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The Genuine Omega-regular Unitary Dual of the Metaplectic Group

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Abstract. We classify all genuine unitary representations of the metaplectic group whose infinitesimal character is real and at least as regular as that of the oscillator representation. In a previous paper we exhibited a certain family of representations satisfying these conditions, obtained by cohomological induction from the tensor product of a one-dimensional representation and an oscillator representation. Our main theorem asserts that this family exhausts the genuine omega-regular unitary dual of the metaplectic group.

1 Introduction

In [4], we formulated a conjecture that provides a classification of all the genuine unitary *omega-regular* representations of the metaplectic group. (Roughly, a representation is called "omega-regular" if its infinitesimal character is real and at least as regular as that of the oscillator representation. See Definition 2.1.) In this paper, we prove that conjecture. In particular, we show that all such representations are obtained by cohomological parabolic induction from the tensor product of a one-dimensional representation and an oscillator representation. The reader may think of the notion of omega-regular representations as a generalization, in the context of genuine representations of the metaplectic group, of the idea of *strongly regular* representations. (Recall that an infinitesimal character is called "strongly regular" if it is at least as regular as the infinitesimal character of the trivial representation.) Then our classification appears, for the case of the metaplectic group, as a generalization of the main result of [10], which asserts that, for real reductive Lie groups, every irreducible unitary representation with strongly regular infinitesimal character is cohomologically induced from a unitary character. For double covers of other linear groups one can define a similar notion of "omega-regular" for which every genuine unitary omega-regular representation should be obtained by cohomological induction from one of a small set of "basic" unitary representations. We make this more explicit after Theorem 1.4.

All representations considered in this paper have real infinitesimal character. Let $Mp(2n)$ be the metaplectic group of rank n, *i.e.*, the two-fold connected cover of the symplectic group $Sp(2n,\mathbb{R})$. We recall the construction of the $A_{\mathfrak{q}}(\Omega)$ representations of $Mp(2n)$. Choose a theta stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ of $\mathfrak{sp}(2n,\mathbb{C})$, and

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let L be the Levi subgroup of $Mp(2n)$ corresponding to $\mathfrak{l}$. We may identify L with a quotient of

$$\overline{L} = \Big[\prod_{i=1}^{u} \widetilde{U}(p_i, q_i)\Big] \times Mp(2m). \tag{1.1}$$

Here, for $i = 1 \dots u$, $\widetilde{U}(p_i, q_i)$ is a connected two-fold cover of $U(p_i, q_i)$. However, we will abuse notation and say $L \simeq \overline{L}$. (See Section 2 for details.)

If $\mathbb{C}_\lambda$ is a genuine (unitary) character of $\prod_{i=1}^{u} \widetilde{U}(p_i, q_i)$, and ω is an irreducible summand of one of the two oscillator representations of $Mp(2m)$, we set $\Omega = \mathbb{C}_\lambda \otimes \omega$, considered as a representation of L. Let $A_\mathfrak{q}(\Omega) := \mathcal{R}_\mathfrak{q}(\Omega)$ (see Section 2 for notation).

Remark 1.1 Our (new) definition of $A_\mathfrak{q}(\Omega)$ is slightly different from the one given in [4], because we do not require Ω to be in the good range for $\mathfrak{q}$.

Definition 1.2 A representation Y of L is in the good range for $\mathfrak{q}$ if its infinitesimal character γ^Y satisfies

$$\left\langle \gamma^Y + \rho(\mathfrak{u}), \alpha \right\rangle > 0, \qquad \forall \alpha \in \Delta(\mathfrak{u}).$$

Here $\rho(\mathfrak{u})$ denotes one half the sum of the roots in $\Delta(\mathfrak{u})$.

For brevity of notation, we call an omega-regular representation "ω-regular". In [4] we proved the following result.

Proposition 1.3 ([4, Proposition 3]) *If the representation Ω of L is in the good range for $\mathfrak{q}$, then the representation $A_\mathfrak{q}(\Omega)$ of $Mp(2n)$ is nonzero, irreducible, genuine, ω-regular, and unitary.*

Our main result is the following converse of this statement.

Theorem 1.4 *Let X be an irreducible, genuine, ω-regular, and unitary representation of $Mp(2n)$. Then $X \cong A_\mathfrak{q}(\Omega)$ for some theta stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ and some representation $\Omega = \mathbb{C}_\lambda \otimes \omega$, in the good range for $\mathfrak{q}$, of the Levi subgroup corresponding to $\mathfrak{q}$.*

Theorem 1.4 suggests that the four oscillator representations form a "basic" set of building blocks for all genuine omega-regular unitary representations of the metaplectic group, via cohomological parabolic induction. Using the results of [14] and [2], one can verify that similar statements hold true for the simply connected split group of type G_2 and the nonlinear double cover of $GL(n, \mathbb{R})$. Here, "omega-regularity" might be defined in terms of $\frac{1}{2}\rho$, and the basic representations are the pseudospherical representations that correspond to the trivial representation of the linear group under the Shimura correspondence of [1]. This suggests the following generalization of "omega-regular" for a double cover G of a *split* linear group. Fix a choice of positive roots with respect to the split Cartan and consider "metaplectic" roots as in [1, Definition 4.4]. Define γ^ω to be one half the sum of the non-metaplectic positive roots plus one fourth the sum of the metaplectic positive roots, and call a representation of G "omega-regular" if its infinitesimal character is at least as regular as γ^ω (as in Definition 2.1). This definition agrees with that for $Mp(2n)$ and the examples considered above, and reduces to the strongly regular definition if G is linear. If G is a nonlinear double cover of a split real group as in [1], then the set of basic representations for G should include the (conjecturally unitary) pseudospherical representations at infinitesimal character γ^ω, *i.e.*, the Shimura lifts of the trivial representation of the corresponding linear group (*e.g.*, the even oscillator representations of $Mp(2n)$). This collection of pseudospherical representations at infinitesimal character γ^ω may or may not exhaust the set of basic representations for G.

We conjecture that a similar notion of "omega-regularity" and a similar small collection $\Pi_\omega(G)$ (finite if G is semisimple) of "basic" genuine, unitary, and omega-regular representations can be defined for any double cover G of a reductive linear real Lie group.

Conjecture 1.5 *For each double cover H of a real reductive linear Lie group there is a set $\Pi_\omega(H)$ of basic representations as above, with the following property. Suppose that G is a double cover of a linear reductive real Lie group. If X is the Harish-Chandra module of a genuine irreducible omega-regular unitary representation of G, then there is a Levi subgroup L of G and a representation $Y \in \Pi_\omega(L)$ such that X is obtained from Y by cohomological parabolic induction.*

Remark 1.6 For linear double covers, such as the trivial double cover, or the square root of the determinant cover of $U(p,q)$, the basic representations are the (genuine) unitary one-dimensional representations. For these groups, the notion of omega-regularity should coincide with strong regularity, and the conjecture recovers Salamanca-Riba's result [10].

In [4], we proved Theorem 1.4 for a metaplectic group of rank 2. The proof was based on a case-by-case calculation. For each (genuine) $\widetilde{U}(2)$-type μ that *is* the lowest K-type of an $A_{\mathfrak{q}}(\Omega)$ representation, we showed that there exists a unique unitary and ω-regular representation of $Mp(4)$ with lowest K-type μ. For each genuine $\widetilde{U}(2)$-type μ that *is not* the lowest K-type of an $A_{\mathfrak{q}}(\Omega)$ representation, we showed that every ω-regular representation of $Mp(4)$ with lowest $\widetilde{U}(2)$-type μ must be non-unitary. The main tool in the proof of both claims was Parthasarathy's Dirac Operator Inequality (*cf.* [6]). This scheme worked for all genuine $\widetilde{U}(2)$-types, except for the (unique) fine $\widetilde{U}(2)$-type that occurs in the genuine non-pseudospherical principal series. In this case, we explicitly computed the intertwining operator that gives the invariant Hermitian form on the representation space and showed that its signature is indefinite.

The case-by-case calculation we used to prove Theorem 1.4 for the case $n = 2$ is not suitable for a generalization to arbitrary n. For the general case, we apply a reduction argument similar to the one used in [9], but we also need some of the non-unitarity results and non-unitarity certificates obtained in [5]. (The question of the unitarity of the ω-regular principal series of $Mp(2n)$ was the motivation for [5].)

We sketch the proof of Theorem 1.4 (for arbitrary n). Let X be a genuine admissible irreducible unitary representation of $Mp(2n)$. The first step is to realize X as the lowest K-type constituent of a module of the form $\mathcal{R}_{\mathfrak{q}}(X_1 \otimes X_0)$, where $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$

is a theta stable subalgebra of $\mathfrak{sp}(2n, \mathbb{C})$ with Levi subgroup $L \simeq \widetilde{U}(r,s) \times Mp(2d)$, X_1 is a genuine strongly regular irreducible representation of $\widetilde{U}(r,s)$, and X_0 is the irreducible Langlands quotient of a genuine ω-regular principal series representation of $Mp(2d)$. If $X_1 \otimes X_0$ *is* unitary, then the results of [9] imply that $X_1 \simeq A_{\mathfrak{q}_2}(\lambda_2)$ for some theta stable parabolic subalgebra $\mathfrak{q}_2$ and some parameter λ_2, and the results of [5] tell us that X_0 must be the even half of an oscillator representation of $Mp(2d)$. In this case, by a version of induction by stages, X must be of the desired form. If $X_1 \otimes X_0$ *is not* unitary, then there must be an $(L \cap K)$-type μ^L that detects non-unitarity (in the sense that the invariant Hermitian form will change sign on μ^L). The results of [9], which rely heavily on Parthasarathy's Dirac Operator Inequality, together with the calculations in [5] (see Lemma 5.2), give us specific information about what μ^L could look like. Recall that, associated with the cohomological induction functor $\mathcal{R}_{\mathfrak{q}}$, there is the bottom layer map, which takes $(L \cap K)$-types (in $(X_1 \otimes X_0)$) to K-types (in $\mathcal{R}_{\mathfrak{q}}(X_1 \otimes X_0)$). Let μ be the image of μ^L under this map. If μ were nonzero, then by a theorem of Vogan, μ would occur in the lowest K-type constituent X of $\mathcal{R}_{\mathfrak{q}}(X_1 \otimes X_0)$ and would carry the same signature as μ^L; hence X would be not unitary. Because we are assuming that X *is* unitary, we deduce that μ^L must be mapped to 0. It turns out that, in this case, there exists a different theta stable parabolic subalgebra $\mathfrak{q}'$ of $\mathfrak{sp}(2n, \mathbb{C})$ with Levi subgroup $L' \simeq \widetilde{U}(r', s') \times Mp(2d+2)$ such that X is the lowest K-type constituent of $\mathcal{R}_{\mathfrak{q}'}(X_1' \otimes X_0')$. Here X_1' is an $A_{\mathfrak{q}_3}(\lambda_3)$ module of $\widetilde{U}(r', s')$, and X_0' is the odd half of an oscillator representation of $Mp(2d+2)$. As in the previous case, using induction by stages, we get the desired result.

In [4], we also classified the non-genuine unitary ω-regular representations of $Mp(4)$, *i.e.*, the ω-regular part of the unitary dual of $Sp(4, \mathbb{R})$. We plan to address the generalization of this result to metaplectic groups of arbitrary rank in a future paper.

The paper is organized as follows. In Section 2, we set up the notation and recall some properties of the cohomological induction construction. We outline the proof of our main theorem in Section 3. The argument is essentially reduced to two main propositions, which we prove in Sections 6 and 7, and a series of technical lemmas that are presented in Section 4 (the casual reader may want to skip this section). Additional results needed for the proof are included in Sections 5 and 8.

2 Definitions and Preliminary Results

We begin with some notation. For the metaplectic group of rank r, we denote by ω an irreducible summand of an oscillator representation, and we write ω^r for the corresponding infinitesimal character. Recall that there are four such summands, namely, the odd and even halves of the holomorphic and antiholomorphic oscillator representations, respectively. For any compact connected group, we often identify irreducible representations of the group with their highest weight.

For $G = Mp(2n)$, set $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, and let $\mathfrak{t}_0$ and $\mathfrak{t}$ be the real and complexified Lie algebra of a compact Cartan subgroup T of G. Let $\langle \cdot, \cdot \rangle$ denote a fixed non-degenerate G-invariant θ-invariant symmetric bilinear form on $\mathfrak{g}_0$, negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$; use the same notation for its complexification and its various restrictions and dualizations.

Definition 2.1 Let $\gamma \in i\mathfrak{t}_0^*$. Choose a positive system $\Delta^+(\gamma) \subseteq \Delta(\mathfrak{g}, \mathfrak{t})$ such that $\langle \alpha, \gamma \rangle \geq 0$ for all $\alpha \in \Delta^+(\gamma)$, and let ω^n be the representative of the infinitesimal character of the oscillator representation of G that is dominant with respect to $\Delta^+(\gamma)$. We call γ *ω-regular* if the following regularity condition is satisfied:

$$\langle \alpha, \gamma - \omega^n \rangle \geq 0, \quad \forall \alpha \in \Delta^+(\gamma).$$

We say that a representation of G is ω-regular if its infinitesimal character is.

Given a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$, write L for the corresponding subgroup of $Mp(2n)$. Then L is a double cover of a Levi subgroup L_d of $Sp(2n, \mathbb{R})$ of the form

$$L_d \cong \prod_{i=1}^{u} U(p_i, q_i) \times Sp(2m, \mathbb{R}). \tag{2.1}$$

For each $i = 1 \dots u$, let $\widetilde{U}(p_i, q_i)$ be the inverse image of $U(p_i, q_i)$ (in (2.1)) under the covering map

$$p\colon Mp(2n) \longrightarrow Sp(2n, \mathbb{R}),$$

and similarly for $\widetilde{Sp}(2m, \mathbb{R})$. Then $\widetilde{U}(p_i, q_i)$ is the (connected) "square root of the determinant cover" of $U(p_i, q_i)$, and $\widetilde{Sp}(2m, \mathbb{R}) \cong Mp(2m)$. The groups $\widetilde{U}(p_i, q_i)$ and $\widetilde{Sp}(2m, \mathbb{R})$ intersect in the kernel of the covering map p, and there is a surjective map

$$\overline{L} = \Big[\prod_{i=1}^{u} \widetilde{U}(p_i, q_i)\Big] \times Mp(2m) \longrightarrow L \tag{2.2}$$

given by multiplication inside $Mp(2n)$. Since the factors in (2.2) commute, we have that genuine irreducible representations of L are in correspondence with tensor products of genuine irreducible representations of the factors of $\overline{L}$. In order to keep our notation simpler, we will identify $\overline{L}$ with L, and just write

$$L \cong \Big[\prod_{i=1}^{u} \widetilde{U}(p_i, q_i)\Big] \times Mp(2m).$$

Now let $(\mathcal{R}_{\mathfrak{q}}^{(\mathfrak{g},K)})^i$ be the functors of cohomological parabolic induction carrying $(\mathfrak{l}, L \cap K)$-modules to $(\mathfrak{g}, K)$-modules (*cf.* [12, Def. 6.3.1]). We will occasionally apply these functors to different settings. When the group and the Lie algebra are clear from the context, we will omit the superscript $(\mathfrak{g}, K)$ and use the more standard notation $\mathcal{R}_{\mathfrak{q}}^i$. In our situation, we only use the degree $i = \dim(\mathfrak{u} \cap \mathfrak{k})$ (usually denoted by S), hence we may omit the superscript i as well.

