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Abstract

The well-known fiber dimension theorem in algebraic geometry says that for every morphism $f\colon X \to Y$ of integral schemes of finite type the dimension of every fiber of f is at least $\dim X - \dim Y$. This has recently been generalized by Brosnan, Reichstein and Vistoli to certain morphisms of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$, where the usual dimension is replaced by essential dimension. We will prove a general version for morphisms of categories fibered in groupoids. Moreover, we will prove a variant of this theorem, where essential dimension and canonical dimension are linked. These results let us relate essential dimension to canonical dimension of algebraic groups. In particular, using the recent computation of the essential dimension of algebraic tori by MacDonald, Meyer, Reichstein and the author, we establish a lower bound on the canonical dimension of algebraic tori.

1. Introduction

A category fibered in groupoids (abbreviated CFG) over a field F is roughly a category $\mathcal{X}$ equipped with a functor $\pi\colon \mathcal{X} \to \mathrm{Sch}_F$ to the category Sch_F of schemes over F for which pullbacks exist and are unique up to canonical isomorphism. See § 2 for a formal definition.

A typical example of a CFG over F is the quotient $[X/G]$ of a scheme X by the action of an algebraic group G; see Example 2.1. CFGs of the form $[X/G]$ often arise in moduli problems. Unlike many quotients in geometric invariant theory, they keep a lot of information about the G-equivariant geometry of X.

To every CFG $\mathcal{X}$ over F we can attach two numbers, $\mathrm{ed}\,\mathcal{X}$ and $\mathrm{cdim}\,\mathcal{X}$ (with $\mathrm{cdim}\,\mathcal{X} \leqslant \mathrm{ed}\,\mathcal{X}$), called the essential dimension, respectively the canonical dimension, of $\mathcal{X}$; see § 2. In the case where $\mathcal{X}$ is representable by a scheme X locally of finite type, the essential dimension of $\mathcal{X}$ coincides with the usual dimension of X, and $\mathrm{cdim}\,\mathcal{X}$ is a number between 0 and $\dim X$, which measures how far X is from having a rational point. Explicitly, if X is regular and complete, $\mathrm{cdim}\,\mathcal{X}$ is the least dimension of a subvariety Y of X admitting a dominant rational map $X \dashrightarrow Y$ over F (see [KM06, Corollary 4.6]).

There are versions of essential and canonical dimension relative to a prime p, written $\mathrm{ed}_p\,\mathcal{X}$ and $\mathrm{cdim}_p\,\mathcal{X}$, which basically neglect effects from passing to prime to p field extensions. We will include the case $p = 0$ for usual dimensions and write $\mathrm{ed}_0\,\mathcal{X}$ and $\mathrm{cdim}_0\,\mathcal{X}$ for $\mathrm{ed}\,\mathcal{X}$ and $\mathrm{cdim}\,\mathcal{X}$, respectively.

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Denote by $\mathbb{P} = \{2, 3, \ldots\}$ the set of all primes. We will prove the following general result on fiber dimensions.

THEOREM 1.1. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of CFGs over F. Then for every $p \in \mathbb{P} \cup \{0\}$*

$$\operatorname{ed}_p \mathcal{X} \leqslant \operatorname{ed}_p \mathcal{Y} + \sup_y \operatorname{ed}_p \mathcal{X}_y$$

and

$$\operatorname{cdim}_p \mathcal{X} \leqslant \operatorname{ed}_p \mathcal{Y} + \sup_y \operatorname{cdim}_p \mathcal{X}_y,$$

where the supremum is taken over all finitely generated field extensions K/F and all morphisms $y\colon \operatorname{Spec} K \to \mathcal{Y}$ of CFGs over F.

Here $\mathcal{X}_y$ (the fiber of f over y) is the 2-fiber product of $\mathcal{X}$ and $\operatorname{Spec} K$ over $\mathcal{Y}$ with respect to f and y; see § 2. It is considered as a CFG over K.

The special case of the first inequality, where both CFGs are represented by schemes locally of finite type over F, is implied by the well-known fiber dimension theorem from algebraic geometry (cf. [Har77, Exercise II.3.22]). The more general case of the same inequality, when $\mathcal{X}$ and $\mathcal{Y}$ are algebraic stacks and all fibers $\mathcal{X}_y$ are representable by quasi-separated algebraic spaces, locally of finite type and of dimension $\leqslant d$ for some fixed $d \in \mathbb{N}_0$ is exactly the result of [BRV11, Theorem 3.2].

The second inequality, where canonical and essential dimension are linked, seems to be completely new and is a key ingredient for establishing results on canonical dimension of algebraic groups later on.

Let G be an algebraic group over a field F. The essential p-dimension of G, denoted $\operatorname{ed}_p G$, is defined as the essential p-dimension of $BG \simeq [\operatorname{Spec} F/G]$, the CFG of G-torsors. It was introduced by Buhler and Reichstein in [BR97] and has been object of study for numerous mathematicians since then. See Reichstein's ICM proceedings [Rei10] for a survey on the topic.

The essential dimension of a G-torsor X over a field extension K of F, viewed as an object of BG, measures how far X is from being defined (up to isomorphism) over the base field F. On the other hand, the canonical dimension of X, introduced by Berhuy and Reichstein in [BR05], is the canonical dimension of the CFG represented by the scheme X and measures how far X is from being split.

Set

$$\operatorname{cdim} G := \sup \operatorname{cdim} X \quad (\text{and } \operatorname{cdim}_p G := \sup \operatorname{cdim}_p X)$$

where X runs over all G-torsors over field extensions. Then for G connected and smooth we have $\operatorname{ed} G = 0$ if and only if $\operatorname{cdim} G = 0$ if and only if G is special; i.e., all G-torsors over field extensions of F are split (see [Mer09, Proposition 4.4] and recall that a geometrically integral variety X over F has strictly positive canonical dimension unless it has a F-rational point). In general $\operatorname{ed} G$ can be much larger than $\operatorname{cdim} G$ (e.g. for spin groups see Corollary 4.10) and vice versa (see Example 5.13).

For *split* simple (affine) algebraic groups G the value of the canonical p-dimension of G has been computed for every prime p. The case of classical G is due to Karpenko and Merkurjev [KM06], and the case of exceptional G is due to Zainoulline [Zai07].

The assumption on G being split, i.e., containing a split maximal torus, is essential in their approach. Let B be a Borel subgroup containing the split maximal torus. Then B is special (i.e., has no non-split torsors over field extensions), and, therefore, for a G-torsor X the varieties X

and X/B have the same splitting fields and in particular the same canonical p-dimension. The variety X/B is smooth, projective and generically split. For these varieties the canonical p-dimension can be expressed through the existence of rational cycles in Chow-groups with $\mathbb{F}_p$-coefficients [KM06, Theorem 5.8]. For a survey on canonical dimension of smooth projective varieties we refer to Karpenko's ICM survey [Kar10].

We will be mainly interested in the canonical dimension of tori. In this case, all we can do with the above approach is to reduce the study of the canonical p-dimension of torsors of an arbitrary torus to the case of an anisotropic torus (mod out the maximal split subtorus).

Our approach to computing the canonical dimension of tori will be very different from the one above used for split simple algebraic groups. We will use Theorem 1.1 to relate, for an algebraic group G, the essential dimension of suitable subgroups D of G with the canonical dimension of the quotient G/D. This approach produces interesting results for algebraic tori T which split over a Galois extension of p-power degree, where p is a prime. Here D is any subgroup of T which contains the (unique) largest subgroup $C(T)$ of T of the form $(\mu_p)^r$, $r \geqslant 0$. The relation we establish in Corollary 5.5 has the following simple form.

Theorem 1.2. *We have* $\operatorname{cdim}_p T/D \geqslant \dim T/D - \operatorname{ed}_p D$.

Its proof makes full use of the computation of the essential p-dimension of T from [LMMR11]. The general statement for arbitrary G is given in Theorem 5.1. In § 5 we then proceed to find algebraic tori S which can be written as quotients $S \simeq T/D$ with $D \supseteq C(T)$ as in Theorem 1.2 and for which we can show that equality holds. This happens, for instance, for every anisotropic algebraic torus S which splits over a cyclic Galois extension of p-power degree (see Example 5.12) and for every direct product of such tori.

The rest of the paper is structured as follows: in § 2 we recall some basics on torsors, twists, CFGs, 2-fiber products, stacks, gerbes etc. and define essential and canonical dimension. Section 3 is devoted to the proof of the general fiber dimension results and to applications in basic situations. In § 4 we introduce and study p-exhaustive subgroups. Roughly speaking these are normal subgroups of an algebraic group G for which the essential p-dimension of G can be expressed via the essential p-dimension of gerbes of the form $[E/G]$ for G/C-torsors E. We then apply the fiber dimension results to spin groups. Finally § 5 contains our results on canonical dimension of algebraic groups, in particular of algebraic tori.

2. Preliminaries

2.1 Conventions

We denote by F a field, which serves as our base field.

We will use the 'Stacks Project' [Stacks] as our main reference for algebraic spaces, stacks, gerbes etc. All these notions are understood with respect to the fppf-topology. As in [Stacks] (and in contrast to [LM00], for instance) we try not to ignore any set-theoretical issues. Thus we will work over any big fppf-site Sch_F as in [Stacks, Definition 021R]. This site is non-canonical but has the advantage that its class of objects is a set. All schemes over F under consideration are assumed to be objects of Sch_F. Note that Sch_F contains, among other objects, for every finitely generated F-algebra A and for every finitely generated field extensions K/F some scheme isomorphic to $\operatorname{Spec} A$ (respectively $\operatorname{Spec} K$); see [Stacks, Lemma 000R]. For notational convenience we will assume that for every finitely generated field extension K/F there exists a field extension K'/F isomorphic to K such that $\operatorname{Spec} K' \in \mathrm{Sch}_F$.

All of our group algebraic spaces and group schemes over a field K under consideration are assumed to be locally of finite type over K.

2.2 Torsors and twists

Let G be a group algebraic space (locally of finite type) over a field F in the sense of [Stacks, 043H] (with $B = S = \operatorname{Spec} F$). Usually G will be an affine group scheme of finite type over F for us. However, more general group algebraic spaces will appear naturally as automorphism group algebraic spaces of points of algebraic stacks. Since we do not assume algebraic spaces to be quasi-separated, there are group algebraic spaces over a field F which are not group schemes (for an example see [Stacks, Lemma 06E4]).

Let U be an algebraic space over F. A G-torsor over U is an algebraic space E over F with a right action of G (in the sense of [Stacks, Definition 043Q]) and a G-invariant morphism $E \to U$ of algebraic spaces which is fppf-locally isomorphic on U to the trivial torsor $U \times G \to U$.

