cap P = B \cap R \cap P$. Let D be $(A \cap R) \cup (B \cap P)$. Then $D \cap R = (A \cap R) \cup (B \cap P \cap R) = (A \cap R) \cup (A \cap P \cap R) = A \cap R$. Therefore $A \mathfrak{r} D$. In the same way, $D \mathfrak{p} B$. Therefore $A \mathfrak{rp} B$, and so $\mathfrak{pr} \subseteq \mathfrak{rp}$. Similarly, $\mathfrak{rp} \subseteq \mathfrak{pr}$.

Note. This theorem was proved for relatively complemented lattices by R. P. Dilworth, *Annals of Mathematics*, 50(1950), 348.

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