Definition 2.2 An $A_{\mathfrak{q}}(\Omega)$ representation is a genuine representation of G of the following form. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a theta stable parabolic subalgebra of $\mathfrak{g}$, with corresponding Levi subgroup L (as in equation (1.1)). Let $\mathbb{C}_\lambda$ be a genuine (on each factor) one-dimensional representation of $[\prod_{i=1}^{u} \widetilde{U}(p_i, q_i)]$ and let ω be an irreducible summand of an oscillator representation of $Mp(2m)$. We define

$$A_{\mathfrak{q}}(\Omega) := \mathcal{R}_{\mathfrak{q}}(\Omega).$$

Remark 2.3 In Definition 2.2, the rank m of the metaplectic factor of L is allowed to be equal to 0 or n; in these cases, the representation $A_{\mathfrak{q}}(\Omega)$ of $Mp(2n)$ is an $A_{\mathfrak{q}}(\lambda)$ representation or an oscillator representation, respectively.

We will prove Theorem 1.4 in Section 3. Now we recall some properties of the functors of cohomological induction. Let X be an irreducible admissible $(\mathfrak{g}, K)$-module. Let

$$(\lambda_a, \mathfrak{q}_a, L_a, X^{L_a}) \tag{2.3}$$

be the "θ-stable parameters" associated with X via Vogan's classification of admissible representations, so that X is the unique lowest K-type constituent of $\mathcal{R}^{S_a}_{\mathfrak{q}_a}(X^{L_a})$, with $S_a = \dim(\mathfrak{u}_a \cap \mathfrak{k})$ and X^{L_a} a minimal principal series of L_a (*cf.* [12]). Recall that the parameter $\lambda_a \in i\mathfrak{t}_0^*$ determines the theta stable parabolic subalgebra $\mathfrak{q}_a = \mathfrak{l}_a + \mathfrak{u}_a$. In particular, the set of roots in $\mathfrak{u}_a$ is given by

$$\Delta(\mathfrak{u}_a) = \left\{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \mid \langle \lambda_a, \alpha \rangle > 0 \right\}.$$

Choose and fix the positive system of compact roots

$$\Delta_c^+ = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n\} \tag{2.4}$$

so that dominant weights, and therefore highest weights of K-types, are given by strings of weakly decreasing half-integers. The parameter λ_a is weakly dominant for Δ_c^+ as well. If

$$\lambda_a = \Big(\underbrace{g_1, \dots, g_1}_{r_1}, \dots, \underbrace{g_t, \dots, g_t}_{r_t} \Big| \underbrace{0, \dots, 0}_{d} \Big| \underbrace{-g_t, \dots, -g_t}_{s_t}, \dots, \underbrace{-g_1, \dots, -g_1}_{s_1} \Big) \tag{2.5}$$

with $g_1 > \cdots > g_t > 0$, then the centralizer of λ_a in G is of the form

$$L_a = \mathrm{Centr}_G(\lambda_a) \simeq \Big[\prod_{i=1}^{t} \widetilde{U}(r_i, s_i) \Big] \times Mp(2d),$$

and each factor is quasisplit.

Note that the parameter

$$\xi = \Big(\underbrace{1, 1, \dots, 1}_{r} \Big| \underbrace{0, 0, \dots, 0}_{d} \Big| \underbrace{-1, -1, \dots, -1}_{s} \Big) \tag{2.6}$$

is a singularization of λ_a (*cf.* [11]). Here $r = \sum_{i=1}^{t} r_i$ and $s = \sum_{i=1}^{t} s_i$. Set

$$L = \mathrm{Centr}_G(\xi) \simeq \widetilde{U}(r, s) \times Mp(2d)$$

and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. Then $L \supseteq L_a$, $\mathfrak{l}_a \subseteq \mathfrak{l}$, $\mathfrak{u}_a = \mathfrak{u} + (\mathfrak{l} \cap \mathfrak{u}_a)$, and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \supseteq \mathfrak{q}_a$.

Proposition 2.4 lists a few results regarding the functors of cohomological induction and their restriction to K (see [3, 12] for proofs). Most of these results are gathered together in [11], but are stated there for the functors $\mathcal{L}^S_{\mathfrak{q}}$ instead of $\mathcal{R}^S_{\mathfrak{q}}$. Note that in our context, the two constructions are isomorphic by a result due to Enright and Wallach (*cf.* [13, Theorem 5.3]).

Proposition 2.4 *Suppose that X is an irreducible admissible $(\mathfrak{g}, K)$-module of $Mp(2n)$. Define L and $\mathfrak{q}$ as above, and let $S = \dim(\mathfrak{u} \cap \mathfrak{k})$.*

(i) *There is a unique irreducible $(\mathfrak{l}, L \cap K)$-module X^L associated with X so that X can be realized as the unique lowest K-type constituent of $\mathcal{R}^S_{\mathfrak{q}}(X^L)$.*

(ii) *If μ^L is (the highest weight of) an irreducible representation V_{μ^L} of $L \cap K$ and $\mu = \mu^L + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k}, \mathfrak{t})$-dominant, then every irreducible constituent of $(\mathcal{R}^{(K,\mathfrak{k})}_{\mathfrak{q}\cap\mathfrak{k}})^S(V_{\mu^L})$ has highest weight μ. If μ is not dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t})$, then $(\mathcal{R}^{(K,\mathfrak{k})}_{\mathfrak{q}\cap\mathfrak{k}})^S(V_{\mu^L}) = 0$.*

(iii) *There is a natural injective map, the bottom layer map, of K-representations*

$$\mathcal{B}_{X^L} \colon \left(\mathcal{R}^{(K,\mathfrak{k})}_{\mathfrak{q}\cap\mathfrak{k}}\right)^S(X^L) \to \mathcal{R}^S_{\mathfrak{q}}(X^L).$$

Moreover, there is a one-to-one correspondence (with multiplicities) between the lowest K-types of X and the lowest $(L \cap K)$-types of X^L.

(iv) *The module X is endowed with a nonzero invariant Hermition form $\langle \cdot , \cdot \rangle^G$ if and only if the module X^L is endowed with a nonzero invariant Hermitian form $\langle \cdot , \cdot \rangle^L$.*

(v) *The bottom layer map is unitary. That is, for X Hermitian, on each K-type in the bottom layer of X (*cf.* Remark 2.6), the signature of $\langle \cdot , \cdot \rangle^G$ matches the signature of $\langle \cdot , \cdot \rangle^L$ on the corresponding $(L \cap K)$-type of X^L.*

(vi) *If $\gamma^{X^L} \in i\mathfrak{t}^*_0$ is a representative of the infinitesimal character of X^L, then*

$$\gamma^X = \gamma^{X^L} + \rho(\mathfrak{u})$$

is a representative of the infinitesimal character of X.

(vii) *If λ^L_a is the Vogan classification parameter associated with X^L, then*

$$\lambda_a = \lambda^L_a + \rho(\mathfrak{u}).$$

Proof Because ξ in (2.6) is a singularization of λ_a, these facts follow from [11, Lemma 2.7 and Theorem 2.13]. More precisely, part (i) and (vii) follow from Theorem 2.13(b); (ii) from Lemma 2.7; (iii) and (v) follow from Theorem 2.13(d); (iv) is proved in more generality in [9, Proposition 5.2 and Corollary 5.3], but it is also [11, Theorem 2.13(c)]. Part (vi) is [12, Proposition 6.3.11]. ■

Remark 2.5 If X has real infinitesimal character, then X^L and X are Hermitian by [9, Lemma 6.5]; the argument given there is easily seen to extend to the case of the metaplectic group. Consequently, we have that in our setting, the forms $\langle \cdot , \cdot \rangle^L$ and $\langle \cdot , \cdot \rangle^G$ of Proposition 2.4(iv) always exist.

Remark 2.6 The image of the bottom layer map $\mathcal{B}_{X^L}$ (as in Proposition 2.4(iii)) is called "the bottom layer of X". We say that an $(L \cap K)$-type μ^L survives in the bottom layer if $\mu = \mu^L + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k}, \mathfrak{t})$-dominant (as in Proposition 2.4(ii)). Note that, in this case, μ is the highest weight of a K-type in the bottom layer of X.

Lemma 2.7 *Retain the notation of Proposition 2.4. Set $L = L_1 \times L_0$ with $L_0 = Mp(2d)$ and $L_1 = \widetilde{U}(r,s)$, and write $X^L \simeq X_1 \otimes X_0$, with X_i an irreducible $(\mathfrak{l}_i, L_i \cap K)$-module ($i = 0, 1$). If X is ω-regular, then X_1 is strongly regular for L_1 and X_0 is ω-regular for L_0.*

Proof Let $\gamma^{X^L} = (\gamma^{X_1}, \gamma^{X_0})$ be (a representative of) the infinitesimal character of X^L, and write

$$\gamma^X = \gamma^{X^L} + \rho(\mathfrak{u})$$

as in Proposition 2.4(vi). Choose a positive system $\Delta^+ \subset \Delta(\mathfrak{g}, \mathfrak{t})$ of roots so that γ^X is dominant with respect to Δ^+, and choose the representative of the infinitesimal character ω^n that is dominant with respect to Δ^+. Because X is ω-regular, we have

$$\langle \gamma^X, \alpha \rangle \geq \langle \omega^n, \alpha \rangle, \quad \forall\, \alpha \in \Delta^+.$$

If we normalize our form so that $\langle \alpha, \alpha \rangle = 2$ for all short roots α, then this is equivalent to saying that

$$\langle \gamma^X, \alpha \rangle \geq 1, \quad \forall\, \alpha \in \Delta^+.$$

Now let

$$\Delta^+(\mathfrak{l}) = \Delta^+ \cap \Delta(\mathfrak{l}, \mathfrak{t}) = \left[\Delta^+ \cap \Delta(\mathfrak{l}_1, \mathfrak{t}) \right] \cup \left[\Delta^+ \cap \Delta(\mathfrak{l}_0, \mathfrak{t}) \right].$$

Because $\rho(\mathfrak{u})$ is orthogonal to the roots of $\mathfrak{l}$, we have

$$\langle \gamma^{X^L}, \alpha \rangle \geq 1, \quad \forall\, \alpha \in \Delta^+(\mathfrak{l}).$$

Note that the roots of $\mathfrak{l}_1$ are orthogonal to γ^{X_0}, hence

$$\langle \gamma^{X_0}, \alpha \rangle \geq 1, \quad \forall\, \alpha \in \Delta^+ \cap \Delta(\mathfrak{l}_0, \mathfrak{t}),$$

and X_0 is ω-regular for L_0. Similarly, the roots of $\mathfrak{l}_0$ are orthogonal to γ^{X_1}, hence

$$\langle \gamma^{X_1}, \alpha \rangle \geq 1, \quad \forall\, \alpha \in \Delta^+(\mathfrak{l}) \cap \Delta(\mathfrak{l}_1, \mathfrak{t}).$$

This implies that X_1 is strongly regular for L_1. ■

3 Proof Of Theorem 1.4

The proof of Theorem 1.4 relies on a series of auxiliary lemmas and propositions. We will state these results as needed along the way and postpone the longer proofs to later sections.

Fix X as in Theorem 1.4, *i.e.*, let X be a genuine, irreducible, ω-regular, unitary representation of $Mp(2n)$. By virtue of Proposition 2.4 and Lemma 2.7, we can assume that our genuine, irreducible, ω-regular, unitary representation X is (the unique lowest K-type constituent of) a representation of the form $\mathcal{R}_{\mathfrak{q}}^S(X_1 \otimes X_0)$ with X_1 an irreducible genuine strongly regular $(\mathfrak{l}_1, L_1 \cap K)$-module for $L_1 = \widetilde{U}(r,s)$, and X_0 an irreducible genuine ω-regular $(\mathfrak{l}_0, L_0 \cap K)$-module for $L_0 = Mp(2d)$. Note that, because L_0 is a factor of L_a (and X^{L_a} is a minimal principal series representation of L_a), X_0 must be a minimal principal series representation of L_0. The following proposition asserts that, in this setting, X_1 is a *good* $A_{\mathfrak{q}_2}(\lambda_2)$ module.

Proposition 3.1 *Let X be an irreducible unitary $(\mathfrak{g}, K)$-module of $Mp(2n)$. Assume that X is genuine and ω-regular; realize X as the unique lowest K-type constituent of $\mathcal{R}_{\mathfrak{q}}^{S}(X^L)$, as in Proposition 2.4, and write*

$$L = L_1 \times L_0 = \widetilde{U}(r,s) \times Mp(2d), \quad X^L \simeq X_1 \otimes X_0,$$

as in Lemma 2.7. Suppose that $r + s \neq 0$. Then there exist a theta stable parabolic subalgebra $\mathfrak{q}_2$ and a representation $\mathbb{C}_{\lambda_2}$ of the Levi factor corresponding to $\mathfrak{q}_2$, with λ_2 in the good range for $\mathfrak{q}_2$, such that $X_1 \simeq A_{\mathfrak{q}_2}(\lambda_2)$.

Using Proposition 3.1, we will prove the following result.

Proposition 3.2 *In the setting of Proposition 3.1, assume $d > 0$ and allow $r + s$ to be possibly equal to zero. Then one of the following (mutually exclusive) options occurs:*

(i) *X_0 is an even oscillator representation with lowest K-type*

$$\mu_0 = \pm(1/2, 1/2, \dots, 1/2).$$

(ii) *There exist a subgroup $L' = L_1' \times L_0' = \widetilde{U}(r', s') \times Mp(2(d+1)) \subset G$ also containing L_a, a theta stable subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \supseteq \mathfrak{q}_a$, and representations $X_i^{'}$ of $L_i^{'}$ $(i = 0, 1)$, such that*
   (a) *$\widetilde{U}(r', s') \subset \widetilde{U}(r, s)$, with either $r = r' + 1$ or $s = s' + 1$,*
   (b) *$X_1^{'}$ is a good $A_{\mathfrak{q}_3}(\lambda_3)$ (or $r' + s' = 0$),*
   (c) *$X_0^{'}$ is an odd oscillator representation with lowest K-type*

   $$\mu_0^{'} = (3/2, 1/2, \dots, 1/2) \text{ or } (-1/2, -1/2, \dots, -1/2. -3/2),$$

   and
   (d) *X can be realized as the unique lowest K-type constituent of $\mathcal{R}_{\mathfrak{q}'}^{S'}(X_1^{'} \otimes X_0^{'})$.*

Putting all these results together, we obtain the following corollary.