A G-torsor over a field extension K/F is a G-torsor E over $\operatorname{Spec} K$. It is trivial if and only if it has a K-rational point. Note that since G is locally of finite type over F every G-torsor E over K becomes trivial over the algebraic closure K_{alg}.

We remark that if G is an affine group scheme (which will usually be the case for us) then every G-torsor over a field extension K/F is representable by a scheme (cf. [Stacks, Remark 049C]).

For any G-torsor X over a field extension K/F we can form the twist

$$^{X}G := \mathbf{Aut}_G(X),$$

the group algebraic space over K of G-equivariant automorphisms of X. If X is trivial we have $^{X}G \simeq G_K$.

More generally, if N is a normal subgroup of G we form the twist ^{X}N as follows. First note that for every morphism $f\colon G \to H$ there is an induced H-torsor $f_*(X)$, defined as the quotient

$$f_*(X) = (X \times H)/G,$$

where G acts by the formula $(x, h)g = (xg, f(g)^{-1}h)$. By descent, the quotient exists as an algebraic space and is an H-torsor; see [Stacks, Lemma 04U0].

Now we apply this construction to the canonical morphism $\pi\colon G \to H := G/N$. Let $Y = \pi_*(X)$. Then we get an induced morphism $^{X}G \to {}^{Y}H$ of group algebraic spaces. The twist of N by X is defined as the kernel

$$^{X}N := \ker(^{X}G \to {}^{Y}H)$$

of this morphism.

If G, N and G/N are smooth affine group schemes over F we associate with a G-torsor X a 1-cocycle $z \in Z^1(\Gamma, G(F_{\mathrm{sep}}))$ (unique up to the choice of a point $x_0 \in X(F_{\mathrm{sep}})$), where $\Gamma := \operatorname{Gal}(F_{\mathrm{sep}}/F)$, and consider the twisted Γ-action on $N(F_{\mathrm{sep}})$ by the cocycle z. This group, denoted ${}_zN(F_{\mathrm{sep}})$ in [Ser02] can be identified Γ-equivariantly with the group of F_{sep}-rational points of the twist ^{X}N. Thus our construction of ^{X}N is equivalent to the twist-construction ${}_zN(F_{\mathrm{sep}})$ in Galois-cohomology for smooth affine group schemes.

2.3 CFGs, stacks and gerbes

Let $\mathcal{C}$ be a category. A *category fibered in groupoids*, abbreviated *CFG*, over $\mathcal{C}$ is a category $\mathcal{A}$ equipped with a functor $\pi\colon \mathcal{A} \to \mathcal{C}$ subject to the following two conditions.

(i) For every morphism $\iota\colon U \to V$ in $\mathcal{C}$ and object $a \in \mathcal{A}$ with $\pi(a) = V$, there exist an object b of $\mathcal{A}$ and a morphism $f\colon b \to a$ in $\mathcal{A}$ such that $\pi(f) = \iota$ (cf. diagram below).

$$\begin{tikzcd}
\exists b \arrow[r, dashed, "f"] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
U \arrow[r, "\iota"] & V
\end{tikzcd}$$

(ii) For every pair of morphisms $f\colon b \to a$ and $g\colon c \to a$ in $\mathcal{A}$ and every morphism $\iota\colon \pi(c) \to \pi(b)$ such that $\pi(f) \circ \iota = \pi(g)$ there exists a unique morphism $h\colon c \to b$ in $\mathcal{A}$ such that $\pi(h) = \iota$ and $f \circ h = g$ (cf. diagram below).

$$\begin{tikzcd}
c \arrow[d, mapsto] \arrow[dr, dashed, "h"'] \arrow[drr, "g"] & & \\
\pi(c) \arrow[dr, "\iota"'] \arrow[drr, "\pi(g)"] & b \arrow[r, "f"] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
& \pi(b) \arrow[r, "\pi(f)"'] & \pi(a)
\end{tikzcd}$$

We will denote by *CFG over F* a CFG over the category Sch_F. Every scheme $X \in \mathrm{Sch}_F$ gives rise to a CFG $\tilde{X}$ over F: its objects are morphisms $T \to X$, where $T \in \mathrm{Sch}_F$, its morphisms are morphisms $T \to S$ compatible with the morphisms to X, and the structure morphism $\tilde{X} \to \mathrm{Sch}_F$ is the projection onto the domain.

Recall that morphisms $X \to Y$ of schemes over F are in canonical one-to-one correspondence with morphisms $\tilde{X} \to \tilde{Y}$ by the Yoneda lemma. In the following, we will use the notation X for the CFG $\tilde{X}$ associated with a scheme X and make it clear from the context whether the scheme X or the CFG X is meant.

By the term '*stack* over F', we mean a CFG $\mathcal{X}$ over F satisfying the additional conditions (2) and (3) of [Stacks, Definition 02ZI] on patching isomorphisms and objects of $\mathcal{X}$. Note that all our stacks are fibered in groupoids.

The CFG associated with a scheme X over F is a stack. More generally, every algebraic space X over F is a stack. The algebraic spaces over F are precisely those stacks over F whose objects do not have any non-trivial automorphisms lying over the identity of their base; see [Stacks, Proposition 04SZ].

Another type of example that we will often use in the following are quotients of algebraic spaces by group actions.

Example 2.1. Let G be a group algebraic space over F and X be an algebraic space over F on which G acts (from the right). The quotient stack $[X/G]$ of X by the G-action is the CFG over F, whose objects are diagrams

$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d] & X \\
U &
\end{tikzcd} \tag{1}$$

where $U \in \mathrm{Sch}_F$, $E \to U$ is a G-torsor and $\varphi\colon E \to X$ is a G-equivariant morphism of algebraic spaces over F.

Morphisms between two such objects (1) are pairs consisting of a morphism $U \to U'$ of schemes and a G-equivariant morphism $E \to E'$ of algebraic spaces, such that the diagram

$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d] \arrow[rr, bend right=15] & X & E' \arrow[l, "\varphi'"'] \arrow[d] \\
U \arrow[rr, bend right=10] & & U'
\end{tikzcd}$$

commutes. The structure map $[X/G] \to \mathrm{Sch}_F$ is projection onto the bottom row. The quotient stack $[X/G]$ is indeed a stack over F; see [Stacks, Lemma 0370].

In the special case when $X = \mathrm{Spec}\, F$ (with trivial G-action) the quotient stack $[X/G]$ can be canonically identified with BG, the *classifying stack* of G. An object of BG is simply a G-torsor $E \to U$.

The construction of quotients $[X/G]$ is functorial with respect to G-equivariant morphisms of algebraic spaces. For G-equivariant morphisms of algebraic spaces $X \to Y$ we write $f_*^G\colon [X/G] \to [Y/G]$ for the induced morphism of quotient stacks. On objects it is simply given by replacing the morphism $E \to X$ by the composition $E \to X \to Y$ in a diagram (1).

The construction of $[X/G]$ is also functorial with respect to morphisms $a\colon G \to H$ of group algebraic spaces. Let H act on X and let G act on X through a. Then we have a morphism, $a_*^X\colon [X/G] \to [X/H]$, which takes a diagram (1) to the diagram

$$\begin{tikzcd}
a_*(E) \arrow[r, "\psi"] \arrow[d] & X \\
U &
\end{tikzcd}$$

where the H-equivariant map $\psi\colon a_*(E) \to X$ is induced by the G-invariant map $E \times H \to X$, $(e, h) \mapsto \varphi(e)h$.

An algebraic stack over F is a stack $\mathcal{X}$ over F whose diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces and such that there exists a smooth and surjective morphism $U \to \mathcal{X}$ for some scheme $U \in \mathrm{Sch}_F$.

A *gerbe* over F is an algebraic stack $\mathcal{X}$ over F satisfying the additional two conditions (2) and (3) of [Stacks, Definition 06NZ], which say that any two objects of $\mathcal{X}$ are locally isomorphic and that objects exist locally. An example of a gerbe is the classifying stack BG for any group algebraic space G (locally of finite type) over F. Any gerbe over F becomes isomorphic over a finite field extension K/F to BG for some group algebraic space G over K.

CFGs $(\mathcal{A}, \pi)$ over F (where π is the structure map $\pi\colon \mathcal{A} \to \mathrm{Sch}_F$) form a 2-category, in which morphisms $(\mathcal{A}, \pi) \to (\mathcal{A}', \pi')$ are functors $\varphi\colon \mathcal{A} \to \mathcal{A}'$ such that $\pi' \circ \varphi = \pi$, and in which 2-morphisms $\varphi_1 \to \varphi_2$ for morphisms $\varphi_1, \varphi_2\colon (\mathcal{A}, \pi) \to (\mathcal{A}', \pi')$ are natural transformations $t\colon \varphi_1 \to \varphi_2$ such that $\pi'(t_a) = \mathrm{id}_{\pi(a)}$ for all objects a of $\mathcal{A}$.

We will use the notion of 2-fiber product in the 2-category of CFGs over F. If $\varphi\colon \mathcal{X} \to \mathcal{Z}$ and $\psi\colon \mathcal{Y} \to \mathcal{Z}$ are two morphisms of CFGs over F a 2-fiber product is a CFG $\mathcal{A}$ over F together with morphisms $p\colon \mathcal{A} \to \mathcal{X}$ and $q\colon \mathcal{A} \to \mathcal{Y}$ such that the square

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "q"] \arrow[d, "p"'] & \mathcal{Y} \arrow[d, "\psi"] \\
\mathcal{X} \arrow[r, "\varphi"'] & \mathcal{Z}
\end{tikzcd}$$

2-commutes (i.e., the two compositions $\mathcal{A} \to \mathcal{Z}$ are 2-isomorphic) and is a final object in the 2-category of 2-commutative squares; see [Stacks, Definition 003Q] for details. In particular, for every other 2-commutative square

$$\begin{tikzcd}
\mathcal{A}' \arrow[r, "q'"] \arrow[d, "p'"'] & \mathcal{Y} \arrow[d, "\psi"] \\
\mathcal{X} \arrow[r, "\varphi"'] & \mathcal{Z}
\end{tikzcd}$$

there exists a morphism $\alpha\colon \mathcal{A}' \to \mathcal{A}$ that makes the diagram

$$\begin{tikzcd}
\mathcal{A}' \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "q'"] \arrow[ddr, bend right, "p'"'] & & \\
& \mathcal{A} \arrow[r, "q"] \arrow[d, "p"] & \mathcal{Y} \arrow[d, "\psi"] \\
& \mathcal{X} \arrow[r, "\varphi"'] & \mathcal{Z}
\end{tikzcd}$$

2-commute.