Corollary 3.3 *Every genuine irreducible ω-regular unitary representation X of $Mp(2n)$ satisfies one of the following two properties:*

(i) *X is (the unique lowest K-type constituent of) a representation of the form $\mathcal{R}_{\mathfrak{q}}^{S}(X^L)$ with $L = \widetilde{U}(r,s) \times Mp(2d) = L_1 \times L_0$ and $X^L = X_1 \otimes X_0$. Here*
   (a) *X_1 is a good $A_{\mathfrak{q}_2}(\lambda_2)$ module for $\widetilde{U}(r,s)$, unless $r + s = 0$, and*
   (b) *X_0 is an even oscillator representation of $Mp(2d)$ with lowest K-type*

   $$\mu_0 = \pm(1/2, 1/2, \dots, 1/2),$$

   unless $d = 0$.

(ii) *X is (the unique lowest K-type constituent of) a representation of the form $\mathcal{R}_{\mathfrak{q}'}^{S'}(X^{L'})$ with $L' = \widetilde{U}(r', s') \times Mp(2(d+1)) = L_1' \times L_0'$ and $X^{L'} = X_1' \otimes X_0'$. Here*
   (a) *X_1' is a good $A_{\mathfrak{q}_3}(\lambda_3)$ module for $\widetilde{U}(r', s')$, unless $r' + s' = 0$, and*

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(b) *X_0' is an odd oscillator representation of $Mp(2(d+1))$ with lowest K-type*

$$\mu_0\prime = (3/2, 1/2, \dots, 1/2) \text{ or } (-1/2, -1/2, \dots, -1/2. -3/2).$$

In order to conclude the proof of Theorem 1.4, we must show that (in both cases) our genuine, irreducible, ω-regular, unitary representation X of $Mp(2n)$ can also be realized as an $A_{\mathfrak{q}}(\Omega)$ representation for some theta stable parabolic subalgebra $\mathfrak{q}$ and some representation $\Omega = \mathbb{C}_\lambda \otimes \omega$ in the good range for $\mathfrak{q}$. For this we need two more results.

Proposition 3.4 *Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ be a theta stable parabolic subalgebra. Assume that $L = N_G(\mathfrak{q})$ is a direct product of two reductive subgroups $L = L_1 \times L_0$. (Here L can be of the form $\widetilde{U}(r,s) \times Mp(2d)$, as in Proposition 2.4, or $\widetilde{U}(r', s') \times Mp(2(d+1))$, as in Proposition 3.2.) Suppose further that we have a representation*

$$X \simeq \mathcal{R}_{\mathfrak{q}}\big(A_{\mathfrak{q}'}(\lambda') \otimes \omega\big),$$

where ω is an irreducible summand of an oscillator representation, $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \subseteq \mathfrak{l}_1$, and $A_{\mathfrak{q}'}(\lambda')$ is good for $\mathfrak{q}'$. Then there is a theta stable parabolic subalgebra $\mathfrak{q}_\omega = \mathfrak{l}_\omega + \mathfrak{u}_\omega$ of $\mathfrak{g}$ such that

$$X \simeq \mathcal{R}_{\mathfrak{q}_\omega}(\mathbb{C}_{\lambda'} \otimes \omega) = A_{\mathfrak{q}_\omega}(\Omega)$$

with $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$.

Proof Assume that $X \simeq \mathcal{R}_{\mathfrak{q}}(A_{\mathfrak{q}'}(\lambda') \otimes \omega)$, where $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \subseteq \mathfrak{l}_1$. Set

$$\mathfrak{q}_b = \mathfrak{q}' \oplus \mathfrak{l}_0 = (\mathfrak{l}' + \mathfrak{u}') \oplus \mathfrak{l}_0.$$

Then $\mathfrak{q}_b$ is a theta stable parabolic subalgebra of $\mathfrak{l}$, and by Lemma 3.5, we have that

$$A_{\mathfrak{q}'}(\lambda') \otimes \omega \simeq \mathcal{R}_{\mathfrak{q}'}^{(\mathfrak{l}_1, L_1 \cap K)}(\mathbb{C}_{\lambda'}) \otimes \mathcal{R}_{\mathfrak{l}_0}^{(\mathfrak{l}_0, L_0 \cap K)}(\omega) \simeq \mathcal{R}_{\mathfrak{q}_b}^{(\mathfrak{l}, L \cap K)}(\mathbb{C}_{\lambda'} \otimes \omega).$$

Therefore,

$$X \simeq \mathcal{R}_{\mathfrak{q}}\big(\mathcal{R}_{\mathfrak{q}_b}^{(\mathfrak{l}, L \cap K)}(\mathbb{C}_{\lambda'} \otimes \omega)\big).$$

Now set

$$\mathfrak{q}_\omega = \mathfrak{q}_b + \mathfrak{u} = \underbrace{(\mathfrak{l}' \oplus \mathfrak{l}_0)}_{\mathfrak{l}_\omega} + \underbrace{(\mathfrak{u}' + \mathfrak{u})}_{\mathfrak{u}_\omega}.$$

Note that $\mathfrak{q}_\omega$ is a parabolic subalgebra of $\mathfrak{g}$. Since $X_1 \cong A_{\mathfrak{q}'}(\lambda')$, by [10, Proposition 3.5], we know that $\mathfrak{l}' \supseteq \mathfrak{l}_a \cap \mathfrak{l}_1$ and $\mathfrak{u}' \subseteq \mathfrak{u}_a \cap \mathfrak{l}_1$. We have

$$\mathfrak{l}_\omega = \mathfrak{l}' \oplus \mathfrak{l}_0 \subseteq \mathfrak{l}, \quad \mathfrak{u} \subseteq \mathfrak{u}' + \mathfrak{u} = \mathfrak{u}_\omega, \quad \text{and} \quad \mathfrak{q}_\omega \subseteq \mathfrak{q}. \tag{3.1}$$

Hence, by induction in stages (*cf.* [12, Proposition 6.3.6]), we find that

$$\mathcal{R}_{\mathfrak{q}}\big(\mathcal{R}_{\mathfrak{q}_b}^{(\mathfrak{l}, L \cap K)}(\mathbb{C}_{\lambda'} \otimes \omega)\big) \simeq \mathcal{R}_{\mathfrak{q}_\omega}(\mathbb{C}_{\lambda'} \otimes \omega) = A_{\mathfrak{q}_\omega}(\Omega),$$

and Proposition 3.4 is proved. ■

Lemma 3.5 *For $i = 1, 2$, let G_i be a reductive Lie group with maximal compact subgroup K_i, complexified Lie algebra $\mathfrak{g}_i$, and a theta stable parabolic subalgebra $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i$. Moreover, let X_i be an $(\mathfrak{l}_i, L_i \cap K_i)$-module, and let $S_i = \dim(\mathfrak{u}_i \cap \mathfrak{k}_i)$. Then*

$$\mathcal{R}_{\mathfrak{q}_1}^{S_1}(X_1) \otimes \mathcal{R}_{\mathfrak{q}_2}^{S_2}(X_2) \simeq \mathcal{R}_{\mathfrak{q}_1 \oplus \mathfrak{q}_2}^{S_1+S_2}(X_1 \otimes X_2).$$

Proof This follows by tracing through the definitions of the various cohomological induction functors, using standard techniques and identities in homological algebra, including an appropriate Künneth formula. ■

Finally, we need to show that our representation Ω is indeed in the good range for $\mathfrak{q}_\omega$.

Proposition 3.6 *In the setting of Proposition 3.4, let X be the irreducible (lowest K-type constituent of the) representation $\mathcal{R}_{\mathfrak{q}_\omega}(\mathbb{C}_{\lambda'} \otimes \omega)$. Assume, moreover, that λ' is in the good range for $\mathfrak{q}'$. Then $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$ is in the good range for $\mathfrak{q}_\omega$.*

This concludes the proof of Theorem 1.4. The proofs of Propositions 3.1, 3.2, and 3.6 will be given in Sections 6, 7, and 8, respectively. Before we turn to those proofs, we need to prove several technical lemmas concerning lowest K-types and to recall some results about genuine minimal principal series representations of $Mp(2d)$. We address these issues in the next two sections.

4 Technical Lemmas

The purpose of this section is to describe lowest K-types of irreducible, genuine, and ω-regular representations of $Mp(2n)$, and to present some results that will be needed for the proofs of Propositions 3.1, 3.2, and 3.6. We begin, in great generality, with irreducible admissible $(\mathfrak{g},K)$-modules $Mp(2n)$.

Lemma 4.1 *Let X be an irreducible admissible $(\mathfrak{g},K)$-module of $Mp(2n)$, and let μ be a lowest $\widetilde{U}(n)$-type of X. Let λ_a and $\mathfrak{q}_a = \mathfrak{l}_a + \mathfrak{u}_a$ be the Vogan classification parameter and the theta stable parabolic subalgebra associated with X, respectively, as in (2.3). Then μ is of the form*

$$\mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \delta^{L_a}, \tag{4.1}$$

where δ^{L_a} is a fine $(L_a \cap K)$-type.

Proof This is a known result due to Vogan. It follows from Proposition 2.4(ii)–(iii) (when $\mathfrak{q} = \mathfrak{q}_a$) and from the fact that (the highest weight of) the representation μ^{L_a} in that proposition is of the form $\lambda_a - \rho(\mathfrak{u}_a) + \delta^{L_a}$, with δ^{L_a} a fine $(L_a \cap K)$-type. See [7, Proposition 3.2.7] for a detailed proof in the case of $G = U(p, q)$. ■

Next, we restrict our attention to irreducible admissible $(\mathfrak{g},K)$-modules of $Mp(2n)$ that are genuine and ω-regular.

Lemma 4.2 *Let X be an irreducible admissible $(\mathfrak{g}, K)$-module of $Mp(2n)$. Assume that X is genuine and ω-regular. Realize X as the unique lowest K-type constituent of a cohomologically induced representation $\mathcal{R}^S_{\mathfrak{q}}(X^L)$, with*

$$L = L_1 \times L_0 = \widetilde{U}(r, s) \times Mp(2d) \quad \text{and} \quad X^L \simeq X_1 \otimes X_0,$$

as in Proposition 2.4. Let

$$\lambda_a = \Big(\underbrace{g_1, \dots, g_1}_{r_1}, \dots, \underbrace{g_t, \dots, g_t}_{r_t} \Big| \underbrace{0, \dots, 0}_{d} \Big| \underbrace{-g_t, \dots, -g_t}_{s_t}, \dots, \underbrace{-g_1, \dots, -g_1}_{s_1}\Big)$$

be the Vogan classification parameter of X, as in equation (2.5).

(i) *If X_1 is an $A_{\mathfrak{q}'}(\lambda')$ module of $\widetilde{U}(r, s)$, then the representation X has a unique lowest $\widetilde{U}(n)$-type μ.*

(ii) *For $i = 1, \dots, t$, if $r_i \neq s_i$, then $g_i \in \mathbb{Z} + \frac{1}{2}$. In this case, the infinitesimal character of X contains an entry $\pm g_i$, and the infinitesimal character of X_0 does not.*

(iii) *Suppose that X has a unique lowest $\widetilde{U}(n)$-type μ.*

(a) *μ is of the form*

$$\mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \mu_0, \tag{4.2}$$

where μ_0 is the lowest $\widetilde{U}(d)$-type of X_0.

(b) *Moreover, if $r_i = s_i$, then $g_i \in \mathbb{Z}$.*

Proof Recall that X_0 is a genuine minimal principal series of $Mp(2d)$, therefore it has a unique lowest K-type (*cf.* [5]). If X_1 is an $A_{\mathfrak{q}'}(\lambda')$ module of $\widetilde{U}(r, s)$, then X_1 has a unique lowest K-type as well. The uniqueness of the lowest K-type of X then follows from Proposition 2.4(iii).

For part (ii), recall from [8, Section 3.1] the relationship between the parameter λ_a, the infinitesimal character, the lowest K-types, and the Langlands parameters of a representation. There, the theory is laid out in detail for representations of the symplectic group. The corresponding statements for genuine representations of $Mp(2n)$ can be obtained by making slight modifications. In the case of ω-regular (hence non-singular) representations, all limits of discrete series representations are discrete series, and all the lowest K-types of a standard module appear in the same irreducible representation. Because X is a genuine representation with associated parameter λ_a as in (2.5), we can realize X as the Langlands subquotient of an induced representation from a cuspidal parabolic subgroup with Levi component MA isomorphic to (a quotient of) $Mp(2a) \times \widetilde{GL}(2, \mathbb{R})^b \times \widetilde{GL}(1, \mathbb{R})^d$, where $a = \sum_{i=1}^t |r_i - s_i|$ and $b = \sum_{i=1}^t \min\{r_i, s_i\}$. If $r_i = s_i + 1$ for some i, then the entry g_i coincides with an entry of the Harish-Chandra parameter of a genuine discrete series representation of $Mp(2a)$. Similarly for $-g_i$, if $r_i = s_i - 1$. Therefore, $g_i \in \mathbb{Z} + \frac{1}{2}$ whenever $r_i \neq s_i$. In this case, we also find that g_i is an entry of the infinitesimal character of X. Then the fact that X_0 does not contain an entry $\pm g_i$ follows from the ω-regular condition.

For part (iii), assume that our representation has a unique lowest $\widetilde{U}(n)$-type μ. Write

$$\mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \delta^{L_a}$$

(as in Lemma 4.1). By Proposition 2.4, this equals $\mu_1 + \mu_0 + 2\rho(\mathfrak{u} \cap \mathfrak{p})$. Now look at the restriction to $L_0 \cap K$:

$$\begin{aligned} \lambda_a \,|_{\mathfrak{l}_0 \cap \mathfrak{k}} &= 0 \\ \mu_1 \,|_{\mathfrak{l}_0 \cap \mathfrak{k}} &= 0 \\ 2\rho(\mathfrak{u} \cap \mathfrak{p}) \,|_{\mathfrak{l}_0 \cap \mathfrak{k}} &= \left[\rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k})\right] \,|_{\mathfrak{l}_0 \cap \mathfrak{k}} . \end{aligned}$$

Hence, $\delta^{L_a} \,|_{\mathfrak{l}_0 \cap \mathfrak{k}} = \mu_0$. Next, we show that $\delta^{L_a} \,|_{\mathfrak{l}_1 \cap \mathfrak{k}} = 0$. Because X has a unique lowest $\widetilde{U}(n)$-type, the representation X_1 must have the same property. In order to have a representation of L_1 with a unique lowest $\widetilde{U}(r) \times \widetilde{U}(s)$-type, every factor $\widetilde{U}(r_i, s_i)$ of L_a with $r_i = s_i$ should carry the trivial fine $\widetilde{U}(r_i) \times \widetilde{U}(s_i)$-type, because non-trivial fine $\widetilde{U}(r_i) \times \widetilde{U}(s_i)$-types on such factors come in pairs. This shows that the restriction of δ^{L_a} to $L_1 \cap K$ is zero, hence $\delta^{L_a} = \mu_0$ and

$$\mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \mu_0,$$

completing the proof of part (a). For part (b), write $r_i = s_i + \varepsilon_i$ for all $i = 1, \dots, t$. Note that $\varepsilon_i = 0$ or ± 1, because each factor of

$$L_a = \Big[\prod_{i=1}^{t} \widetilde{U}(r_i, s_i)\Big] \times Mp(2d)$$

is quasisplit. Then

$$\rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) = \Big(\dots, \underbrace{f_i, \dots, f_i}_{r_i}, \cdots \,\Big|\, \underbrace{c, \dots, c}_{d} \,\Big|\, \dots, \underbrace{h_i, \dots, h_i}_{s_i}, \dots\Big),$$

with

$$\begin{aligned} f_i &= \sum_{j<i}(r_j - s_j) + \frac{\varepsilon_i + 1}{2} = \sum_{j<i}\varepsilon_j + \frac{\varepsilon_i + 1}{2}, \\ h_i &= \sum_{j<i}(r_j - s_j) + \frac{\varepsilon_i - 1}{2} = \sum_{j<i}\varepsilon_j + \frac{\varepsilon_i - 1}{2}, \qquad (4.3) \\ c &= r - s. \end{aligned}$$

If $r_i = s_i$ for some $i = 1, \dots, t$, then $\varepsilon_i = 0$. The corresponding entry f_i of $\rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k})$ is then in $\mathbb{Z} + \frac{1}{2}$. Because genuine $\widetilde{U}(n)$-types have half-integral entries, this implies that $g_i \in \mathbb{Z}$. The proof of Lemma 4.2 is now complete. ■

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Remain in the setting of Lemma 4.2. Let μ be a lowest K-type of X. Recall that μ satisfies equation (4.1):

$$\mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \delta^{L_a}.$$

Write

$$\mu = \Big(\underbrace{a_1,\dots,a_1}_{r_1},\dots,\underbrace{a_t,\dots,a_t}_{r_t} \;\Big|\; \underbrace{c_1,\dots,c_d}_{d} \;\Big|\; \underbrace{b_t,\dots,b_t}_{s_t},\dots,\underbrace{b_1,\dots,b_1}_{s_1}\Big), \tag{4.4}$$

according to how the coordinates break into the factors of the subgroup L_a. Then the fine $(L \cap K)$-type δ^{L_a} is of the form

$$\delta^{L_a} = \Big(\underbrace{y_1,\dots,y_1}_{r_1},\dots,\underbrace{y_t,\dots,y_t}_{r_t} \;\Big|\; \underbrace{z_1,\dots,z_d}_{d} \;\Big|\; \underbrace{y_t,\dots,y_t}_{s_t},\dots,\underbrace{y_1,\dots,y_1}_{s_1}\Big),$$

with $y_i = 0$ or $\pm\frac{1}{2}$, and $z_i = \pm\frac{1}{2}$ (*cf.* [8, Proposition 6]). Note that the $\{z_i\}$ are weakly decreasing.

Remark 4.3 If $r_i = 0$, then a_i does not occur as a coordinate of μ, but it is still convenient to define

$$a_i = g_i + f_i + y_i,$$

with f_i as in equation (4.3). (Because $s_i > 0$ in this case, the quantities y_i and $(-)g_i$ can be determined by the coordinates of δ^{L_a} and λ_a, respectively.) Similarly, if $s_i = 0$ (and $r_i > 0$), we define b_i by $b_i = -g_i + h_i + y_i$. We obtain sequences $\{a_i\}$ and $\{b_i\}$ of half-integers satisfying

$$a_1 \ge a_2 \ge \dots \ge a_t \quad \text{and} \quad b_t \ge b_{t-1} \ge \dots \ge b_1. \tag{4.5}$$

Lemma 4.4 *Retain all the previous notation.*

(i) $a_t - b_t \ge 2$.

(ii) *Assume that the K-type μ satisfies the additional condition*

$$a_t = c_1. \tag{4.6}$$

(a) $\varepsilon_t \ge 0$ *and* $g_t = \frac{1}{2}$.

(b) *If* $\varepsilon_t = 0$, *then* δ^{L_a} (cf. (4.1)) *must be of the form*

$$\delta^{L_a} = \Big(\dots,\underbrace{-\tfrac{1}{2},\dots,-\tfrac{1}{2}}_{s_t} \;\Big|\; \underbrace{\tfrac{1}{2},\dots,\tfrac{1}{2}}_{p},\underbrace{-\tfrac{1}{2},\dots,-\tfrac{1}{2}}_{q} \;\Big|\; \underbrace{-\tfrac{1}{2},\dots,-\tfrac{1}{2}}_{s_t},\dots\Big) \tag{4.7}$$

for some $p > 0$ *and* $q = d - p$.

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(c) *If* $\varepsilon_t = 1$, *then* δ^{L_a} *(*cf. (4.1)*) must be of the form*

$$\delta^{L_a} = \Big(\dots, \underbrace{0, \dots, 0}_{s_t+1} \;\Big|\; \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q} \;\Big|\; \underbrace{0, \dots, 0}_{s_t}, \dots\Big)$$

for some $p > 0$ *and* $q = d - p$.

(d) $c_d > b_t$.

Proof For part (i), observe that

$$a_t = g_t + (r-s) - (r_t - s_t) + \frac{\varepsilon_t + 1}{2} + y_t = g_t + (r-s) + \frac{1}{2}(1-\varepsilon_t) + y_t,$$

$$c_1 = (r-s) + z_1,$$

$$c_d = (r-s) + z_d, \text{ and}$$

$$b_t = -g_t + (r-s) - (r_t - s_t) + \frac{\varepsilon_t - 1}{2} + y_t = -g_t + (r-s) + \frac{1}{2}(-1-\varepsilon_t) + y_t.$$

Hence, we obtain

$$a_t - c_1 = g_t + \frac{1}{2}(1-\varepsilon_t) + y_t - z_1, \tag{4.8}$$

$$c_d - b_t = g_t + \frac{1}{2}(1+\varepsilon_t) + z_d - y_t, \tag{4.9}$$

and

$$a_t - b_t = 2g_t + 1 \geq 2,$$

because $g_t \geq \frac{1}{2}$. This proves (i).

Now assume that μ satisfies equation (4.6). The coordinate g_t of λ_a is either an integer or a half-integer. We consider the two cases separately.

First assume that $g_t \in \mathbb{Z}$. By Lemma 4.2, $\varepsilon_t = 0$, so

$$a_t \in \mathbb{Z} + \frac{1}{2} + y_t \quad \text{and} \quad c_1 \in \mathbb{Z} + z_1.$$

In order for μ to be genuine, the coordinates of the fine $(L_a \cap K)$-type δ^{L_a} must satisfy

$$\begin{cases} y_t \in \mathbb{Z} \\ z_1 \in \mathbb{Z} + \frac{1}{2} \end{cases} \iff \begin{cases} y_t = 0 \\ z_1 = \pm\frac{1}{2}. \end{cases}$$

This says that the representation δ^{L_a} of $(L_a \cap K)$ is trivial on the $\widetilde{U}(s_t, s_t)$-factor of L_a, and non-trivial on the $Mp(2d)$-factor:

$$\delta^{L_a} = \Big(\dots, \underbrace{0, \dots, 0}_{s_t} \;\Big|\; \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q} \;\Big|\; \underbrace{0, \dots, 0}_{s_t}, \dots\Big).$$

Equation (4.8) then gives

$$a_t - c_1 = \begin{cases} g_t & \text{if } p > 0, \\ g_t + 1 & \text{if } p = 0. \end{cases}$$

Because $g_t > 0$, this contradicts the fact that μ satisfies condition (4.6). Hence g_t cannot be an integer.

Next, assume that $g_t \in \mathbb{Z}+\frac{1}{2}$. We need to show that $g_t = \frac{1}{2}$. Recall that ε_t is either 0 or ± 1. We consider the two cases separately. If $\varepsilon_t = 0$, the fine representation δ^{L_a} of $(L_a \cap K)$ must be non-trivial on both the $\widetilde{U}(s_t, s_t)$-factor and the $Mp(2d)$-factor of L_a. Then either

$$\text{(4.10)} \quad \delta^{L_a} = \Big(\dots, \underbrace{-\tfrac{1}{2}, \dots, -\tfrac{1}{2}}_{s_t} \,\Big|\, \underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{p}, \underbrace{-\tfrac{1}{2}, \dots, -\tfrac{1}{2}}_{q} \,\Big|\, \underbrace{-\tfrac{1}{2}, \dots, -\tfrac{1}{2}}_{s_t}, \dots\Big)$$

or

$$\text{(4.11)} \quad \delta^{L_a} = \Big(\dots, \underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{s_t} \,\Big|\, \underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{p}, \underbrace{-\tfrac{1}{2}, \dots, -\tfrac{1}{2}}_{q} \,\Big|\, \underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{s_t}, \dots\Big)$$

for some $p, q \geq 0$ such that $p + q = d$. Equation (4.11) contradicts condition (4.6), because the difference $a_t - c_1$ is always positive; hence equation (4.10) must hold. We find

$$a_t - c_1 = 0 \iff g_t = \frac{1}{2} \quad \text{and} \quad p > 0.$$

Note that, in this case, $c_d - b_t > 0$ by equation (4.9). This proves (b) (and also (d) for the case $\varepsilon_t = 0$). If $\varepsilon_t \neq 0$, the fine representation δ^{L_a} of $(L_a \cap K)$ must be trivial on the $\widetilde{U}(s_t + \varepsilon_t, s_t)$-factor of L_a and non-trivial on the $Mp(2d)$-factor of L_a. Hence

$$\delta^{L_a} = \Big(\dots, \underbrace{0, \dots, 0}_{s_t} \,\Big|\, \underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{p}, \underbrace{-\tfrac{1}{2}, \dots, -\tfrac{1}{2}}_{q} \,\Big|\, \underbrace{0, \dots, 0}_{s_t}, \dots\Big).$$

Just as above, we get that

$$a_t - c_1 = 0 \iff g_t = \frac{1}{2}, \ \varepsilon_t = 1, \text{ and } p > 0.$$

Moreover, $c_d - b_t > 0$ always in this case. This concludes the proofs of (a), (c), and (d). The proof of Lemma 4.4 is now complete. ■

Finally, we look at the case in which the irreducible, admissible, genuine, and ω-regular $(\mathfrak{g}, K)$-module X of $Mp(2n)$ is also unitary. Realize X as the unique lowest K-type constituent of a representation $\mathcal{R}_{\mathfrak{q}}^S(X^L)$, with

$$L = L_1 \times L_0 = \widetilde{U}(r, s) \times Mp(2d) \quad \text{and} \quad X^L \simeq X_1 \otimes X_0,$$

as in Proposition 2.4. Proposition 3.1 then implies that X_1 is a good $A_{\mathfrak{q}_2}(\lambda_2)$ module. The next lemma describes the coordinates of the lowest K-type of X under some technical assumptions that are needed for the proof of Claim (A) in Proposition 3.2.

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Lemma 4.5 *Let X be an irreducible, admissible, genuine, and ω-regular $(\mathfrak{g}, K)$-module of $Mp(2n)$. Assume that X_1 (as above) is a good $A_{\mathfrak{q}_2}(\lambda_2)$ module. Let μ be the unique lowest K-type of X and let λ_a be its Vogan classification parameter. Write the coordinates of μ and λ_a as in equations (4.4) and (2.5), respectively. Set*

$$x = \max\{i < t \mid r_i > 0\}, \quad y = \max\{i < t \mid s_i > 0\}.$$

(i) *If $g_t = \frac{1}{2}$, $\varepsilon_t = 1$, and $s_t > 0$, then $b_t - b_y \geq 2$;*
(ii) *If $g_{t-1} \in \mathbb{Z}$ or if $g_{t-1} \in \mathbb{Z} + \frac{1}{2}$ and $g_{t-1} \geq \frac{7}{2}$, then $a_x - a_t \geq 2$.*

Proof Recall that, by Lemma 4.2, X has a unique lowest $\widetilde{U}(n)$-type μ (of the form (4.2)). Equations (4.3) give

$$\begin{aligned}(4.12)\qquad b_t - b_{t-1} &= -g_t + h_t - (-g_{t-1} + h_{t-1}) = -g_t + g_{t-1} + \frac{\varepsilon_{t-1} + \varepsilon_t}{2} \\ &= g_{t-1} + \frac{\varepsilon_{t-1}}{2}\end{aligned}$$

(because $g_t = 1/2$ and $\varepsilon_t = 1$). Note that g_{t-1} is a half-integer greater than $g_t = 1/2$, and ε_{t-1} is 0 or ± 1. Then $b_t - b_{t-1} \geq 1$. (This is obvious if $g_{t-1} \in \mathbb{Z} + \frac{1}{2}$; when g_{t-1} is an integer, it follows from the fact that $\varepsilon_{t-1} = 0$, because $b_t - b_{t-1}$ must be an integer.) Since the entries $\{b_i\}_{i=1}^t$ are weakly increasing, we also find that $b_t - b_y \geq 1$.

In order to show that the difference $b_t - b_y$ is, in fact, at least 2, we consider the lowest $(K \cap L_1)$-type μ_1 of X_1. This is a $(\widetilde{U}(r) \times \widetilde{U}(s))$-type of the form

$$\mu_1 = \big(\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})\big)\,|_{\widetilde{U}(r) \times \widetilde{U}(s)}.$$

Note that $2\rho(\mathfrak{u} \cap \mathfrak{p})$ is constant on $\widetilde{U}(r)$ and $\widetilde{U}(s)$. Therefore, to understand the difference $b_t - b_y$, it is sufficient to look at the (difference among) coordinates of μ_1.

Write

$$(4.13)\qquad L_2 = \prod_{i=1}^{u} \widetilde{U}(p_i, q_i).$$

Section 8 of [9] gives a number of properties of lowest K-types of (good) $A_{\mathfrak{q}}(\lambda)$ modules. Assuming we have chosen $\mathfrak{q}_2$ so that L_2 is maximal (there is a unique such choice), μ_1 must be of the form

(4.14)

$$\mu_1 = \Big(\underbrace{n_1, \dots, n_1}_{p_1}, \dots, \underbrace{n_u, \dots, n_u}_{p_u} \,\Big|\, \underbrace{0, \dots, 0}_{d} \,\Big|\, \underbrace{m_u, \dots, m_u}_{q_u}, \dots, \underbrace{m_1, \dots, m_1}_{q_1}\Big)$$

with $n_j > n_k$ and $m_k > m_j$ for all $1 \leq j < k \leq u$. Moreover, [9, Proposition 8.6] implies that

$$(4.15)\qquad n_j - n_k \geq q_i + q_{i+1} \qquad \text{and} \qquad m_k - m_j \geq p_i + p_{i+1}$$

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whenever $j \le i$ and $k \ge i+1$. Let $z = \max\{i < u \mid q_i > 0\}$, so that m_z exists. Then we get

$$b_t - b_y = m_u - m_z \ge p_u + p_{u-1}$$

(by (4.15) for $i = u-1$). By Proposition 2.4(vii), and [10, Proposition 3.5], $L_a \cap L_1 \subset L_2$, therefore $\mathfrak{u}(r_t, s_t) \subset \mathfrak{u}(p_u, q_u)$, so that

$$r_t \le p_u \qquad \text{and} \qquad s_t \le q_u.$$

In fact, $s_t = q_u$, because $2\rho(\mathfrak{u} \cap \mathfrak{p})$ is constant on $\widetilde{U}(r)$ and $\widetilde{U}(s)$, and $b_t - b_{t-1} > 0$. Hence we can write

$$b_t - b_y \ge p_u + p_{u-1} \ge r_t = s_t + \varepsilon_t \ge 2.$$

This proves (i).