A 2-fiber product is unique up to unique equivalence. A 2-fiber product of $\varphi\colon \mathcal{X} \to \mathcal{Z}$ and $\psi\colon \mathcal{Y} \to \mathcal{Z}$ can be constructed as in [Stacks, Proposition 0040] as a category whose objects are quadruples (U, x, y, f) where $U \in \mathrm{Sch}_F$, x and y are objects of $\mathcal{X}$ and $\mathcal{Y}$, respectively, over U, and $f\colon \varphi(x) \xrightarrow{\sim} \psi(y)$ is an isomorphism in $\mathcal{Z}$ lying over the identity of U.

In some concrete situations, as in the following two examples, which will be used later on, 2-fiber products have simpler alternative descriptions.

Example 2.2. Let X, Y and Z be algebraic spaces with a G-action from the right and let $f\colon X \to Z$ and $g\colon Y \to Z$ be G-equivariant morphisms of algebraic spaces. Then G acts diagonally on the (usual) fiber-product $X \times_Z Y$ in the category of algebraic spaces and the following diagram is 2-cartesian.

$$\begin{tikzcd}
{[(X \times_Z Y)/G]} \arrow[r, "(\pi_Y)_*^G"] \arrow[d, "(\pi_X)_*^G"'] & {[Y/G]} \arrow[d, "g_*^G"] \\
{[X/G]} \arrow[r, "f_*^G"'] & {[Z/G]}
\end{tikzcd}$$

For a proof, see e.g. [Wan11, Lemma 2.3.2].

Example 2.3. Let $a\colon G \to H$ be a morphism of group algebraic spaces and $f\colon X \to Y$ an H-equivariant morphism of algebraic spaces. Then the following diagram is 2-cartesian.

$$\begin{tikzcd}
{[X/G]} \arrow[r, "f_*^G"] \arrow[d, "a_*^X"'] & {[Y/G]} \arrow[d, "a_*^Y"] \\
{[X/H]} \arrow[r, "f_*^H"'] & {[Y/H]}
\end{tikzcd} \tag{2}$$

This fact is probably well known. For lack of a reference we outline a proof using the construction of the 2-fiber product in [Stacks, Proposition 0040]. Take an object of the 2-fiber product $[X/H] \times_{[Y/H]} [Y/G]$ over $U \in \mathrm{Sch}_F$. It is given by a G-torsor E over U with a G-equivariant map $\varphi\colon E \to Y$, a H-torsor E' over U with a H-equivariant map $\psi\colon E' \to X$ and an H-equivariant isomorphism $\alpha\colon E' \to a_*(E)$ over U fitting into the commutative diagram

$$\begin{tikzcd}
E' \arrow[r, "\alpha"] \arrow[d, "\psi"] & a_*(E) \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where the vertical map on the right is induced by the G-invariant map $E \times H \to Y\colon (e, h) \mapsto \varphi(e)h$. We associate with this object the G-torsor E with the G-equivariant map $\psi \circ \alpha^{-1} \circ \iota\colon E \to X$, where ι is the map $E \to a_*(E)$, $e \to [e, 1]$. This construction yields a morphism $[X/H] \times_{[Y/H]} [Y/G] \to [X/G]$ of CFGs.

On the other hand the 2-commutativity of diagram (2) induces a morphism $[X/G] \to [X/H] \times_{[Y/H]} [Y/G]$. The two morphisms are easily seen to be mutually quasi-inverse. It follows that diagram (2) is 2-cartesian as claimed.

2.4 Essential and canonical dimension of CFGs

We will define essential and canonical dimension for CFGs over F, in particular of algebraic stacks. The essential dimension of algebraic stacks has been introduced by Brosnan *et al.* in [BRV11] (see also [BRV07]). Since then, several authors have worked on the essential dimension of algebraic stacks. The definitions of essential dimension that we give below are equivalent to those in the literature; see e.g. [Mer09] or [BRV11]. However, the definitions below will be more suitable for our purposes.

Definition 2.4. Let $\mathcal{X}$ be a CFG over F. For a finitely generated field extension K/F, a field K_0 with a morphism $\operatorname{Spec} K \to \operatorname{Spec} K_0$ over F and a morphism $x\colon \operatorname{Spec} K \to \mathcal{X}$ over F we say that:

- x *is defined over* K_0 (or that K_0 is a *field of definition* of x) if there exists a morphism $x_0\colon \operatorname{Spec} K_0 \to \mathcal{X}$ over F such that the diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "x"] \arrow[d] & \mathcal{X} \\
\operatorname{Spec} K_0 \arrow[ur, "x_0"'] &
\end{tikzcd}$$

  2-commutes;
- x *is detected over* K_0 (or that K_0 is a *detection field* of x) if there exists a morphism $x_0\colon \operatorname{Spec} K_0 \to \mathcal{X}$ over F.

We define

$$\operatorname{ed} x := \min_{K_0} \operatorname{tdeg}_F K_0 \in \mathbb{N}_0, \quad \operatorname{cdim} x := \min_{K_0'} \operatorname{tdeg}_F K_0' \in \mathbb{N}_0$$

where the minimum is taken over all fields of definition K_0 of x, respectively over all detection fields K_0' of x. For $p \in \mathbb{P} \cup \{0\}$ we define

$$\operatorname{ed}_p x := \min \operatorname{ed} x_L \in \mathbb{N}_0, \quad \operatorname{cdim}_p x := \min \operatorname{cdim} x_L \in \mathbb{N}_0,$$

where L runs over all prime to p extensions of K such that $\operatorname{Spec} L \in \mathrm{Sch}_F$ and $x_L\colon \operatorname{Spec} L \to \mathcal{X}$ is the composite $\operatorname{Spec} L \to \operatorname{Spec} K \xrightarrow{x} \mathcal{X}$. Here, and in the following, 'prime to 0 extension' means 'trivial extension', as usual, so that $\operatorname{ed}_0 x = \operatorname{ed} x$ and $\operatorname{cdim}_0 x = \operatorname{cdim} x$.

We set

$$\operatorname{ed}_p \mathcal{X} := \sup_x \operatorname{ed}_p x \in \mathbb{N}_0 \cup \{-\infty, \infty\}, \quad \operatorname{cdim}_p \mathcal{X} := \sup_x \operatorname{cdim}_p x \in \mathbb{N}_0 \cup \{-\infty, \infty\},$$

where the supremum runs over all (finitely generated) field extensions K/F and morphisms $x\colon \operatorname{Spec} K \to \mathcal{X}$ over F, and $\operatorname{ed} \mathcal{X} := \operatorname{ed}_0 \mathcal{X}$, $\operatorname{cdim} \mathcal{X} := \operatorname{cdim}_0 \mathcal{X}$. We have $\operatorname{ed}_p \mathcal{X} = -\infty$ (or equivalently $\operatorname{cdim}_p \mathcal{X} = -\infty$) if and only if $\mathcal{X}$ is empty.

Note that the condition that $x\colon \operatorname{Spec} K \to \mathcal{X}$ is detected over K_0 depends on x only through the field K, since K_0 is required to be an intermediate field of K/F. Thus $\operatorname{cdim} \mathcal{X}$ is determined by the class of finitely generated field extensions K/F with $\mathcal{X}(K) \neq \emptyset$.

If G is a group algebraic space over F, the essential p-dimension of G for $p \in \mathbb{P} \cup \{0\}$ is defined via its classifying stack $BG \simeq [\operatorname{Spec} F/G]$:

$$\operatorname{ed}_p G := \operatorname{ed}_p BG.$$

Moreover,

$$\operatorname{cdim}_p G := \sup \operatorname{cdim}_p X,$$

where X runs over all G-torsors over field extensions K of F with $\operatorname{Spec} K \in \operatorname{Sch}_F$.

We will sometimes tacitly use the following fact.

Lemma 2.5 [BRV11, Example 2.4]. *Let X be a scheme or a quasi-separated algebraic space locally of finite type over F. Then $\operatorname{ed}_p X = \dim X$ for every $p \in \mathbb{P} \cup \{0\}$.*

For every CFG $\mathcal{X}$ over F we have $\operatorname{cdim}_p \mathcal{X} \leqslant \operatorname{ed}_p \mathcal{X}$ for every $p \in \mathbb{P} \cup \{0\}$. However, note that $\operatorname{cdim}_p G$ has nothing to do with $\operatorname{cdim}_p BG$, which is zero (since F is a detection field for all morphisms $x\colon \operatorname{Spec} K \to BG$), and $\operatorname{ed}_p G = \operatorname{ed}_p BG$ has nothing to do with the essential p-dimension of the algebraic space G, which is equal to $\dim G$. Thus, *a priori*, there are no relations between the values of $\operatorname{ed}_p G$ and $\operatorname{cdim}_p G$. However, when G is quasi-separated we always have

$$\operatorname{cdim}_p G \leqslant \dim G$$

since the canonical p-dimension of a G-torsor X is always less or equal to the essential p-dimension of the algebraic space X, which is $\dim X = \dim G$ by Lemma 2.5.

Let Fields_F be the category of finitely generated field extensions K/F with $\operatorname{Spec} K \in \operatorname{Sch}_F$. We can define a functor $\mathcal{F}_{\mathcal{X}}\colon \operatorname{Fields}_F \to \operatorname{Sets}$ by taking $\mathcal{F}_{\mathcal{X}}(K)$ to be the set of isomorphism classes in $\mathcal{X}(K)$. Then $\operatorname{ed} \mathcal{X}$ is easily seen to coincide with $\operatorname{ed} \mathcal{F}_{\mathcal{X}}$ as defined in [BF03]. Similarly $\operatorname{cdim} \mathcal{X}$ coincides with the essential dimension of the detection functor

$$D_{\mathcal{X}}\colon \operatorname{Fields}_F \to \operatorname{Sets} \quad K \mapsto \begin{cases} \{\emptyset\} & \text{if } \mathcal{X}(K) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

3. Fibers for morphisms of CFGs

We start this section by proving our version of the fiber dimension theorem.

Proof of Theorem 1.1. Let $x\colon \operatorname{Spec} K \to \mathcal{X}$ be a morphism over F for some finitely generated field extensions K/F. Let $y = f \circ x\colon \operatorname{Spec} K \to \mathcal{X} \to \mathcal{Y}$. By the definition of $\operatorname{ed}_p y$, there exist a prime to p extension L/K and an intermediate field L_0 of L/F with $\operatorname{tdeg}_F L_0 = \operatorname{ed}_p y$ together with a 2-commutative diagram, as follows.