For the proof of (ii), we distinguish two cases. If g_{t-1} is an integer, then Lemma 4.2 implies that $\varepsilon_{t-1} = 0$, hence $s_{t-1} = r_{t-1} > 0$ and $x = y = t-1$. Furthermore, because $L_a \cap L_1 \subset L_2$, we have $\mathfrak{u}(r_{t-1}, s_{t-1}) \subset \mathfrak{u}(p_{u-1}, q_{u-1})$. This implies that

$$(p_u, q_u) = (r_t, s_t) \quad \text{and} \quad q_{u-1} \ge s_{t-1} \ge 1.$$

Then

$$a_{t-1} - a_t = n_{u-1} - n_u \ge q_u + q_{u-1} \ge 1 + 1 = 2,$$

proving (ii). If $g_{t-1} \ge \frac{7}{2}$, then using equations (4.3), we get

$$\begin{aligned} a_{t-1} - a_t &= g_{t-1} - g_t + f_{t-1} - f_t = g_{t-1} - g_t - \frac{\varepsilon_{t-1} + \varepsilon_t}{2} \\ &= g_{t-1} - \frac{\varepsilon_{t-1}}{2} - 1 \ge 2. \end{aligned}$$

This concludes the proof of the lemma. ■

5 Genuine Principal Series of $Mp(2d)$

Genuine minimal principal series of $Mp(2d)$ were studied in detail in [5]. In this section, we summarize the non-unitarity results that are needed for this paper. We refer the reader to [5] for more details.

Lemma 5.1 (*cf.* [5]) (i) *Every genuine minimal principal series representation of $Mp(2d)$ has a unique lowest $\widetilde{U}(d)$-type of the form*

$$\mu_{\delta_{p,q}} = \Big(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q}\Big)$$

with $p + q = d$. Here $\delta_{p,q}$ is the character of the finite subgroup M of $Mp(2d)$ used to construct the induced representation.

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(ii) *For every pair of non-negative integers p and q with $p+q=d$, the Langlands quotients of genuine minimal principal series of $Mp(2d)$ with lowest $\widetilde{U}(d)$-type $\mu_{\delta_{p,q}}$ are parametrized by d-tuples of real numbers*

$$\nu = (\nu_1, \dots, \nu_d) := (\nu^p | \nu^q),$$

where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p \geq 0$ and $\nu_{p+1} \geq \nu_{p+2} \geq \cdots \geq \nu_{p+q} \geq 0$. The infinitesimal character of the corresponding representation is equal to ν.

(iii) *Irreducible genuine pseudospherical representations of $Mp(2d)$,* i.e., *those with lowest $\widetilde{U}(d)$-type $\mu_{\delta_{d,0}}$ or $\mu_{\delta_{0,d}}$, are uniquely determined by their infinitesimal character.*

Lemma 5.2 *Let $J(\delta_{p,q}, \nu)$ be the Langlands quotient of a genuine minimal principal series representation of $Mp(2d)$, with lowest $\widetilde{U}(d)$-type $\mu_{\delta_{p,q}}$ and parameter $\nu = (\nu^p|\nu^q)$ as in Lemma* 5.1*.*

(i) *If $\nu_p > \frac{1}{2}$ or if $\nu_i - \nu_{i+1} > 1$ for some $1 \leq i \leq p-1$, then $J(\delta_{p,q}, \nu)$ is not unitary, and the $\widetilde{U}(d)$-type*

$$\delta_1 = \Big(\underbrace{\frac{1}{2}, \dots \frac{1}{2}}_{p-1}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q}, -\frac{3}{2}\Big)$$

detects non-unitarity.

(ii) *If $\nu_{p+q} > \frac{1}{2}$ or if $\nu_i - \nu_{i+1} > 1$ for some $p+1 \leq i \leq p+q-1$, then $J(\delta_{p,q}, \nu)$ is not unitary, and the $\widetilde{U}(d)$-type*

$$\delta_2 = \Big(\frac{3}{2}, \underbrace{\frac{1}{2}, \dots \frac{1}{2}}_{p}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q-1}\Big)$$

detects non-unitarity.

(iii) *If $J(\delta_{p,q}, \nu)$ is ω-regular, $pq \neq 0$, $\nu^p \neq \omega^p$, and $\nu^q \neq \omega^q$, then both δ_1 and δ_2 detect non-unitarity. As usual, ω^r denotes the infinitesimal character of the oscillator representation of $Mp(2r)$.*

Proof The main ingredients for the proof of Lemma 5.2 are certain non-unitarity results contained in [1, 5]. Note that the assumptions in Lemma 5.2(i) are precisely conditions (1) and (3) of [5, Proposition 7.7]; similarly, the assumptions in part (ii) coincide with conditions (2) and (4) of that proposition. Therefore, in both cases, the non-unitarity of the Langlands quotient $J(\delta_{p,q}, \nu)$ of $Mp(2d)$ follows directly from [5, Proposition 7.7].

We are left with the problem of identifying a $\widetilde{U}(d)$-type on which the intertwining operator changes sign. Recall from [5, §5] that each $\widetilde{U}(d)$-type μ in $J(\delta_{p,q}, \nu)$ carries a representation ψ_μ of the stabilizer

$$W^{\delta_{p,q}} = W(C_p) \times W(C_q)$$

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of $\delta_{p,q}$. This group is isomorphic to the Weyl group of

$$G^{\delta_{p,q}} = SO(p+1,p) \times SO(q+1,q);$$

hence, associated with ψ_μ, there is an intertwining operator for spherical Langlands quotients of $G^{\delta_{p,q}}$. If μ is petite, the $Mp(2d)$-intertwining operator on μ with parameters $\{\delta = \delta_{p,q},\ \nu = (\nu^p|\nu^q)\}$ coincides with the $G^{\delta_{p,q}}$-intertwining operator on ψ_μ with parameters $\{\delta = \text{triv},\ \nu = (\nu^p|\nu^q)\}$. If $\nu_p > \frac{1}{2}$ or if $\nu_i - \nu_{i+1} > 1$ for some $1 \le i \le p-1$, the spherical Langlands quotient of $SO(p+1,p)$ with parameter ν^p is not unitary; the reflection representation $\sigma_R = (p-1) \times (1)$ of $W(C_p)$ detects non-unitarity (see [1, Lemma 14.6]). For all choices of ν^q, the spherical Langlands quotient of $SO(p+1,p) \times SO(q+1,1)$ with parameter $(\nu^p|\nu^q)$ is also not unitary, and the representation $\sigma_R \times \text{triv}$ of $W^{\delta_{p,q}}$ detects non-unitarity. The computations in [5, §10.1] show that the $\widetilde{U}(d)$-type

$$\delta_1 = \Big(\underbrace{\tfrac{1}{2}, \dots \tfrac{1}{2}}_{p-1}, \underbrace{-\tfrac{1}{2}, \dots, -\tfrac{1}{2}}_{q}, -\tfrac{3}{2}\Big)$$

carries the representation $\sigma_R \times \text{triv}$ of $W^{\delta_{p,q}}$. Because δ_1 is petite, the $Mp(2d)$-intertwining operator on δ_1 with parameters $\{\delta_{p,q},\ (\nu^p|\nu^q)\}$ matches the $G^{\delta_{p,q}}$-spherical operator with parameter $(\nu^p|\nu^q)$, hence it is not positive semi-definite. This shows that $J(\delta_{p,q}, (\nu^p|\nu^q))$ is not unitary, and concludes the proof of part (i) of the lemma.

The proof of part (ii) is analogous: the $\widetilde{U}(d)$-type δ_2 carries the representation $\text{triv} \otimes \sigma_R$ of $W^{\delta_{p,q}}$, hence the $Mp(2d)$-intertwining operator on δ_2 with parameters $\{\delta_{p,q},\ (\nu^p|\nu^q)\}$ is not positive semi-definite and the Langlands quotient $J(\delta_{p,q}, (\nu^p|\nu^q))$ of $Mp(2d)$ is not unitary.

For part (iii) of the lemma, note that if $\nu = (\nu^p|\nu^q)$ is ω-regular and $\nu^p \neq \omega^p$, then

$$\nu_p > \frac{1}{2} \text{ or } \nu_i - \nu_{i+1} > 1 \text{ for some } 1 \le i \le p-1.$$

Similarly, if $\nu = (\nu^p|\nu^q)$ is ω-regular and $\nu^q \neq \omega^q$, then

$$\nu_{p+q} > \frac{1}{2} \text{ or } \nu_i - \nu_{i+1} > 1 \text{ for some } p+1 \le i \le p+q-1.$$

Therefore, the assumptions of both part (i) and part (ii) of the lemma must hold, and both δ_1 and δ_2 detect non-unitarity. ■

6 Proof of Proposition 3.1

In this section, we give the proof of Proposition 3.1. For convenience, we begin by restating the result.

Proposition 3.1 *Let X be an irreducible unitary $(\mathfrak{g}, K)$-module of $Mp(2n)$. Assume that X is genuine and ω-regular. Realize X as the unique lowest K-type constituent of $\mathcal{R}^S_{\mathfrak{q}}(X^L)$, with*

$$L = L_1 \times L_0 = \widetilde{U}(r,s) \times Mp(2d) \quad \textit{and} \quad X^L \simeq X_1 \otimes X_0,$$

as in Proposition 2.4*. Suppose that* $r + s \neq 0$*. Then there exist a theta stable parabolic subalgebra* $\mathfrak{q}_2$ *and a representation* $\mathbb{C}_{\lambda_2}$ *of the Levi factor corresponding to* $\mathfrak{q}_2$*, with* λ_2 *in the good range for* $\mathfrak{q}_2$*, such that* $X_1 \simeq A_{\mathfrak{q}_2}(\lambda_2)$*.*

Proof Note that, by Proposition 2.4, the $(\mathfrak{l}, L \cap K)$-module X^L is irreducible, and so are the Harish-Chandra modules X_i of L_i $(i = 0, 1)$. Let μ be a lowest K-type of X, and let μ^L be the $(L \cap K)$-type of X^L corresponding to μ via the bottom layer map (*cf.* Proposition 2.4(ii)–(iii)); then

$$\mu^L = \mu_1 + \mu_0 \tag{6.1}$$

with $\mu_i = \mu^{L_i}$ a lowest $(L_i \cap K)$-type of X_i. By Lemma 2.7, the representation X_1 of $L_1 = \widetilde{U}(r, s)$ is strongly regular (and irreducible).

Assume, by way of contradiction, that X_1 is not a good $A_{\mathfrak{q}}(\lambda)$ module. We will show that X must be non-unitary, reaching a contradiction. By [9, Theorem 1.2] for the case $G = SU(p, q)$, if X_1 is not a good $A_{\mathfrak{q}}(\lambda)$ module, then X_1 is not unitary, and there exists an $(L_1 \cap K)$-type η_1 such that $\eta_1 = \mu_1 + \beta$ for some $\beta \in \Delta(\mathfrak{l}_1 \cap \mathfrak{p})$, and the form

$$\langle\,\cdot\,,\,\cdot\,\rangle^{L_1}|_{V(\mu_1)\oplus V(\eta_1)}$$

is indefinite. (In [9], the infinitesimal character of X_1 is assumed to be integral; however, the proof of Theorem 1.2 only uses the fact that it is real and strongly regular.) If

$$\eta = \mu + \beta = (\mu_1 + \beta) + \mu_0 + 2\rho(\mathfrak{u} \cap \mathfrak{p})$$

is also dominant, then, by the bottom layer argument, η occurs in X and the Hermitian form on $V(\mu) \oplus V(\eta)$ is indefinite (*cf.* Proposition 2.4(iii) and (v)). This implies that X is not unitary, reaching a contradiction.

So we may assume that $\eta = \mu + \beta$ is not dominant. Moreover, we can assume that μ is weakly dominant with respect to our fixed choice of Δ_c^+ in (2.4). Because β is a short non-compact root, if $\eta_1 = \mu_1 + \beta$ is dominant but $\eta = \mu + \beta$ is *not* dominant, then one of the following two options must occur:

$$d = 0 \quad \text{and} \quad a_t = b_t, \tag{6.2}$$

or

$$a_t = c_1 \quad \text{or} \quad c_d = b_t. \tag{6.3}$$

Remark 6.1 Lemma 4.4 shows that (6.2) is not possible and that the two identities in (6.3) cannot hold simultaneously.

Without loss of generality, we may then assume that $d > 0$ and $a_t = c_1$. We first look at the case in Lemma 4.4 when $\varepsilon_t = 1$ and $g_t = \frac{1}{2}$. By Lemma 4.2(ii), the infinitesimal character γ^X of X contains an entry $\frac{1}{2}$, but γ^{X_0} does not. Recall that X_0 is a genuine principal series representation of $L_0 = Mp(2d)$. We claim that in this case, X_0 is not unitary, and that a $\widetilde{U}(d)$-type detecting non-unitarity survives under

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the bottom layer map, leading to a contradiction. We make use of the non-unitarity results for genuine minimal principal series of $Mp(2d)$ contained in Section 5.

Write $X_0 \cong J(\delta_{p,q}, \nu)$ as in Lemma 5.2; note that $p > 0$, by Lemma 4.4. Then, by ω-regularity, ν_p is strictly greater than $\frac{1}{2}$. By Lemma 5.2(i), we know that the representation X_0 is not unitary, and the signature of the Hermitian form changes on δ_1. If μ_1 is as in equation (6.1), the weight

$$\eta = \mu_1 + \delta_1 + 2\rho(\mathfrak{u} \cap \mathfrak{p})$$

is K-dominant. By Proposition 2.4(iii) and (v), our original representation X is not unitary, which contradicts our assumption.

Now assume that $\varepsilon_t = 0$. Recall that the fine K-type δ^{L_a} is of the form (4.7); in particular, its restriction to the subgroup $\widetilde{U}(s_t, s_t)$ of L_a is given by

$$\delta^{L_a}\,|_{\widetilde{U}(s_t,s_t)} = \Big(0, \dots, 0, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t} \;\Big|\; \underbrace{0, \dots, 0}_{d} \;\Big|\; \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t}, 0, \dots, 0\Big).$$

Hence, the representation $\mu_1|_{\widetilde{U}(s_t,s_t)}$, inside $Mp(2n)$, is of the form

$$\Big(0, \dots, 0, \underbrace{a, \dots, a}_{s_t} \;\Big|\; \underbrace{0, \dots, 0}_{d} \;\Big|\; \underbrace{-a-1, \dots, -a-1}_{s_t}, 0, \dots, 0\Big)$$

for some $a \in \frac{1}{2}\mathbb{Z}$. By [9, Lemmas 8.8 and 6.3(b)], if $\beta = (1, 0 \dots 0; 0, \dots 0, -1)$ is the root $\epsilon_1 - \epsilon_{2s_t}$ in $U(s_t, s_t)$, then $\mu_1|_{U(s_t,s_t)} + \beta$ detects the signature change and survives in the bottom layer. Note that the root in $Mp(2n, \mathbb{R})$ corresponding to β is

$$\Big(0, \dots, 0, \underbrace{1, 0, \dots, 0}_{s_t} \;\Big|\; \underbrace{0, \dots, 0}_{d} \;\Big|\; \underbrace{1, 0, \dots, 0}_{s_t}, 0, \dots, 0\Big).$$

Therefore, X is not unitary, once again a contradiction. The proof of Proposition 3.1 is now complete. ∎

7 Proof of Proposition 3.2

We now restate and prove Proposition 3.2.

Proposition 3.2 *Let X be an irreducible unitary $(\mathfrak{g}, K)$-module of $Mp(2n)$. Assume that X is genuine and ω-regular; realize X as the unique lowest K-type constituent of $\mathcal{R}_{\mathfrak{q}}^S(X^L)$, with*

$$L = L_1 \times L_0 = \widetilde{U}(r, s) \times Mp(2d) \quad \textit{and} \quad X^L \simeq X_1 \otimes X_0,$$

as in Proposition 2.4. Suppose that $d > 0$. Then one of the following (mutually exclusive) options occurs:

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(i) *X_0 is an even oscillator representation with lowest $\widetilde{U}(d)$-type*

$$\mu_0 = \pm(1/2, 1/2, \dots, 1/2).$$

(ii) *There exist a subgroup $L' = L'_1 \times L'_0 = \widetilde{U}(r', s') \times Mp(2(d+1)) \subset G$ also containing L_a, a theta stable subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \supseteq \mathfrak{q}$ and representations $X_i^{'}$ of $L_i^{'}$ ($i = 0, 1$), such that*

(a) *$\widetilde{U}(r', s') \subset \widetilde{U}(r, s)$, with either $r = r' + 1$ or $s = s' + 1$,*

(b) *$X_1^{'}$ is a good $A_{\mathfrak{q}_3}(\lambda_3)$ (or $r' + s' = 0$),*

(c) *$X_0^{'}$ is an odd oscillator representation with lowest K-type*

$$\mu_0^{'} = (3/2, 1/2, \dots 1/2) \text{ or } (-1/2, -1/2, \dots - 1/2. - 3/2),$$

and

(d) *X can be realized as the unique lowest K-type constituent of $\mathcal{R}_{\mathfrak{q}'}^{S'}(X_1^{'} \otimes X_0^{'})$.*

Proof Let λ_a be the Vogan classification parameter of X. Recall that, by Proposition 3.1, X_1 is a good $A_{\mathfrak{q}_2}(\lambda_2)$ module; then, by Lemma 4.2, X has a unique lowest $\widetilde{U}(n)$-type μ.

To prove Proposition 3.2, we first show that the representation X_0 of $Mp(2d)$ is pseudospherical, *i.e.*, that $X_0 \simeq J(\delta_{p,q}, \nu)$ with $pq = 0$. Assume, by way of contradiction, that $X_0 \simeq J(\delta_{p,q}, \nu)$ with $p > 0$ and $q > 0$, and set $\nu = (\nu^p | \nu^q)$ (as in Lemma 5.1). Write the lowest K-type μ of X as in equation (4.4) and the Vogan classification parameter λ_a as in (2.5).

Recall that the half integers a_i and b_i are defined even if $r_i = 0$ or $s_i = 0$, and that they satisfy (4.5). By an argument similar to the proof of Lemma 4.4, (at least) one of the following conditions must hold:

$$a_t > c_1 \quad \text{or} \quad c_d > b_t.$$

Notice that, since $p > 0$ and $q > 0$, the $\widetilde{U}(d)$-types

$$\delta_1 = \Big(\underbrace{\frac{1}{2}, \dots \frac{1}{2}}_{p-1}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q}, -\frac{3}{2}\Big) \quad \text{and} \quad \delta_2 = \Big(\frac{3}{2}, \underbrace{\frac{1}{2}, \dots \frac{1}{2}}_{p}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q-1}\Big)$$

(of Lemma 5.2) are obtained from the lowest $\widetilde{U}(d)$-type μ_0 of X_0 by adding or subtracting a short root. One can check that if $a_t > c_1$, then δ_2 survives in the bottom layer; if $c_d > b_t$, then δ_1 survives in the bottom layer.

We distinguish two cases. First, suppose that $\nu^p \neq \omega^p$ and $\nu^q \neq \omega^q$; then (by Lemma 5.2(iii)) X_0 is not unitary and both δ_1 and δ_2 detect non-unitarity. Because (at least) one of the two $\widetilde{U}(d)$-types survives in the bottom layer, we conclude that X is not unitary, and we reach a contradiction. Next, assume that $\nu^p = \omega^p$ or $\nu^q = \omega^q$. (Note that the two options can not hold at the same time, because X_0 is ω-regular.) If $\nu^p = \omega^p$, then the entries of ν^q are all greater than or equal to $p + \frac{1}{2}$; in particular, $\nu_{p+q} > \frac{1}{2}$. Hence, we are in the situation of Lemma 5.2(ii): X_0 is not unitary and

δ_2 detects non-unitarity. Similarly, if $\nu^q = \omega^q$, then X_0 is not unitary and δ_1 detects non-unitarity. In either case, the infinitesimal character of X_0 contains an entry $\frac{1}{2}$. Then, by Lemma 4.2, $g_t \geq 1$. Because $g_t \neq \frac{1}{2}$, Lemma 4.4(ii)(a) implies that $a_t > c_1$; a similar argument shows that $c_d > b_t$ (because if $c_d = b_t$ then $-g_t = -\frac{1}{2}$). We conclude that the coordinates of μ satisfy both conditions

$$a_t > c_1 \quad \text{and} \quad c_d > b_t$$

and that both $\widetilde{U}(d)$-types δ_1 and δ_2 survive in the bottom layer. Hence X is not unitary, contradicting our assumption.

Now we know that X_0 is pseudopsherical. Without loss of generality, we may assume that X_0 has lowest $\widetilde{U}(d)$-type $\mu_0 = (-\frac{1}{2}, \dots, -\frac{1}{2})$. If X_0 is the even antiholomorphic oscillator representation (*i.e.,* if X_0 has infinitesimal character ω^d), then we are in Proposition 3.2(i), and we are done. So suppose not. Recall that, by Lemma 2.7, X_0 is ω-regular. Therefore, if the infinitesimal character of X_0 is not equal to ω^d, then it must satisfy one of the conditions of Lemma 5.2(ii). We conclude that X_0 is not unitary, and the $\widetilde{U}(d)$-type δ_2 detects non-unitarity. Note that the $\widetilde{U}(d)$-type δ_2 may or may not survive in the bottom layer. If δ_2 survives in the bottom layer, then X is not unitary, and we reach a contradiction. Hence, we may assume that δ_2 does not survive in the bottom layer. In this case, the lowest K-type μ satisfies the condition $a_t - c_1 \leq 1$. By equation (4.8),

$$a_t - c_1 = g_t + \tfrac{1}{2}(1 - \varepsilon_t) + y_t - z_1.$$

Here $z_1 = -\frac{1}{2}$ and $y_t = 0$ because, by equation (4.2), the fine K-type δ^{L_a} in (4.1) is trivial except on $Mp(2d)$. Therefore, we obtain

$$a_t - c_1 = g_t + \frac{1}{2}(1 - \varepsilon_t) + \frac{1}{2} \geq \frac{1}{2} + \frac{1}{2} = 1$$

(because $g_t \geq \frac{1}{2}$). This forces $a_t - c_1 = 1$, so $g_t = \frac{1}{2}$, $\varepsilon_t = 1$, and $r_t - s_t = 1$.

Use the (new) singularization

$$\xi' = \Big(\underbrace{1, 1, \dots, 1}_{r - r_t} \Big| \underbrace{0, 0, \dots, 0}_{d + r_t + s_t} \Big| \underbrace{-1, -1, \dots, -1}_{s - s_t} \Big)$$

of λ_a to construct a (new) parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}'$, just as at the beginning of Section 3, and let

$$L' = L'_1 \times L'_0 \cong \widetilde{U}(r - r_t, s - s_t) \times Mp\big(2(d + r_t + s_t)\big)$$

be the subgroup corresponding to the Levi factor of $\mathfrak{q}'$. Then X may be realized as the unique lowest K-type constituent of a representation

$$\mathcal{R}_{\mathfrak{q}'}^{S'}(X^{L'}),$$

as in Proposition 2.4. Using notation analogous to the one in Lemma 2.7, write $X^{L'} = X'_1 \otimes X'_0$. Note that X'_0 is still ω-regular, and X'_1 is still strongly regular, as representations of L'_0 and L'_1, respectively. To finish the proof of our proposition, we need to show that, if X is unitary, then:

(A) $s_t = 0$ (hence $r_t = 1$);
(B) X_0' is an odd oscillator representation with lowest $\widetilde{U}(d+1)$-type

$$\mu_0' = \Big(\frac{3}{2}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{d}\Big);$$

and
(C) X_1' is a good $A_{\mathfrak{q}_3}(\lambda_3)$ module.

We begin with the proof of Claim (A). Assume, to the contrary, that $c := s_t > 0$, and let μ_0' be the lowest $\widetilde{U}(d+2c+1)$-type of X_0'. By Proposition 2.4(ii), the lowest $(L' \cap \widetilde{U}(n))$-type μ' of $X^{L'}$ satisfies $\mu' = \mu - 2\rho(\mathfrak{u}' \cap \mathfrak{p})$. Therefore, μ_0' must be of the form

$$\mu_0' = \Big(\underbrace{a_t - r' + s', \dots, a_t - r' + s'}_{c+1}, \\ \underbrace{c_1 - r' + s', \dots, c_d - r' + s'}_{d}, \underbrace{b_t - r' + s', \dots, b_t - r' + s'}_{c}\Big)$$

with $r' = r - r_t$ and $s' = s - s_t$. Using $g_t = \frac{1}{2}$ and $\varepsilon_t = 1$, we obtain

$$\mu_0' = \Big(\underbrace{\frac{3}{2}, \dots, \frac{3}{2}}_{c+1}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{d}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{c}\Big).$$

We will show, using Parthasarathy's Dirac Operator Inequality, that X_0' is not unitary, and that it contains a $\widetilde{U}(d+2c+1)$-type η_0' on which the signature of the Hermitian form changes. Because η_0' survives under the bottom layer map, this will imply that X is not unitary, reaching a contradiction and thus completing the proof of Claim (A).

We recall some results from [6], and from [9, Lemmas 6.1 and 6.3]. Let $\Delta^+(\mathfrak{l}_0')$ be a choice of positive roots for $Mp(2(d+2c+1))$ that is compatible with our fixed system of positive compact roots, and let ρ_n be the corresponding half sum of noncompact positive roots. Choose a Weyl group element w such that $w^{-1}(\mu_0' - \rho_n)$ is K-dominant, and denote by $\gamma^{X_0'}$ the infinitesimal character of X_0'. If

$$\langle \mu_0' - \rho_n + w\rho_c, \mu_0' - \rho_n + w\rho_c \rangle < \langle \gamma^{X_0'}, \gamma^{X_0'} \rangle, \tag{7.1}$$

then X_0' is not unitary. In this case, there is a non-compact root $\beta \in \Delta(\mathfrak{l}_0')$ such that $\eta_0' = \mu_0' - \beta$ occurs in X_0' and detects non-unitarity (in the sense that the signature of the Hermitian form on X_0', restricted to $\mu_0' \oplus \eta_0'$, is indefinite). If the conditions of [9, Lemma 6.3(b)] are satisfied, then the root β may be chosen from $\Delta^+(\mathfrak{l}_0')$.

We will prove that, in our setting, we can always choose $\Delta^+(\mathfrak{l}_0')$ so that equation (7.1) holds and the $\widetilde{U}(d+2c+1)$-type $\eta_0' = \mu_0' - \beta$ survives under the bottom layer map. The choice of $\Delta^+(\mathfrak{l}_0')$ will depend on the Vogan classification parameter λ_a of X; in particular, we will distinguish two cases, according to the possible values of the entry g_{t-1}.

Recall that g_{t-1} is a half integer (strictly) greater than $g_t = \frac{1}{2}$. First, assume that

$$g_{t-1} \in \mathbb{Z} \quad \text{or} \quad g_{t-1} \in \mathbb{Z} + \frac{1}{2} \quad \text{and} \quad g_{t-1} \geq \frac{7}{2}.$$

In this case, we let $\Delta^+(\mathfrak{l}_0')^{(1)}$ be such that

$$\rho(\mathfrak{l}_0')^{(1)} = (2c + d + 1, 2c + d, \ldots, c + 1, -1, -2, \ldots, -c)$$

and

$$\rho_n^{(1)} = \Bigl(\underbrace{\frac{d}{2} + c + 1, \ldots, \frac{d}{2} + c + 1}_{c+d+1}, \underbrace{\frac{d}{2}, \ldots, \frac{d}{2}}_{c}\Bigr).$$

Then

$$\mu_0' - \rho_n^{(1)} = \Bigl(\underbrace{-\frac{d}{2} - c + \frac{1}{2}, \ldots, -\frac{d}{2} - c + \frac{1}{2}}_{c+1}, \underbrace{-\frac{d}{2} - c - \frac{1}{2}, \ldots, -\frac{d}{2} - c - \frac{1}{2}}_{d}, \underbrace{-\frac{d}{2} - \frac{1}{2}, \ldots, -\frac{d}{2} - \frac{1}{2}}_{c}\Bigr).$$

The element $w^{(1)}\rho_c$ can be chosen to be

$$\Bigl(\underbrace{\frac{d}{2}, \frac{d}{2} - 1, \ldots, \frac{d}{2} - c}_{c+1}, \underbrace{\frac{d}{2} - c - 1, \frac{d}{2} - c - 2, \ldots, -c - \frac{d}{2}}_{d}, \underbrace{c + \frac{d}{2}, c + \frac{d}{2} - 1, \ldots, \frac{d}{2} + 1}_{c}\Bigr).$$

Hence, we obtain

$$\mu_0' - \rho_n^{(1)} + w^{(1)}\rho_c = \Bigl(\underbrace{-c + \frac{1}{2}, -c - \frac{1}{2}, \ldots, -2c + \frac{1}{2}}_{c+1}, \underbrace{-2c - \frac{3}{2}, -2c - \frac{5}{2}, \ldots, -2c - d - \frac{1}{2}}_{d}, \underbrace{c - \frac{1}{2}, c - \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}}_{c}\Bigr). \tag{7.2}$$

Because X_0' is ω-regular, its infinitesimal character $\gamma^{X_0'}$ must satisfy the condition

$$\langle \gamma^{X_0'}, \gamma^{X_0'} \rangle \geq \langle \omega^{d+2c+1}, \omega^{d+2c+1} \rangle. \tag{7.3}$$

Writing ω^{d+2c+1} in coordinates, we get

$$\omega^{d+2c+1} = \Big(d + 2c + \frac{1}{2}, d + 2c - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \Big). \tag{7.4}$$

Rearranging the entries of $\mu_0' - \rho_n^{(1)} + w^{(1)}\rho_c$ in equation (7.2) and comparing them with the entries of ω^{d+2c+1} in equation (7.4), we can see that

$$\begin{aligned} \langle \omega^{d+2c+1}, \omega^{d+2c+1} \rangle - \langle \mu_0' - \rho_n^{(1)} + w^{(1)}\rho_c, \mu_0' - \rho_n^{(1)} + w^{(1)}\rho_c \rangle &= (2c + \frac{1}{2})^2 - (c - \frac{1}{2})^2 \\ &= 3c^2 + 3c. \end{aligned}$$

This quantity is strictly positive if $c > 0$. Hence, by (7.3), we find

$$\langle \gamma^{X_0'}, \gamma^{X_0'} \rangle - \langle \mu_0' - \rho_n^{(1)} + w^{(1)}\rho_c, \mu_0' - \rho_n^{(1)} + w^{(1)}\rho_c \rangle > 0. \tag{7.5}$$

We conclude that X_0' is not unitary and there is a non-compact root $\beta \in \Delta(\mathfrak{l}_0')$ such that the $\widetilde{U}(d+2c+1)$-type $\eta_0' = \mu_0' - \beta$ (is dominant and) detects non-unitarity. Set

$$x = \max\{ i < t \mid r_i > 0 \}, \quad y = \max\{ i < t \mid s_i > 0 \}.$$

By Lemma 4.5, the coordinates of μ (in equation (4.4)) satisfy the conditions

$$a_x - a_t \geq 2 \qquad \text{and} \qquad b_t - b_y \geq 2. \tag{7.6}$$

Then the $\widetilde{U}(n)$-type $\mu - \beta$ is dominant for Δ_c^+, and the $\widetilde{U}(d + 2c + 1)$-type $\eta_0' = \mu_0' - \beta$ survives in the bottom layer. This implies that X is not unitary, and gives a contradiction. (If x or y do not exist, then the corresponding condition in (7.6) is empty.)