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r, "x"] & \mathcal{X} \arrow[d, "f"] \\
\operatorname{Spec} L_0 \arrow[rr, "y_0"'] & & \mathcal{Y}
\end{tikzcd}$$

By the universal property of 2-fibered products there exists a morphism $z\colon \operatorname{Spec} L \to \mathcal{X}_{y_0}$ such that the diagram

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[r] \arrow[dr, "z"] \arrow[ddr, bend right] & \operatorname{Spec} K \arrow[dr, "x"] & \\
 & \mathcal{X}_{y_0} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
 & \operatorname{Spec} L_0 \arrow[r, "y_0"'] & \mathcal{Y}
\end{tikzcd}$$

2-commutes. We will now argue for essential and canonical dimension separately.

- *Essential dimension*. By the definition of $\operatorname{ed}_p z$ there exist a prime to p extension M/L and an intermediate field M_0 of M/L_0 with $\operatorname{tdeg}_{L_0} M_0 = \operatorname{ed}_p z$ together with a morphism $z_0\colon \operatorname{Spec} M_0 \to \mathcal{X}_{y_0}$ over L_0 such that the above diagram can be completed to a 2-commutative diagram.

$$\begin{tikzcd}
\operatorname{Spec} M \arrow[r] \arrow[d] & \operatorname{Spec} L \arrow[r] \arrow[dr, "z"] \arrow[ddr, bend right] & \operatorname{Spec} K \arrow[dr, "x"] & \\
\operatorname{Spec} M_0 \arrow[rr, "z_0"] & & \mathcal{X}_{y_0} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
 & & \operatorname{Spec} L_0 \arrow[r, "y_0"'] & \mathcal{Y}
\end{tikzcd}$$

  Therefore x_M is defined over M_0. It follows that $\operatorname{ed}_p x \leqslant \operatorname{tdeg}_F M_0 = \operatorname{tdeg}_F L_0 + \operatorname{tdeg}_{L_0} M_0 = \operatorname{ed}_p y + \operatorname{ed}_p z \leqslant \operatorname{ed}_p \mathcal{Y} + \operatorname{ed}_p \mathcal{X}_{y_0}$. Hence the first inequality follows.

- *Canonical dimension*. By the definition of $\operatorname{cdim}_p z$ there exist a prime to p extension M'/L and an intermediate field M'_0 of M'/L_0 with $\operatorname{tdeg}_{L_0} M'_0 = \operatorname{cdim}_p z$ together with a morphism $z'_0\colon \operatorname{Spec} M'_0 \to \mathcal{X}_{y_0}$ over L_0. Hence there exists a morphism $\operatorname{Spec} M'_0 \to \mathcal{X}$, which shows that $\operatorname{cdim}_p x \leqslant \operatorname{tdeg}_F M'_0$. Now the second inequality follows as above. □

The following lemma on the essential dimension of gerbes $\mathcal{X}$ will be useful in the following. The case where $\mathcal{X}$ is banded by a commutative group scheme is [Mer08, Proposition 4.9]. Recall that for any algebraic stack $\mathcal{X}$ over F there exists, for every morphism $y\colon \operatorname{Spec} K \to \mathcal{X}$ over F, a group algebraic space $\mathbf{Aut}_K(y)$ over K of automorphisms of y (cf. [Stacks, Lemmas 04YP and 04XR]). Its T-rational points for $T \in \mathrm{Sch}_K$ are the automorphisms $y_T \xrightarrow{\simeq} y_T$ over T.

LEMMA 3.1. *Let $\mathcal{X}$ be a gerbe over F. Then, for every $p \in \mathbb{P} \cup \{0\}$,*

$$\operatorname{ed}_p \mathcal{X} \leqslant \operatorname{cdim}_p \mathcal{X} + \sup \operatorname{ed}_p \mathbf{Aut}_K(y),$$

where the supremum is taken over all field extensions K/F and all morphisms $y\colon \operatorname{Spec} K \to \mathcal{X}$.

Proof. Let $x\colon \operatorname{Spec} K \to \mathcal{X}$ be a morphism. By the definition of $\operatorname{cdim}_p x$ there exist a prime to p extension L/K, an intermediate extension L_0/F of L/F and a morphism $x_0\colon \operatorname{Spec} L_0 \to \mathcal{X}$ such that $\operatorname{tdeg}_F L_0 = \operatorname{cdim}_p x \leqslant \operatorname{cdim}_p \mathcal{X}$. Then $\mathcal{X}_{L_0}$ is equivalent to BG, where $G := \mathbf{Aut}_{L_0}(x_0)$ (cf. [LM00, Lemme 3.21]). We get a morphism $y\colon \operatorname{Spec} L \to BG$ such that $x_L\colon \operatorname{Spec} L \to \mathcal{X}$ and the composition $\operatorname{Spec} L \xrightarrow{y} BG \xrightarrow{\simeq} \mathcal{X}_{L_0} \to \mathcal{X}$ are 2-isomorphic. By the definition of $\operatorname{ed}_p y$ there exist a prime to p extension M/L, an intermediate field M_0 of the extension M/L_0 with $\operatorname{tdeg}_{L_0} M_0 = \operatorname{ed}_p y$ and a morphism $y_0\colon \operatorname{Spec} M_0 \to BG$ such that the diagram

$$\begin{tikzcd}
\operatorname{Spec} M \arrow[r] \arrow[d] & \operatorname{Spec} L \arrow[r] \arrow[d, "y"] & \operatorname{Spec} K \arrow[drr, "x"] & & \\
\operatorname{Spec} M_0 \arrow[r, "y_0"] & BG \arrow[r, "\simeq"] & \mathcal{X}_{L_0} \arrow[rr] & & \mathcal{X}
\end{tikzcd}$$

2-commutes. Hence

$$\begin{aligned}\operatorname{ed}_p x &\leqslant \operatorname{tdeg}_F M_0 = \operatorname{tdeg}_F L_0 + \operatorname{tdeg}_{L_0} M_0 = \operatorname{cdim}_p x + \operatorname{ed}_p y \\ &\leqslant \operatorname{cdim}_p \mathcal{X} + \operatorname{ed}_p G,\end{aligned}$$

and the claim follows. □

Corollary 3.2. *Let $1 \to C \to G \to H \to 1$ be an exact sequence of group algebraic spaces over F. Let E be an H-torsor over some field extension K/F. Then*

$$\operatorname{ed}_p[E/G] \leqslant \operatorname{cdim}_p[E/G] + \sup \operatorname{ed}_p {}^X C,$$

where X runs over all lifts of E to a G-torsor over field extensions L/K.

In particular, when C is central in G then

$$\operatorname{ed}_p[E/G] \leqslant \operatorname{cdim}_p[E/G] + \operatorname{ed}_p C_K.$$

Proof. A morphism $y\colon \operatorname{Spec} L \to [E/G]$ corresponds to a lifting of E to a G-torsor X, and $\mathbf{Aut}_L(y)$ is isomorphic to the twist ${}^X C$. Hence the first inequality follows from Lemma 3.1.

If C is central in G then ${}^X C$ is isomorphic to C_L. Hence the second inequality follows from the first one and the fact $\sup_L \operatorname{ed}_p C_L \leqslant \operatorname{ed}_p C_K$. □

We will apply Theorem 1.1 in the following cases.

Example 3.3. Let G be a group algebraic space over F and $a\colon X \to Y$ be a G-equivariant morphism of algebraic spaces over F. A morphism $y\colon \operatorname{Spec} K \to [Y/G]$ corresponds to a G-torsor E over K with a G-equivariant morphism $E \to Y$. By Example 2.2 the fiber of the morphism $a_*^G\colon [X/G] \to [Y/G]$ over y is equivalent to $[E \times_Y X/G]$. Thus, for every $p \in \mathbb{P} \cup \{0\}$,

$$\operatorname{ed}_p[X/G] \leqslant \operatorname{ed}_p[Y/G] + \sup \operatorname{ed}_p[E \times_Y X/G]$$

(cf. [BRV11, Example 3.1]), and

$$\operatorname{cdim}_p[X/G] \leqslant \operatorname{ed}_p[Y/G] + \sup \operatorname{cdim}_p[(E \times_Y X)/G],$$

where the supremum is taken over all field extensions K/F and all G-torsors E over K with a G-equivariant morphism $E \to Y$.

Note that $[E \times_Y X/G]$ is an algebraic space. Thus if it is quasi-separated or a scheme then we can replace $\operatorname{ed}_p[E \times_Y X/G]$ by $\dim[E \times_Y X/G]$ in the first inequality above.

Now we apply this to the following situation. Let $g\colon G \to H$ be a morphism of group schemes over F. Let X be an H-torsor over some field extension L/F. Then G acts on X via g, and $[(E \times X)/G]$ is an $\mathbf{Aut}_H(X)$-torsor over $K \in \mathrm{Fields}_L$, which is quasi-separated. Thus,

$$\mathrm{ed}_p[X/G] \leqslant \mathrm{ed}_p G + \dim H \tag{3}$$

(cf. [Mer09, Theorem 4.8] and [BRV11, Corollary 3.3]) and

$$\mathrm{cdim}_p[X/G] \leqslant \mathrm{ed}_p G + \mathrm{cdim}_p \mathbf{Aut}_H(X). \tag{4}$$

More generally, suppose we are given morphisms $g\colon G \to H$ and $h\colon H \to Q$ of group schemes over F. Let X be an H-torsor over some field extension L/F and let $Y = h_*(X)$ be the induced Q-torsor. Then G acts on X and Y via g and $h \circ g$, respectively, and $X \to Y$ is G-equivariant. In this situation $[(E \times_Y X)/G]$ is a torsor over $K \in \mathrm{Fields}_L$ for the group scheme

$$U := \ker(\mathbf{Aut}_H(X) \to \mathbf{Aut}_Q(Y))$$

over L (that becomes isomorphic to $\ker(h\colon H \to Q)$ over L_{alg}). Thus,

$$\mathrm{ed}_p[X/G] \leqslant \mathrm{ed}_p[Y/G] + \dim(\ker h), \tag{5}$$

$$\mathrm{cdim}_p[X/G] \leqslant \mathrm{ed}_p[Y/G] + \mathrm{cdim}_p U. \tag{6}$$

Note that in the case, h is surjective, U is simply the twist $U = {}^X C$ of the kernel $C := \ker h$ by X.

Example 3.4. Let $f\colon G \to H$ be a morphism of group algebraic spaces over F and let H act on an algebraic space X. A morphism $y\colon \operatorname{Spec} K \to [X/H]$ corresponds to an H-torsor E over K with an H-equivariant morphism $E \to X$. By Example 2.2, the fiber of the morphism $f_*^X\colon [X/G] \to [X/H]$ over y is isomorphic to $[E/G]$. Thus,

$$\mathrm{ed}_p[X/G] \leqslant \mathrm{ed}_p[X/H] + \sup \mathrm{ed}_p[E/G],$$
$$\mathrm{cdim}_p[X/G] \leqslant \mathrm{ed}_p[X/H] + \sup \mathrm{cdim}_p[E/G],$$

where the supremum runs over all field extensions K/F and all H-torsors E over K admitting an H-equivariant morphism $E \to X$.

We have the following interesting special cases.

(i) This case was independently discovered by Chernousov and Merkurjev and used for split spin groups (cf. § 4). For $X = \operatorname{Spec} F$,

$$\mathrm{ed}_p G \leqslant \mathrm{ed}_p H + \sup \mathrm{ed}_p[E/G],$$

where the supremum is taken over all field extensions K/F and all H-torsors E over K.

When f is surjective, $[E/G]$ is a gerbe. Applying Lemma 3.1 yields, with $C = \ker f$,

$$\mathrm{ed}_p G \leqslant \mathrm{ed}_p H + \sup \mathrm{cdim}_p[E/G] + \sup \mathrm{ed}_p {}^T C,$$

where the suprema are taken over all H-torsors E, respectively all G-torsors T, over field extensions K of F.

(ii) For G trivial (and X, H quasi-separated for the first inequality),

$$\dim X \leqslant \mathrm{ed}_p[X/H] + \dim H,$$
$$\mathrm{cdim}_p X \leqslant \mathrm{ed}_p[X/H] + \sup \mathrm{cdim}_p E \leqslant \mathrm{ed}_p[X/H] + \mathrm{cdim}_p H,$$

where the supremum runs over all field extensions K/F and all H-torsors E over K admitting an H-equivariant morphism $E \to X$.

The following result has been proven by Nguyen (for smooth group schemes) and can be seen as a special case of Example 3.4(3.4). The case where U is commutative is due to Tossici and Vistoli.

Corollary 3.5 (c.f. [Ngu10, Proposition 2.2] and [TV10, Lemma 3.4]). *Let $1 \to U \to G \to H \to 1$ be an exact sequence of group schemes over F with U unipotent. Then*

$$\mathrm{ed}_p\, G \leqslant \mathrm{ed}_p\, H + \sup \mathrm{ed}_p\, {}^X U,$$

where X runs over all G-torsors over field extensions K/F.

In particular, when U is central in G then

$$\mathrm{ed}_p\, G \leqslant \mathrm{ed}_p\, H + \mathrm{ed}_p\, U.$$

Proof. Since U is unipotent, every H-torsor lifts to a G-torsor [Oes84] (the reference assumes algebraic groups to be smooth; however, the same argument still works in the general case with Galois-cohomology replaced by fppf-cohomology). Therefore every gerbe $[E/G]$ over K from Example 3.4(3.4) has $\mathrm{cdim}_p[E/G] = 0$. The claim follows. □

Note that the same argument works for semi-direct products.

Corollary 3.6. *Let $G = N \rtimes H$ be a semidirect product of group schemes N and H over F. Then*

$$\mathrm{ed}_p\, H \leqslant \mathrm{ed}_p\, G \leqslant \mathrm{ed}_p\, H + \sup \mathrm{ed}_p\, {}^T N,$$

where T runs over all G-torsors over field extensions of F.

Example 3.7. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks over F and $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ its relative inertia stack, whose objects are pairs (ξ, α) where $\xi \in \mathrm{Ob}(\mathcal{X})$ and α is an automorphism of ξ with $f(\alpha) = \mathrm{id}_{f(\xi)}$. The fibers of the canonical morphism $\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ over points $x\colon \operatorname{Spec} K \to \mathcal{X}$ are the group algebraic spaces given by the kernels of the morphisms $\mathbf{Aut}_K(x) \to \mathbf{Aut}_K(f(x))$; see [Stacks, Lemma 050Q] and its proof. We will assume that the morphism $\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is quasi-separated, so that all the group algebraic spaces $\ker(\mathbf{Aut}_K(x) \to \mathbf{Aut}_K(f(x)))$ are quasi-separated. Then

$$\mathrm{ed}_p\, \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \leqslant \mathrm{ed}_p\, \mathcal{X} + \sup \dim \ker(\mathbf{Aut}_K(x) \to \mathbf{Aut}_K(f(x))),$$

where the supremum is taken over all field extensions K/F and all morphisms $x\colon \operatorname{Spec} K \to \mathcal{X}$.

We also have

$$\mathrm{ed}_p\, \mathcal{X} \leqslant \mathrm{ed}_p\, \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

since the morphism $\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is surjective on K-rational points for every $K \in \mathrm{Fields}_F$. In particular, if $\mathcal{X}$ has finite relative inertia over $\mathcal{Y}$, then $\mathrm{ed}_p\, \mathcal{X} = \mathrm{ed}_p\, \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$. A stack over F for which all automorphism groups $\mathbf{Aut}_K(x)$ are finite has

$$\mathrm{ed}_p\, \mathcal{X} = \mathrm{ed}_p\, \mathcal{I}_{\mathcal{X}}.$$

Here $\mathcal{I}_{\mathcal{X}} = \mathcal{I}_{\mathcal{X}/\operatorname{Spec} F}$ denotes the absolute inertia stack.

For a group scheme G and a normal subgroup N, the relative inertia stack with respect to the canonical morphism $f\colon BG \to B(G/N)$ is equivalent to $[N/G]$, where G acts by conjugation. The kernels $\ker(\mathbf{Aut}_K(x) \to \mathbf{Aut}_K(f(x)))$ are the twists ${}^X N$ of N by G-torsors X. Therefore

$$\mathrm{ed}_p\, G \leqslant \mathrm{ed}_p[N/G] \leqslant \mathrm{ed}_p\, G + \dim N.$$

4. p-exhaustive subgroups

From now on all group schemes under consideration are assumed to be affine.

Definition 4.1. Let $p \in \mathbb{P} \cup \{0\}$. Let G be a group scheme over F and C be a normal subgroup scheme. Set $H = G/C$. We say that an H-torsor X over some extension $K \in \mathrm{Fields}_F$ is *p-exhaustive* (with respect to C and G) if the inequality

$$\mathrm{ed}_p[X/G] \leqslant \mathrm{ed}_p G + \dim H$$

from Example 3.3 is an equality.

We say that C is a *p-exhaustive* (normal) subgroup of G if a p-exhaustive H-torsor X exists.

Clearly G itself is always a p-exhaustive subgroup, for any $p \in \mathbb{P} \cup \{0\}$. However, there may exist smaller p-exhaustive subgroups. We make the following observation.

Lemma 4.2. *Let G be a group scheme over F. Let C be a p-exhaustive subgroup of G. Then every normal subgroup D of G containing C is p-exhaustive as well.*

Proof. Set $H := G/C$ and $Q := G/D$. Let X be a p-exhaustive H-torsor; i.e.,

$$\mathrm{ed}_p[X/G] = \mathrm{ed}_p G + \dim H.$$

Let $h\colon H \to Q$ the canonical surjective morphism. We will show that the induced Q-torsor $Y = h_*(X)$ is p-exhaustive. By inequality (5) of Example 3.3 we have

$$\mathrm{ed}_p[X/G] \leqslant \mathrm{ed}_p[Y/G] + \dim H - \dim Q.$$

Therefore

$$\mathrm{ed}_p[Y/G] \geqslant \mathrm{ed}_p G + \dim Q.$$

Since the opposite inequality always holds the claim follows. □

If C is a central subgroup of G isomorphic to $(\mu_p)^r$ for some $r \geqslant 0$, we can use a result of Karpenko and Merkurjev [KM08] to compute, at least in principle, the essential p-dimension of $[E/G]$ for every $H = G/C$-torsor E over some field extension K. Denote by $\beta^E\colon \mathrm{Hom}(C, \mu_p) \to \mathrm{Br}(K)$ the group homomorphism, which takes a character χ to the image of the class of E under the map

$$H^1(K, H) \to H^2(K, C(C)) \xrightarrow{\chi_*} H^2(K, \mu_p) = \mathrm{Br}_p(K).$$

Then, by [KM08] (cf. [Mer09, Example 3.7]),

$$\mathrm{ed}_p[E/G] = \min\left\{\sum_{\chi \in B} \mathrm{ind}\, \beta^E(\chi)\right\},$$

where the minimum runs over all bases B of $\mathrm{Hom}(C, \mu_p) \simeq (\mathbb{Z}/p\mathbb{Z})^r$.

The case $r = 1$ is due to [BRV11]. Note that, in this case, $\mathrm{ed}_p[E/G]$ is the index of $\beta^E(\chi)$ for any generator χ of $\mathrm{Hom}(C, \mu_p) \simeq \mathbb{Z}/p\mathbb{Z}$.

We remark that by [KM08, Theorem 4.4 and Remark 4.5] the indices arising in these formulas can be expressed in representation theoretic terms. Namely, $\mathrm{ind}\, \beta^E(\chi)$ is the greatest common divisor $\gcd \dim \rho$ taken over all (irreducible) representations ρ of G for which C acts via multiplication by χ. However, we will not use this description in the following.

In several recent papers about essential dimension, p-exhaustive central subgroups of the form $(\mu_p)^r$ have been used (implicitly) to compute the exact value of the essential p-dimension

$\mathrm{ed}_p\, G$ for some classes of group schemes G whose center is of multiplicative type. Recall from [LMMR11, p. 4] that we can associate with G a subgroup $C(G)$ which is the (uniquely determined) largest central subgroup of G of the form $(\mu_p)^r$, $r \geqslant 0$. The center $Z(G)$ of G or even the subgroup $C(G)$ is p-exhaustive in several cases, summarized in the following list.

Example 4.3. Let p be a prime. For the following group schemes G the subgroup $C(G)$ (exists and) is p-exhaustive.

(i) Let G be a group scheme of multiplicative type which splits over a Galois field extension of p-power degree. See [LMMR11, Theorem 1.1] and its proof.

(ii) Let G be a p-group over a field of characteristic not equal to p and which contains a primitive pth root of unity such that G becomes constant over a Galois field extension of p-power degree. See [LMMR11, Theorem 7.1] and its proof. The case where G is constant is contained in [KM08].

(iii) Let $G = \mathbf{Spin}_n$ (for $p = 2$) over a field of characteristic 0, where $n \geqslant 15$. This is due to [BRV10, Mer09] and a result of Chernousov and Merkurjev. We will give more details below.