Next, we consider the case in which g_{t-1} is either $\frac{3}{2}$ or $\frac{5}{2}$, and we choose $\Delta^+(\mathfrak{l}_0')^{(2)}$ such that

$$\rho(\mathfrak{l}_0')^{(2)} = (2c + d + 1, 2c + d, \dots, 2, 1)$$

and

$$\rho_n^{(2)} = \Big(\underbrace{\frac{d}{2} + c + 1, \dots, \frac{d}{2} + c + 1}_{2c+d+1} \Big).$$

Note that this choice of positive roots for $Mp(2(d + 2c + 1))$ satisfies the conditions of [9, Lemma 6.3(b)]. Because $\mu_0' - \rho_n^{(2)}$ is dominant for our fixed set of positive

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compact roots, we can choose $w^{(2)} = 1$. Then we obtain

$$\mu_0' - \rho_n^{(2)} + \rho_c = \Big(\underbrace{\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \dots, -c + \frac{1}{2}}_{c+1}, \underbrace{-c - \frac{3}{2}, -c - \frac{5}{2}, \dots, -c - d - \frac{1}{2}}_{d}, \underbrace{-c - d - \frac{5}{2}, -c - d - \frac{7}{2}, \dots, -2c - d - \frac{3}{2}}_{c}\Big).$$

Let

$$\widetilde{\gamma} := \Big(d + 2c + \frac{3}{2}, d + 2c + \frac{1}{2}, \dots, \frac{9}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2}\Big).$$

The quantity

$$\langle \widetilde{\gamma}, \widetilde{\gamma} \rangle - \langle \mu_0' - \rho_n^{(2)} + \rho_c, \mu_0' - \rho_n^{(2)} + \rho_c \rangle = \Big(c + \frac{1}{2}\Big)^2 + \Big(c + d + \frac{3}{2}\Big)^2 - \Big(\frac{1}{2}\Big)^2 - \Big(\frac{5}{2}\Big)^2$$

is strictly positive if $c > 0$. The assumptions that $g_{t-1} = \frac{3}{2}$ (or $\frac{5}{2}$), along with Lemma 4.2, imply that the infinitesimal character $\gamma^{X_0}{}'$ of X_0' does not contain an entry $\frac{3}{2}$ (or $\frac{5}{2}$). Then, by ω-regularity, we obtain that $\langle \gamma^{X_0'}, \gamma^{X_0'} \rangle \geq \langle \widetilde{\gamma}, \widetilde{\gamma} \rangle$ and

$$\langle \gamma^{X_0'}, \gamma^{X_0'} \rangle - \langle \mu_0' - \rho_n^{(2)} + \rho_c, \mu_0' - \rho_n^{(2)} + \rho_c \rangle > 0.$$

We conclude that X_0' is not unitary and there is a non-compact root $\beta \in \Delta^+(\mathfrak{l}_0')^{(2)}$ such that $\eta_0' = \mu_0' - \beta$ (is dominant and) detects non-unitarity. Note that β must be of the form $\epsilon_i + \epsilon_j$ or $2\epsilon_j$ for some $1 \leq i < j \leq 2c + d + 1$. Because

$$b_t - b_y \geq 2$$

(by Lemma 4.5), the $\widetilde{U}(d + 2c + 1)$-type $\eta_0' = \mu_0' - \beta$ survives in the bottom layer, leading to a contradiction.

This concludes the proof of Claim (A).

Now we know that $c = s_t = 0$, and that X_0' is an irreducible genuine representation of $Mp(2(d + 1))$ with lowest $\widetilde{U}(d + 1)$-type

$$\mu_0' = \Big(\frac{3}{2}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{d}\Big).$$

Let us prove Claim (B). We apply the same argument used for the proof of Claim (A); note that, because $c = 0$ in this case, our two choices of positive roots $\Delta^+(\mathfrak{l}_0')^{(i)}$ coincide (hence the the conditions of [9, Lemma 6.3(b)] are met). If the inequality (7.3) is strict, *i.e.,* if

$$\langle \gamma^{X_0'}, \gamma^{X_0'} \rangle > \langle \omega^{d+1}, \omega^{d+1} \rangle,$$

then equation (7.5) still holds:

$$\langle \gamma^{X_0'}, \gamma^{X_0'} \rangle - \langle \mu_0' - \rho_n + w\rho_c, \mu_0' - \rho_n + w\rho_c \rangle > 0.$$

Again, we conclude that X_0' is not unitary, and there is a non-compact root $\beta \in \Delta^+(\mathfrak{l}_0')$ such that $\eta_0' = \mu_0' - \beta$ (is dominant and) detects non-unitarity. The choices for β are $\epsilon_1 + \epsilon_{d+1}$, $\epsilon_d + \epsilon_{d+1}$, and $2\epsilon_{d+1}$. In each case, $\eta_0' = \mu_0' - \beta$ survives in the bottom layer because

$$c_d - b_{t-1} = \frac{1}{2} + g_{t-1} + \frac{\varepsilon_{t-1} + 1}{2} \geq 2. \tag{7.7}$$

This follows from the fact that $g_{t-1} \geq 1$ and from a calculation similar to the one in (4.12). (Note that if $g_{t-1} = 1$, then $\varepsilon_{t-1} = 0$ by Lemma 4.2.) Next, assume that the infinitesimal character $\gamma^{X_0'}$ satisfies

$$\langle \gamma^{X_0'}, \gamma^{X_0'} \rangle = \langle \omega^{d+1}, \omega^{d+1} \rangle,$$

so that

$$\gamma^{X_0'} = \omega^{d+1}. \tag{7.8}$$

We want to prove that X_0' is an odd oscillator representation. It is sufficient to show that X_0' is uniquely determined by its infinitesimal character. Because the Vogan classification parameter of X_0' is given by

$$\lambda_a' = \left(\frac{1}{2}, 0, \dots, 0\right),$$

we can rewrite the infinitesimal character of X_0' in the form

$$\gamma^{X_0'} = \left(\frac{1}{2} | \nu\right),$$

where ν is the continuous parameter on the pseudospherical principal series of $Mp(2d)$ (*cf.* Lemma 5.1). Then equation (7.8) gives

$$\nu = \left(d + \frac{1}{2}, d - \frac{1}{2}, \dots, \frac{3}{2}\right). \tag{7.9}$$

This implies that X_0' is indeed the appropriate odd oscillator representation, proving Claim (B).

Finally, we turn to the proof of Claim (C), and show that X_1' is a good $A_{\mathfrak{q}_3}(\lambda_3)$ module. As usual, write the coordinates of μ as in equation (4.4); this time, $r_t = 1$, $s_t = 0$, $a_t - c_1 = 1$, and $c_d - b_{t-1} \geq 2$. Note that it suffices to show that $a_{t-1} - a_t \geq 1$. If this is the case, then our Claim (C) follows by the same kind of argument used in the first part of the proof of Proposition 3.1: if X_1' is not a good $A_{\mathfrak{q}_3}(\lambda_3)$ module,

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then it is not unitary, and we can use Parthasarathy's Dirac Operator Inequality and a bottom layer argument to show that X is also non-unitary, reaching a contradiction.

We now show that $a_{t-1} - a_t \geq 1$. By contradiction, suppose that $a_{t-1} = a_t$. Then, by Lemma 4.2 and some calculations similar to those in the proof of Lemma 4.4, we must have $g_{t-1} = 1$ and $r_{t-1} = s_{t-1} > 0$. Roughly, a jump of $\frac{3}{2}$ or more in the entries of λ_a results in a corresponding jump of 1 or more in the coordinates of μ. Note that, by Lemma 4.2(iii), we cannot have $g_{t-1} = \frac{3}{2}$, because the infinitesimal character of X_0, given by (7.9), contains an entry $\frac{3}{2}$.

As in the proof of Lemma 4.5, consider the lowest $(\widetilde{U}(r) \times \widetilde{U}(s))$-type μ_1 of the $A_{\mathfrak{q}_2}(\lambda_2)$ module X_1. Here L_2 is given in (4.13), and μ_1 has the form

$$\mu_1 = \big(\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})\big)\,|_{\widetilde{U}(r)\times\widetilde{U}(s)}$$

with

$$(7.10)\quad 2\rho(\mathfrak{u} \cap \mathfrak{p}) = \Big(\underbrace{r+d+1, \dots, r+d+1}_{r} \;\Big|\; \underbrace{r-s, \dots, r-s}_{d} \;\Big|\; \underbrace{-s-d-1, \dots, -s-d-1}_{s}\Big).$$

The shape of μ_1 forces the last factor $\widetilde{U}(p_u, q_u)$ of L_2 to satisfy $p_u \geq 2$. Because $L_a \subseteq L_2 \times L_0$ (hence $\widetilde{U}(r_{t-1}, s_{t-1}) \times \widetilde{U}(r_t, s_t) \subseteq L_2$), this implies that $q_u \geq 1$.

Now look at the infinitesimal character of X_1:

$$\gamma^{X_1} = \lambda_2 + \rho(\mathfrak{l}_2) + \rho(\mathfrak{u}_2).$$

Recall that $\lambda_2 = \mu_1 - 2\rho(\mathfrak{u}_2 \cap \mathfrak{p})$. Write the highest weight of μ_1 as in equation (4.14). Because $a_t = r - s + \frac{1}{2}$ and $b_{t-1} = r - s - \frac{5}{2}$ (by equation (7.7) with $g_{t-1} = 1$ and $\varepsilon_{t-1} = 0$), we find

$$\mu_1 = \Big(\dots, \underbrace{-s-d-\frac{1}{2}, \dots, -s-d-\frac{1}{2}}_{p_u} \;\Big|\; \underbrace{0, \dots, 0}_{d} \;\Big|\; \underbrace{r+d-\frac{3}{2}, \dots, r+d-\frac{3}{2}}_{q_u}, \dots\Big)$$

and

$$(7.11)\quad 2\rho(\mathfrak{u}_2 \cap \mathfrak{p}) = \Big(\dots, \underbrace{-s+q_u, \dots, -s+q_u}_{p_u} \;\Big|\; \underbrace{0, \dots, 0}_{d} \;\Big|\; \underbrace{r-p_u, \dots, r-p_u}_{q_u}, \dots\Big).$$

Hence

$$\lambda_2 = \mu_1 - 2\rho(\mathfrak{u}_2 \cap \mathfrak{p}) =$$
$$\Big(\dots, \underbrace{-d-q_u-\frac{1}{2},\dots,-d-q_u-\frac{1}{2}}_{p_u} \;\Big|\; \underbrace{0,\dots,0}_{d} \;\Big|\; \underbrace{d+p_u-\frac{3}{2},\dots,d+p_u-\frac{3}{2}}_{q_u},\dots\Big).$$

Because λ_2 must be constant on $\widetilde{U}(p_u,q_u)$ (with twist because of the embedding into $\mathfrak{g}$), we can conclude that $p_u = q_u + 2$. Recall that here $q_u > 0$. Write

$$\rho(\mathfrak{u}_2) = \Big(\dots, \underbrace{\frac{-r-s+p_u+q_u}{2},\dots,\frac{-r-s+p_u+q_u}{2}}_{p_u} \;\Big|\; \underbrace{0,\dots,0}_{d} \;\Big|$$
$$\underbrace{\frac{r+s-p_u-q_u}{2},\dots,\frac{r+s-p_u-q_u}{2}}_{q_u},\dots\Big),$$

and choose

$$\rho(\mathfrak{l}_2) = \Big(\dots, \underbrace{\frac{p_u+q_u-1}{2},\frac{p_u+q_u-3}{2},\dots,\frac{q_u-p_u+1}{2}}_{p_u} \;\Big|\; \underbrace{0,\dots,0}_{d} \;\Big| \tag{7.12}$$
$$\underbrace{\frac{p_u+q_u-1}{2},\frac{p_u+q_u-3}{2},\dots,\frac{p_u-q_u+1}{2}}_{q_u},\dots\Big).$$

Then

$$\gamma^{X_1} = \lambda_2 + \rho(\mathfrak{u}_2) + \rho(\mathfrak{l}_2) =$$
$$\Big(\dots, \underbrace{-d-\frac{r+s}{2}+p_u-1,-d-\frac{r+s}{2}+p_u-2\dots,-d-\frac{r+s}{2}}_{p_u} \;\Big|\; \underbrace{0,\dots,0}_{d} \;\Big|$$
$$\underbrace{d+\frac{r+s}{2}+p_u-2,d+\frac{r+s}{2}+p_u-3,\dots,d+\frac{r+s}{2}+p_u-q_u-1}_{q_u},\dots\Big).$$

Finally, consider the infinitesimal character of X. By Proposition 2.4(vi), we have

$$\gamma^X = \gamma^{X^L} + \rho(\mathfrak{u}) = \gamma^{X_1} + \gamma^{X_0} + \rho(\mathfrak{u}).$$

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Here the infinitesimal characters of X_1 and X_0 are known, and $\rho(u)$ is given by

$$\rho(\mathfrak{u}) = \Big(\underbrace{d + \frac{r+s+1}{2}, \dots, d + \frac{r+s+1}{2}}_{r} \Big| \underbrace{0, \dots, 0}_{d} \Big| \underbrace{-d - \frac{r+s+1}{2}, \dots, -d - \frac{r+s+1}{2}}_{s}\Big).$$

We obtain

$$\gamma^X = \gamma^{X_1} + \gamma^{X_0} + \rho(\mathfrak{u}) = \Big(\dots, \underbrace{p_u - \frac{1}{2}, p_u - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}}_{p_u} \Big| d + \frac{1}{2}, d - \frac{1}{2}, \dots, \frac{5}{2}, \frac{3}{2} \Big| \underbrace{p_u - \frac{5}{2}, p_u - \frac{7}{2}, \dots, p_u - q_u - \frac{3}{2}}_{q_u}, \dots\Big).$$

This contradicts our assumption that γ^X is nonsingular and concludes the proof of Proposition 3.2. ■

8 Proof of Proposition 3.6

Finally, we give the proof of Proposition 3.6. For convenience, we restate the result.