(iv) Let $G = \mathbf{HSpin}_n$ (for $p = 2$) over a field of characteristic 0, where $n \geqslant 20$ is divisible by 4. See [BRV10].

Let A be a division-algebra of p-power degree over its center. For the following group schemes G the center $Z(G)$ is p-exhaustive.

(v) Let G be the normalizer $G = N_{\mathbf{GL}_1(A)}(\mathbf{GL}_1(B))$ where B is a separable subalgebra of A and $Z(A) = F$. See [Lot11]. Here $Z(G) \simeq \mathbf{G}_m$.

(vi) Let $G = \mathbf{Sim}(A, \sigma)$ (with $p = 2$), where σ is an involution on A with $Z(A)^\sigma = F$. See [Lot11]. Here $Z(G) \simeq \mathbf{G}_m$ if σ is of the first kind, and $Z(G) \simeq R_{K/F}(\mathbf{G}_m)$ with $K = Z(A)$ separable of degree 2 over F if σ is of the second kind.

(vii) Let $G = \mathbf{Iso}(A, \sigma)$ (with $p = 2$), where $Z(A) = F$ and σ is an involution of the first kind on A. See [Lot11].

(viii) Let $G = \mathbf{GO}(A, \sigma, f)$, $\mathbf{O}(A, \sigma, f)$ or, if $r \geqslant 2$, $\mathbf{GO}^+(A, \sigma, f)$ or $\mathbf{O}^+(A, \sigma, f)$, where (σ, f) is a quadratic pair on A and $Z(A) = F$, $p = 2$, $\mathrm{char}\, F = 2$.

Remark 4.4. For a general normal subgroup C of G we do not know if the maximal value of $\mathrm{ed}_p[E/G]$ is reached for a *versal* $H = G/C$-torsor (in the sense of [BF03]). However, if C is central and $C \simeq (\mu_p)^r$ for a prime p, this is true. In particular, C is p-exhaustive if and only if for any versal torsor E we have $\mathrm{ed}_p[E/G] = \mathrm{ed}_p\, G + \dim G$. Let us prove this. By the above formula it suffices to show that for E versal

$$\mathrm{ind}\, \beta^E(\chi) \geqslant \mathrm{ind}\, \beta^{E'}(\chi)$$

for every $\chi \in \mathrm{Hom}(C, \mu_p)$ and every other H-torsor E' over some field extension of F.

Recall that a versal torsor E over some field extension K is defined as the generic fiber of a *classifying* H-torsor $\pi\colon X \to Y$ (here Y is an irreducible scheme over F). The H-torsor $\pi\colon X \to Y$ is, by definition, classifying if, for every field extension F'/F with F' infinite and for every H-torsor E' over F', the set of points $y \in Y(F')$ such that $X_y \simeq E'$ is dense in Y.

There is an Azumaya algebra $\mathcal{A}$ over Y such that, for $K \in \mathrm{Fields}_F$ and $y \in Y(K)$, $E' = X_y$, the class of $\beta^{E'}(\chi)$ is represented by the Azumaya K-algebra $\mathcal{A}_y$; see [KM08, Lemma 4.3] and its proof. In particular, $\beta^E(\chi)$ is represented by the generic fiber of $\mathcal{A}$.

Now let D be a central division $F(Y)$-algebra representing the class of $\beta^E(\chi)$. Then we can lift D to some Azumaya-algebra $\mathcal{B}$ of constant degree equal to $\deg D = \mathrm{ind}\, \beta^E(\chi)$ over some non-empty open subset U of Y. Shrinking U if necessary allows us to assume that $\mathcal{B}$ is Brauer-equivalent to $\mathcal{A}_U$.

Let E' be another H-torsor over some extension $K \in \mathrm{Fields}_F$. In order to prove the claim, we can replace E' by $E' \times \operatorname{Spec} K(T)$ if necessary and thus assume that K is infinite. Since π is classifying, there exists a point $y \in U(K)$ such that the fiber of π over y is isomorphic to E'. Therefore $\mathcal{B}_y$ is Brauer equivalent to $\mathcal{A}_y$, which represents the class $\beta^{E'}(\chi)$ in $\mathrm{Br}(K)$. This implies that $\operatorname{ind} \beta^{E'}(\chi) \leqslant \deg \mathcal{B}_y = \operatorname{ind} \beta^{E}(\chi)$. The claim follows.

We will prove two general lemmas on the behavior of p-exhaustive central subgroups of the form $(\mu_p)^r$. The first one generalizes the additivity theorem from [LMMR12, Theorem 8.1].

Lemma 4.5. *Let p be a prime and G_1, G_2 be group schemes over F. Let $C_1 \simeq \mu_p^{r_1}$ and $C_2 \simeq \mu_p^{r_2}$ be central subgroups of G_1 and G_2, respectively. Set $H_1 = G_1/C_1$, $H_2 = G_2/C_2$. Let E be a versal $H_1 \times H_2$-torsor over some extension $K \in \mathrm{Fields}_F$. Write $E \simeq E_1 \times E_2$, where E_i is an H_i-torsor, for $i = 1, 2$. Then*

$$\mathrm{ed}_p[E/(G_1 \times G_2)] = \mathrm{ed}_p[E_1/G_1] + \mathrm{ed}_p[E_2/G_2].$$

In particular, if C_1 and C_2 are p-exhaustive, then $C_1 \times C_2$ is p-exhaustive as well and

$$\mathrm{ed}_p\, G_1 \times G_2 = \mathrm{ed}_p\, G_1 + \mathrm{ed}_p\, G_2.$$

Proof. Set $G = G_1 \times G_2$, $H = H_1 \times H_2$, $C = C_1 \times C_2$. Choose a basis B of $\mathrm{Hom}(C, \mu_p)$ such that

$$\mathrm{ed}_p[E/G] = \sum_{\chi \in B} \operatorname{ind} \beta^E(\chi).$$

By elementary linear algebra there exists a partition $B = B_1 \coprod B_2$ such that the image of B_j under the projection $\pi_j \colon \mathrm{Hom}(C, \mu_p) = \mathrm{Hom}(C_1, \mu_p) \times \mathrm{Hom}(C_2, \mu_p) \to \mathrm{Hom}(C_j, \mu_p)$ is a basis of $\mathrm{Hom}(C_j, \mu_p)$, for both $j = 1, 2$.

Let T_1 denote the trivial H_1-torsor. Then $\beta^{T_1 \times E_2}(\chi) = \beta^{E_2}(\pi_2(\chi))$. Therefore by Remark 4.4, $\operatorname{ind} \beta^E(\chi) \geqslant \operatorname{ind} \beta^{E_2}(\pi_2(\chi))$, for every $\chi \in \mathrm{Hom}(C, \mu_p)$. Similarly $\operatorname{ind} \beta^E(\chi) \geqslant \operatorname{ind} \beta^{E_1}(\pi_1(\chi))$. We conclude that

$$\begin{aligned}\mathrm{ed}_p[E/G] &\geqslant \sum_{\varphi \in \pi_1(B)} \operatorname{ind} \beta^{E_1}(\varphi) + \sum_{\psi \in \pi_2(B)} \operatorname{ind} \beta^{E_2}(\psi) \\ &\geqslant \mathrm{ed}_p[E_1/G_1] + \mathrm{ed}_p[E_2/G_2] \geqslant \mathrm{ed}_p([E_1/G_1] \times [E_2/G_2]) = \mathrm{ed}_p[E/G].\end{aligned}$$

Therefore $\mathrm{ed}_p[E/G] = \mathrm{ed}_p[E_1/G_1] + \mathrm{ed}_p[E_2/G_2]$ as claimed.

Now assume that C_j is p-exhaustive in G_j, for $j = 1, 2$. It is easy to see that E_j is a versal H_j-torsor, for $j = 1, 2$. Therefore in view of Remark 4.4, $\mathrm{ed}_p[E_j/G_j] = \mathrm{ed}_p\, G_j + \dim H_j$. Hence

$$\mathrm{ed}_p[E/G] = \mathrm{ed}_p\, G_1 + \mathrm{ed}_p\, G_2 + \dim H \geqslant \mathrm{ed}_p\, G + \dim H.$$

It follows that C is p-exhaustive and $\mathrm{ed}_p\, G_1 + \mathrm{ed}_p\, G_2 = \mathrm{ed}_p\, G$. □

Lemma 4.6. *Let G be a group scheme over F and $C \simeq (\mu_p)^r$ a central subgroup of rank $r \geqslant 1$. Assume that for all but at most $r - 1$ index p subgroups D of C the subgroup C/D of G/D is p-exhaustive. Then C is a p-exhaustive subgroup of G.*

Proof. Set $H = G/C$. Let E be a versal $H = G/C$-torsor over some extension $K \in \mathrm{Fields}_F$. Choose a basis B of $\mathrm{Hom}(C, \mu_p) \simeq (\mathbb{Z}/p\mathbb{Z})^r$ such that

$$\mathrm{ed}_p[E/G] = \sum_{\chi \in B} \operatorname{ind} \beta^E(\chi). \tag{7}$$

We will first show that, for any $D = \ker \chi_0$ with $\chi_0 \in B$,

$$\mathrm{ed}_p[E/G] = \mathrm{ed}_p[E/(G/D)] + \sup_{X \text{ lifting } E} \mathrm{ed}_p[X/G] = \mathrm{ed}_p[E/(G/D)] + \sup_{X} \mathrm{ed}_p[X/G], \tag{8}$$

where, on the right, X runs over all G/D-torsors over field extensions of K and, on the left, X runs only over all G/D-torsors over field extensions of K that lift E.

Note that the leftmost expression in equation (8) is less than or equal to the middle expression, by Example 3.4, and therefore less than or equal to the rightmost expression.

For every field extension L/F we have a commutative diagram as follows.

$$\begin{tikzcd}
H^1(L, G/D) \arrow[r] \arrow[d] & H^2(L, D) \arrow[d] & \\
H^1(L, H) \arrow[r] \arrow[d, equal] & H^2(L, C) \arrow[r, "(\chi_0)_*"] \arrow[d] & H^2(L, \mu_p) \arrow[d, equal] \\
H^1(L, H) \arrow[r] & H^2(L, C/D) \arrow[r, "\simeq"] & H^2(L, \mu_p)
\end{tikzcd}$$

In particular, it follows that

$$\mathrm{ed}_p[E/(G/D)] = \mathrm{ind}\, \beta^E(\chi_0). \tag{9}$$

Let X be a G/D-torsor over $L \in \mathrm{Fields}_K$ and let $\bar{X}$ be the induced H-torsor. For $\chi \in \mathrm{Hom}(C, \mu_p)$, the image of $\chi|_D$ under $\beta^X : \mathrm{Hom}(D, \mu_p) \to \mathrm{Br}(L)$ coincides with $\beta^{\bar{X}}(\chi)$. Since the characters $\chi|_D$ with $\chi \in B \backslash \{\chi_0\}$ form a basis of $\mathrm{Hom}(D, \mu_p)$,

$$\mathrm{ed}_p[X/G] \leqslant \sum_{\chi \in B \backslash \{\chi_0\}} \mathrm{ind}\, \beta^{\bar{X}}(\chi) \leqslant \sum_{\chi \in B \backslash \{\chi_0\}} \mathrm{ind}\, \beta^{E}(\chi). \tag{10}$$

The combination of (7), (9) and (10) implies that the rightmost expression in (8) is less than or equal to the leftmost expression. Therefore we have proven (8).

By assumption, there is at least one subgroup $D = \ker \chi_0$ with $\chi_0 \in B$ such that the subgroup C/D of G/D is p-exhaustive. For such D we get, with Remark 4.4,

$$\mathrm{ed}_p[E/(G/D)] = \mathrm{ed}_p\, G/D + \dim G/D = \mathrm{ed}_p\, G/D + \dim H. \tag{11}$$

Example 3.4(3.4) implies

$$\mathrm{ed}_p\, G \leqslant \mathrm{ed}_p\, G/D + \sup \mathrm{ed}_p[X/G], \tag{12}$$

where the supremum is taken over all G/D-torsors X over field extensions of K. Combining (8), (11) and (12) shows

$$\mathrm{ed}_p\, G + \dim H \leqslant \mathrm{ed}_p[E/G].$$

Hence the claim follows. □

We will now consider spin groups for application. The essential dimension of spin groups has been the subject of investigation in several articles, including [BRV11, CS06, Mer09, Ros99]. Assume char $F \neq 2$. Let $\mathbf{Spin}_n$ denote the spin group for a maximally isotropic non-degenerate quadratic form of dimension n. The essential dimension of $\mathbf{Spin}_n$ for $n \leqslant 14$ has been computed by Rost [Ros99]; see also [Gar09]. Then came Brosnan *et al.* [BRV11] who established a strong lower bound on $\mathbf{Spin}_n$ for any $n \geqslant 15$ using essential dimension of algebraic stacks, basically applying inequality (3) from Example 3.3 to the surjective homomorphism $\mathbf{Spin}_n \to \mathbf{O}_n^+$ with kernel μ_2. For fields of characteristic 0 they also proved an upper bound using generically free representations. In the case $n \not\equiv 0 \pmod 4$ their lower bound matched the upper bound.

Then came Merkurjev [Mer09], who improved the lower bound in the case $n \equiv 0 \pmod 4$, by considering the surjective homomorphism $\mathbf{Spin}_n \to \mathbf{PGO}_n^+$ with kernel $\mu_2 \times \mu_2$ instead. This bound matched the upper bound from [BRV11] when n is a power of 2. At the RAGE conference in Atlanta 2011 Merkurjev also showed how to improve the upper bound in the case $n \equiv 0 \pmod 4$ by relating the essential dimension of $\mathbf{Spin}_n$ with the essential dimension of the semi-spinor group $\mathbf{HSpin}_n$. As communicated to the author by Merkurjev, this result will appear in a joint preprint with Chernousov; see [CM12]. This upper bound can be seen as a special case of Example 3.4(3.4) for the morphism $f\colon \mathbf{Spin}_n \to \mathbf{HSpin}_n$. Again the two bounds match. Thus $\operatorname{ed} \mathbf{Spin}_n$ is known for any field of characteristic 0. We refer to [Mer09, § 4.3] for the list of values.

Since the new upper bound of Chernousov and Merkurjev for $\operatorname{ed} \mathbf{Spin}_n$, $n \geqslant 20$ divisible by 4, is such a natural application of Theorem 1.1 we will reproduce it below. Also we feel that non-split spin groups have been excluded unnecessarily for investigation so far, so we would like to fill this gap.

We will entirely focus on the case $n \equiv 0 \pmod 4$, since the other cases can be treated with published results. Moreover, we will always assume that (σ, f) has trivial discriminant. The case where $n \equiv 0 \pmod 4$ and (σ, f) has non-trivial discriminant looks more difficult.

Let (A, σ, f) be a quadratic pair over F with $n := \deg A$ divisible by 4. We assume that (σ, f) has a trivial discriminant. In other words the center $Z = Z(C(A, \sigma, f))$ of the Clifford algebra of (A, σ, f) is isomorphic to $F \times F$. We have an inclusion

$$\mathbf{Spin}(A, \sigma, f) \subseteq R_{Z/F}(\mathbf{GL}_1(C(A, \sigma, f))) = \mathbf{GL}_1(C^+(A, \sigma, f)) \times \mathbf{GL}_1(C^-(A, \sigma, f)).$$

The center of $\mathbf{Spin}(A, \sigma, f)$ is $\mu_2 \times \mu_2$. We denote the image of $\mathbf{Spin}(A, \sigma, f)$ in the first (respectively second) component by $\mathbf{Spin}^+(A, \sigma, f)$ (respectively $\mathbf{Spin}^-(A, \sigma, f)$). In other words $\mathbf{Spin}^+(A, \sigma, f)$ is the quotient of $\mathbf{Spin}(A, \sigma, f)$ by the central subgroup $\{1\} \times \mu_2$. Similarly $\mathbf{Spin}^-(A, \sigma, f)$ is the quotient of $\mathbf{Spin}(A, \sigma, f)$ by $\mu_2 \times \{1\}$. Note that, unlike the split case, these two groups do not need to be isomorphic.

The quotient of $\mathbf{Spin}(A, \sigma, f)$ by the diagonal subgroup of $\mu_2 \times \mu_2$ is $\mathbf{O}^+(A, \sigma, f)$. The quotient of $\mathbf{Spin}(A, \sigma, f)$ by the full center $\mu_2 \times \mu_2$ is $\mathbf{PGO}^+(A, \sigma, f)$.

PROPOSITION 4.7. *Assume* $\operatorname{char} F \neq 2$. *Then*

$$\begin{aligned} d^+ &:= \sup \operatorname{ed}_2[E/\mathbf{Spin}^+(A, \sigma, f)] = 2^{(n-2)/2} \operatorname{ind} C^+(A, \sigma, f), \\ d^- &:= \sup \operatorname{ed}_2[E/\mathbf{Spin}^-(A, \sigma, f)] = 2^{(n-2)/2} \operatorname{ind} C^-(A, \sigma, f), \\ d &:= \sup \operatorname{ed}_2[E/\mathbf{O}^+(A, \sigma, f)] = 2^{\nu_2(n)+\nu_2(\operatorname{ind} A)}, \\ &\sup \operatorname{ed}_2[E/\mathbf{Spin}(A, \sigma, f)] = \min\{d + d^+, d + d^-, d^+ + d^-\}, \end{aligned}$$

where E runs over all $\mathbf{PGO}^+(A, \sigma, f)$-torsors E over field extensions of F. These values are attained for a versal $\mathbf{PGO}^+(A, \sigma, f)$-torsor E.

Furthermore, if $n \geqslant 20$ then

$$\sup \operatorname{ed}_2[E/\mathbf{Spin}(A, \sigma, f)] = \min\{d^+, d^-\} + d.$$

Proof. For a field extension K/F the fppf-cohomology set $H^1(K, \mathbf{PGO}^+(A, \sigma, f))$ is in natural bijection with isomorphism classes of quadruples (B, τ, g, φ) where B is a central simple

K-algebra of degree $\deg B = \deg A$, (τ, g) is a quadratic pair on B, and φ is an isomorphism $Z(C(B,\tau,g)) \xrightarrow{\sim} K \times K$. The connecting map associated with the exact sequence $1 \to \mu_2 \times \mu_2 \to \mathbf{Spin}(A,\sigma,f) \to \mathbf{PGO}^+(A,\sigma,f) \to 1$ takes the isomorphism class of (B,τ,g,φ) to the element

$$([C^+(B,\tau,g)] - [C^+(A,\sigma,f)_K], [C^-(B,\tau,g)] - [C^-(A,\sigma,f)_K])$$

in $\mathrm{Br}_2(K) \times \mathrm{Br}_2(K) = H^2(K, \mu_2 \times \mu_2)$, where $C^+(B,\tau,g) = C(B,\tau,g)\varphi^{-1}(1,0)$ and $C^-(B,\tau,g) = C(B,\tau,g)\varphi^{-1}(0,1)$ are the two components of $C(B,\tau,g)$ labeled with respect to φ (cf. [KMRT98, Exercise VII.15]).

Similarly, the connecting maps associated with the exact sequences

$$1 \to \mu_2 \to G \to \mathbf{PGO}^+(A,\sigma,f) \to 1,$$

for $G = \mathbf{Spin}^+(A,\sigma,f), \mathbf{Spin}^-(A,\sigma,f), \mathbf{O}^+(A,\sigma,f)$, takes the class of (B,τ,g,φ) to $[C^+(B,\tau,g)] - [C^+(A,\sigma,f)_K]$, $[C^-(B,\tau,g)] - [C^-(A,\sigma,f)_K]$ and $[B] - [A_K]$, respectively.

We always have $\operatorname{ind} B \leqslant 2^{\nu_2(n)}$ and $\operatorname{ind} C^\delta(B,\tau,g) \leqslant 2^{(n-2)/2}$. By [MPW96, (5.49)] (here we use the assumption $\operatorname{char} F \neq 2$), there exists a quadruple (B,τ,g,φ) as above such that, for every central simple F-algebra D,

$$\begin{aligned}\operatorname{ind}(D \otimes_F C^\delta(B,\tau,g)) &= 2^{(n-2)/2} \operatorname{ind} D, \quad \forall \delta \in \{+,-\},\\ \operatorname{ind}(D \otimes_F B) &= 2^{\nu_2(n)} \operatorname{ind} D.\end{aligned}$$

In particular,

$$\begin{aligned}d^\delta &= \operatorname{ind}(C^\delta(A,\sigma,f)^{\mathrm{op}} \otimes_F C^\delta(B,\tau,g)) = 2^{(n-2)/2} \operatorname{ind}(C^\delta(A,\sigma,f)), \quad \forall \delta \in \{+,-\},\\ d &= \operatorname{ind}(A^{\mathrm{op}} \otimes_F B) = 2^{\nu_2(n)+\nu_2(\operatorname{ind} A)}.\end{aligned}$$

Moreover, it follows that

$$\sup \mathrm{ed}_2[E/\mathbf{Spin}(A,\sigma,f)] = \min\{d + d^+, d + d^-, d^+ + d^-\}.$$

Now assume $n \geqslant 20$. Then $4\nu_2(n) \leqslant n-2$, and hence

$$d \leqslant 2^{2\nu_2(n)} \leqslant 2^{(n-2)/2} \leqslant \min\{d^+, d^-\}.$$

Thus $\min\{d + d^+, d + d^-, d^+ + d^-\} = \min\{d^+, d^-\} + d$. □

Proposition 4.8. *Assume* $\operatorname{char} F \neq 2$.