Proposition 3.6 *In the setting of Proposition* 3.4*, let X be the irreducible (lowest K-type constituent of the) representation $\mathcal{R}_{\mathfrak{q}_\omega}(\mathbb{C}_{\lambda'} \otimes \omega)$. Assume, moreover, that λ' is in the good range for $\mathfrak{q}'$. Then $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$ is in the good range for $\mathfrak{q}_\omega$.*

Proof Recall from equation (3.1) that $\mathfrak{q}_\omega$ has Levi component $\mathfrak{l}_\omega = \mathfrak{l}' + \mathfrak{l}_0$ and nilpotent part $\mathfrak{u}_\omega = \mathfrak{u}' + \mathfrak{u}$. Let γ^Ω be (a representative of) the infinitesimal character of $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$. We need to show that Ω is in the good range for $\mathfrak{q}_\omega$, *i.e.,* that

$$\langle \gamma^\Omega + \rho(\mathfrak{u}_\omega), \alpha \rangle > 0$$

for all α in $\Delta(\mathfrak{u}_\omega)$. Note that $\rho(\mathfrak{u}_\omega) = \rho(\mathfrak{u}') + \rho(\mathfrak{u})$. Let $d_0(= d \text{ or } d+1)$ be the rank of L_0, so that the infinitesimal character of the oscillator representation of L_0 is ω^{d_0}. Then $\gamma^\Omega = \lambda' + \rho(\mathfrak{l}') + \omega^{d_0}$, and

$$\gamma^\Omega + \rho(\mathfrak{u}_\omega) = \lambda' + \rho(\mathfrak{l}') + \omega^{d_0} + \rho(\mathfrak{u}') + \rho(\mathfrak{u}).$$

If $\alpha \in \Delta(\mathfrak{u}')$, then α is orthogonal to ω^{d_0} and $\rho(\mathfrak{u})$, and

$$\big\langle \gamma^\Omega + \rho(\mathfrak{u}_\omega), \alpha \big\rangle = \big\langle \lambda' + \rho(\mathfrak{l}') + \rho(\mathfrak{u}'), \alpha \big\rangle > 0$$

because λ' is assumed to be in the good range for $\mathfrak{q}'$.

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It remains to consider the case $\alpha \in \Delta(\mathfrak{u})$. In order to keep the notation simple and familiar, we will compute $\gamma^\Omega + \rho(\mathfrak{u}_\omega)$ in our usual coordinates, for the case that X_0 is an even oscillator representation. If X_0 is an odd oscillator representation, the result will follow in exactly the same way. Let λ_a, μ and $L = L_1 \times L_0$ be as in the proof of Proposition 3.1, and write $L' = \prod_{i=1}^{u} \widetilde{U}(p_i, q_i)$ as in (4.13) (it was called L_2 there). In these coordinates, the roots of $\mathfrak{u}$ are those that are positive on

$$\xi = \Big(\underbrace{a, a, \dots, a}_{r} \,\Big|\, \underbrace{0, 0, \dots, 0}_{d} \,\Big|\, \underbrace{-a, -a, \dots, -a}_{s}\Big)$$

for $a > 0$. The roots of $\mathfrak{u}_\omega$ are those that are positive on

$$\xi_\omega = \Big(\underbrace{a_1, \dots, a_1}_{p_1}, \dots, \underbrace{a_u, \dots, a_u}_{p_u} \,\Big|\, \underbrace{0, \dots, 0}_{d} \,\Big|\, \underbrace{-a_u, \dots, -a_u}_{q_u}, \dots, \underbrace{-a_1, \dots, -a_1}_{q_u}\Big),$$

for $a_1 > a_2 > \cdots > a_u > 0$. From [9], we know that the parameter λ' is of the form

$$\lambda' = \Big(\underbrace{\lambda_1, \dots, \lambda_1}_{p_1}, \dots, \underbrace{\lambda_u, \dots, \lambda_u}_{p_u} \,\Big|\, \underbrace{0, \dots, 0}_{d} \,\Big|\, \underbrace{-\lambda_u, \dots, -\lambda_u}_{q_u}, \dots, \underbrace{-\lambda_1, \dots, -\lambda_1}_{q_u}\Big)$$

with $\lambda_i \in \mathbb{Z} + \frac{1}{2}$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_u$. Note that if we choose a system of positive roots Δ^+_ω for $\mathfrak{l}_\omega = \mathfrak{l}' + \mathfrak{l}_0$, then $\Delta^+ = \Delta^+_\omega \cup \Delta(\mathfrak{u}') \cup \Delta(\mathfrak{u})$ is a system of positive roots for $\mathfrak{g}$. Let $\Pi \subset \Delta^+$ be the corresponding set of simple roots. If ω^d is the representative of the infinitesimal character of the oscillator representation determined by Δ^+_ω, then

$$\gamma^\Omega + \rho(\mathfrak{u}_\omega) = \lambda' + \rho(\mathfrak{l}') + \omega^{d_0} + \rho(\mathfrak{u}') + \rho(\mathfrak{u})$$

is automatically dominant for all roots in $\Delta^+_\omega \cup \Delta(\mathfrak{u}')$. Hence, we only need to show dominance for the simple roots $\Pi \cap \Delta(\mathfrak{u})$ in $\Delta(\mathfrak{u})$; this set will turn out to be a single root.

Without loss of generality, we may assume that $q_u > 0$. Set

$$\omega^d = (0, \dots, 0 | d - 1/2, d - 3/2, \dots, 1/2 | 0, \dots, 0)$$

and choose the restriction of $\rho(\mathfrak{l}')$ on each factor in a standard way; the choice is given explicitly (for the last factor $\widetilde{U}(p_u, q_u)$ of L' only) in (7.12). Then

$$\Pi \cap \Delta(\mathfrak{u}) = \{-\epsilon_{r+1} - \epsilon_{r+d+1}\}.$$

If we write

$$\gamma^\Omega + \rho(\mathfrak{u}_\omega) = (\gamma_1, \gamma_2, \dots, \gamma_r | d - 1/2, d - 3/2, \dots, 1/2 | \gamma_{r+d+1}, \dots, \gamma_{r+d+s}),$$

then the dominance condition reduces to

$$\gamma_{r+d+1} + d - 1/2 < 0.$$

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Note that, given the ω-regularity condition and the shape of ω^d, it suffices to prove that

$$\gamma_{r+d+1} \leq d - \tfrac{1}{2}. \tag{8.1}$$

We consider the entries of $\gamma^{\Omega} + \rho(\mathfrak{u}_{\omega})$ corresponding to the last factor $\widetilde{U}(p_u, q_u)$ of L'. These entries were essentially computed in the proof of (Claim (C) in) Proposition 3.2, with the same choice of positive roots, but with different assumptions on the entries of μ and a different notation for the parabolic subalgebra of L_1 (prime algebras were replaced by algebras with subscript 2). By equation (4.2) (in Lemma 4.2), the r-th coordinate of μ is the same as the r-th coordinate of $\lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k})$, which, in turn, is of the form

$$r - s + k = r - s + \Big[g_j + \sum_{i=j+1}^{t} (s_i - r_i) + \frac{1}{2}(s_j - r_j + 1) \Big]$$

for some $1 \leq j \leq t$ (by equations (2.5) and (4.3)). Note that $k \in \mathbb{Z} + \frac{1}{2}$ and $k \geq \frac{1}{2}$, because $g_j \geq \frac{1}{2}$, $r_i = 0$ (hence $s_i = 1$) for all $i > j$, and $|s_j - r_j| \leq 1$. Similarly, the $(r + d + 1)$-th coordinate of μ can be written in the form

$$r - s - l,$$

for some $l \in \mathbb{Z} + \frac{1}{2}$ and $l \geq \frac{1}{2}$. Hence, we can write

$$\mu = \Big(\dots, \underbrace{r - s + k, \dots, r - s + k}_{p_u} \Big| \underbrace{c_1, \dots, c_d}_{d} \Big| \underbrace{r - s - l, \dots, r - s - l}_{q_u}, \dots \Big)$$

with $k, l \in \mathbb{Z} + \frac{1}{2}$ with $k, l \geq \frac{1}{2}$. (This formula also holds true for $d = 0$.)

Then, using the equation $\mu = \mu_1 + \mu_0 + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ and the expression for $2\rho(\mathfrak{u} \cap \mathfrak{p})$ given by (7.10), we find that

$$\mu_1 = \Big(\dots, \underbrace{-s - d - 1 + k, \dots, -s - d - 1 + k}_{p_u} \Big| \underbrace{0, \dots, 0}_{d} \Big| \underbrace{r + d + 1 - l, \dots, r + d + 1 - l}_{q_u}, \dots \Big).$$

Using (7.11), we get

$$\begin{aligned} \lambda' &= \mu_1 - 2\rho(\mathfrak{u}' \cap \mathfrak{p}) \\ &= \Big(\dots, \underbrace{-q_u - d - 1 + k, \dots, -q_u - d - 1 + k}_{p_u} \Big| \underbrace{0, \dots, 0}_{d} \Big| \underbrace{p_u + d + 1 - l, \dots, p_u + d + 1 - l}_{q_u}, \dots \Big), \end{aligned}$$

hence

$$\gamma^{\Omega} + \rho(\mathfrak{u}_\omega) = \lambda' + \rho(\mathfrak{u}') + \rho(\mathfrak{l}') + \rho(\mathfrak{u})$$
$$= \Big(\dots, \underbrace{p_u - 1 + k, p_u - 2 + k, \dots, k + 1, k}_{p_u} \Big| \; d - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \Big|$$
$$\underbrace{p_u - l, p_u - l - 1, \dots, p_u - q_u - l + 1}_{q_u}, \dots \Big).$$

If $p_u = 0$, then $\gamma_{r+d+1} = -l < 0$, which implies (8.1), so we are done. Therefore, we may assume that p_u and q_u are both nonzero. Because λ' is constant on $\widetilde{U}(p_u, q_u)$, we must have $p_u + d + 1 - l = q_u + d + 1 - k$, so that $l = k + p_u - q_u$. Then

$$\gamma^{\Omega} + \rho(\mathfrak{u}_\omega) = \Big(\dots, \underbrace{p_u - 1 + k, p_u - 2 + k, \dots, k + 1, k}_{p_u} \Big| \; d - \frac{1}{2}, \dots$$
$$\dots, \frac{1}{2} \; \Big| \; \underbrace{q_u - k, q_u - k - 1, \dots, -k + 1}_{q_u}, \dots \Big).$$

By ω-regularity and the fact that $k > 0$, we must have $k \geq d + \frac{1}{2}$, so the last entry in this $\widetilde{U}(p_u, q_u)$ factor is

$$-k + 1 \leq -d + \frac{1}{2}.$$

Because the entries $q_u - k, q_u - k - 1, \dots, -k + 1$ form a sequence of half integers decreasing by steps of 1, ω-regularity implies that

$$\gamma_{r+s+1} = q_u - k \leq -d - \frac{1}{2},$$

and we are done.

Notice that this argument also applies in the case $d = 0$. This concludes the proof of Proposition 3.6. ∎

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References

[1] J. Adams, D. Barbasch, A. Paul, P. Trapa, and D. Vogan, *Unitary Shimura correspondences for split real groups.* J. Amer. Math. Soc. **20**(2007), no. 3, 701–751. http://dx.doi.org/10.1090/S0894-0347-06-00530-3

[2] J.-S. Huang, *The unitary dual of the universal covering group of* $GL(n, \mathbb{R})$. Duke Math. J. **61**(1990), no. 3, 705–745. http://dx.doi.org/10.1215/S0012-7094-90-06126-5

[3] A. Knapp and D. Vogan, *Cohomological induction and unitary representations.* Princeton Mathematical Series, 45, Princeton University Press, Princeton, NJ, 1995.

[4] A. Pantano, A. Paul, and S. Salamanca-Riba, *The omega-regular unitary dual of the metaplectic group of rank 2.* In: Council for African American researchers in the mathematical sciences, V, Contemp. Math., 467, American Mathematical Society, Providence, RI, 2008, pp. 1–47.

[5] ———, *Unitary genuine principal series of the metaplectic group.* Represent. Theory **14**(2010), 201–248. http://dx.doi.org/10.1090/S1088-4165-10-00367-5

[6] R. Parthasaraty, *Criteria for the uniterizability of some highest weight modules.* Proc. Indian Acad. Sci. Sect. A Math. Sci. **89**(1980), no. 1, 1–24. http://dx.doi.org/10.1007/BF02881021

[7] A. Paul, *Howe correspondence for real unitary groups.* J. Funct. Anal. **159**(1998), no. 2, 384–431. http://dx.doi.org/10.1006/jfan.1998.3330

[8] ———, *On the Howe correspondence for symplectic-orthogonal dual pairs.* J. Funct. Anal. **228**(2005), no. 2, 270–310. http://dx.doi.org/10.1016/j.jfa.2005.03.015

[9] S. Salamanca-Riba, *On the unitary dual of some classical Lie groups.* Compositio Math. **68**(1988), no. 3, 251–303.

[10] ———, *On the unitary dual of real reductive Lie groups and the $A_{\mathfrak{q}}(\lambda)$-modules: the strongly regular case.* Duke Math. J. **96**(1999), no. 3, 521–546. http://dx.doi.org/10.1215/S0012-7094-99-09616-3

[11] S. Salamanca-Riba and D. A. Vogan Jr., *On the classification of unitary representations of reductive Lie groups.* Ann. of Math. **148**(1998), no. 3, 1067–1133. http://dx.doi.org/10.2307/121036

[12] D. A. Vogan Jr., *Representations of real reductive lie groups.* Progress in Mathematics, 15, Birkhäuser, Boston, MA, 1981.

[13] ———, *Unitarizability of certain series of representations.* Ann. of Math. **120**(1984), no. 1, 141–187. http://dx.doi.org/10.2307/2007074

[14] ———, *The unitary dual of G_2.* Invent. Math. **116**(1994), no. 1–3, 677–791. http://dx.doi.org/10.1007/BF01231578

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