(i) *Let* $\delta \in \{+,-\}$ *and let* $m = \operatorname{ind}(C^\delta(A,\sigma,f))$. *Suppose that the* m*-fold direct sum of the canonical representation of* $\mathbf{HSpin}_n$ *is generically free. Then the center* μ_2 *of* $\mathbf{Spin}^\delta(A,\sigma,f)$ *is a 2-exhaustive subgroup of* $\mathbf{Spin}^\delta(A,\sigma,f)$. *Moreover,*

$$\mathrm{ed}_2\, \mathbf{Spin}^\delta(A,\sigma,f) = 2^{(n-2)/2} m - \frac{n(n-1)}{2}.$$

(ii) *Suppose that* A *is a division algebra. Then the center* μ_2 *of* $\mathbf{O}^+(A,\sigma,f)$ *is a 2-exhaustive subgroup of* $\mathbf{O}^+(A,\sigma,f)$. *Moreover,*

$$\mathrm{ed}_2\, \mathbf{O}^+(A,\sigma,f) = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Proof. (i) Let D be a division F-algebra, representing the Brauer class of $C^\delta(A,\sigma,f)$. We have a representation arising from the composition

$$\rho\colon \mathbf{Spin}^\delta(A,\sigma,f) \hookrightarrow \mathbf{GL}_1(C^\delta(A,\sigma,f)) \hookrightarrow \mathbf{GL}_1(C^\delta(A,\sigma,f) \otimes_F D^{\mathrm{op}}) \xrightarrow{\simeq} \mathbf{GL}_N$$

with $N = 2^{(n-2)/2}m$. Over F_{sep}, this representation decomposes as the m-fold direct sum of the canonical $\mathbf{HSpin}_n$-representation, which is generically free by assumption. Hence ρ is generically free as well. Therefore $\mathrm{ed}_2\, \mathbf{Spin}^{\delta}(A, \sigma, f) \leqslant N - \dim \mathbf{Spin}^{\delta}(A, \sigma, f)$ by [BF03, Proposition 4.11]. Combining this inequality with Proposition 4.7 shows that the center of $\mathbf{Spin}^{\delta}(A, \sigma, f)$ is 2-exhaustive and gives us the value of $\mathrm{ed}_2\, \mathbf{Spin}^{\delta}(A, \sigma, f)$.

(ii) Since $\mathbf{O}^+(A, \sigma, f)$ is a subgroup of the group $\mathbf{GL}_1(A)$ of essential dimension 0 we have $\mathrm{ed}_2\, \mathbf{O}^+(A, \sigma, f) \leqslant \dim \mathbf{GL}_1(A) - \dim \mathbf{O}^+(A, \sigma, f) = n(n+1)/2$ (by [BF03, Theorem 6.19] or Example 3.4(3.4)). Now the claim follows again from Proposition 4.7. □

Remark 4.9. Let $m \geqslant 1$. In the following cases, the m-fold direct sum of the canonical representation of $\mathbf{HSpin}_n$ is generically free:

(i) $m \geqslant 2^{(n-2)/2}$, which is the dimension of the canonical representation of $\mathbf{HSpin}_n$;

(ii) for char $F = 0$ and $m \geqslant 8$;

(iii) for char $F = 0$ and $m \geqslant 2$ if $n \geqslant 16$;

(iv) for char $F = 0$ and m arbitrary if $n \geqslant 20$.

The first case is obvious, since $\mathbf{GL}_m$ acts generically freely on the m-fold direct sum of its canonical m-dimensional representation. The other cases follow from [PV94, Theorems 8.8 and 8.9]. We do not know if the assumption char $F = 0$ can be dropped or not.

Combining Proposition 4.8 and Remark 4.9 with Lemma 4.6 we can compute the essential 2-dimension of $\mathbf{Spin}(A, \sigma, f)$ in many cases. In particular, we get the following result.

COROLLARY 4.10. *Assume* char $F \neq 2$. *Set* $d = 2^{\nu_2(n)+\nu_2(\mathrm{ind}(A))}$, $d^+ = 2^{(n-2)/2}\, \mathrm{ind}\, C^+(A, \sigma, f)$ *and* $d^- = 2^{(n-2)/2}\, \mathrm{ind}\, C^+(A, \sigma, f)$ *as in Proposition* 4.7. *In the following cases, the center* $\mu_2 \times \mu_2$ *of* $\mathbf{Spin}(A, \sigma, f)$ *is 2-exhaustive and*

$$\mathrm{ed}_2\, \mathbf{Spin}(A, \sigma, f) = \min\{d + d^+, d + d^-, d^+ + d^-\} - \frac{n(n-1)}{2}.$$

(i) *At least two of the algebras* A, $C^+(A, \sigma, f)$ *and* $C^-(A, \sigma, f)$ *are division.*

(ii) char $F = 0$ *and* $n \geqslant 20$. *Here the formula simplifies to*

$$\mathrm{ed}_2\, \mathbf{Spin}(A, \sigma, f) = \min\{d^+, d^-\} + d - \frac{n(n-1)}{2}.$$

(iii) char $F = 0$ *and both* $C^+(A, \sigma, f)$ *and* $C^-(A, \sigma, f)$ *have index at least* 8.

(iv) char $F = 0$, $n \geqslant 16$, *and* A *is a division algebra or none of* $C^+(A, \sigma, f)$ *and* $C^-(A, \sigma, f)$ *is split.*

Remark 4.11. All results from Proposition 4.8 and Corollary 4.10 hold with ed_2 replaced by ed. For the lower bounds this is clear and for the upper bound only very slight modifications in the proofs are needed.

Remark 4.12. In the case $n = 8$ the result of Corollary 4.10(i) can be improved. It suffices that two of the three algebras (all of degree 8) have index greater than or equal to 4. This follows from the fact that the 4-fold direct sum of the representation

$$\mathbf{Spin}_8 \hookrightarrow \mathbf{O}_8^+ \times \mathbf{O}_8^+ \hookrightarrow \mathbf{GL}_{16}$$

is generically free, which can easily be checked.

Moreover, in the case, char $F = 0$, $n = 16$, no assumptions on the indices are really needed. This follows from the fact that the representation

$$\mathbf{Spin}_{16} \hookrightarrow \mathbf{HSpin}_{16} \times \mathbf{O}^+_{16} \hookrightarrow \mathbf{GL}_{128} \times \mathbf{GL}_{16} \hookrightarrow \mathbf{GL}_{144}$$

is already generically free; see [BRV10, p. 5].

5. Canonical dimension of group schemes

In the following theorem, we reveal a relation between canonical and essential dimension of group schemes for p-exhaustive subgroups, introduced in § 4.

Theorem 5.1. *Let $p \in \mathbb{P} \cup \{0\}$. Let G be a group scheme over F and let C be a p-exhaustive subgroup of G. Let $H = G/C$ and X be a p-exhaustive H-torsor over some field extension K/F. Then*

$$\operatorname{cdim}_p \mathbf{Aut}_H(X) \geqslant \dim H - \sup \operatorname{ed}_p {}^Z C,$$

where the supremum is taken over all field extensions L/K and all lifts of X to a G-torsor Z over L.

In particular, if C is central then

$$\operatorname{cdim}_p \mathbf{Aut}_H(X) \geqslant \dim H - \operatorname{ed}_p C,$$

and if H is abelian then

$$\operatorname{cdim}_p H \geqslant \dim H - \operatorname{ed}_p {}^X C,$$

and if C is central and H abelian then

$$\operatorname{cdim}_p H \geqslant \dim H - \operatorname{ed}_p C.$$

Proof. Since X is p-exhaustive we have

$$\operatorname{ed}_p[X/G] = \operatorname{ed}_p G + \dim H. \tag{13}$$

By inequality (4) of Example 3.3,

$$\operatorname{cdim}_p[X/G] \leqslant \operatorname{ed}_p G + \operatorname{cdim}_p \mathbf{Aut}_H(X). \tag{14}$$

Corollary 3.2 yields the inequality

$$\operatorname{ed}_p[X/G] \leqslant \operatorname{cdim}_p[X/G] + \sup \operatorname{ed}_p {}^Z C. \tag{15}$$

Combining (13)–(15) yields the desired inequality. ☐

Remark 5.2. Suppose, given a group scheme G over F and a prime p, we want to study the question of whether the subgroup $C(G) \simeq (\mu_p)^r$ (from above) is p-exhaustive. Theorem 5.1 gives an obstruction to an affirmative answer to this question. Namely $C(G)$ can only be p-exhaustive if one of the twisted inner forms $H' = \mathbf{Aut}_H(X)$ of H has $\operatorname{cdim}_p H' \geqslant \dim H - r$.

Combining Theorem 5.1 with items (iii), (v) and (vi) of Example 4.3 we get the following results.

Corollary 5.3. (i) *Let $n \geqslant 15$ with $n \not\equiv 0 \pmod 4$. Assume* char $F = 0$. *Then there exists an n-dimensional quadratic form q of trivial discriminant over some field extension of F such that*

$$\operatorname{cdim}_2 \mathbf{O}^+(q) \geqslant \dim \mathbf{O}^+(q) - 1 = \frac{n(n-1)}{2} - 1.$$

(ii) *Let $n \geqslant 15$ with $n \equiv 0 \pmod 4$. Assume* char $F = 0$. *Then there exist a central simple algebra of degree n over some field extension of F and an orthogonal involution σ on A such that*

$$\operatorname{cdim}_2 \mathbf{PGO}^+(A,\sigma) \geqslant \dim \mathbf{PGO}^+(A,\sigma) - 2 = \frac{n(n-1)}{2} - 2.$$

(iii) *Let p be a prime and let $a, b, n \geqslant 0$ be integers with $a+b \leqslant n$. Then there exist a central simple algebra A of degree p^n over some field extension K of F and a separable subalgebra B of A such that $B \otimes_K K_{\mathrm{sep}} \simeq M_{p^a}(K_{\mathrm{sep}}) \times \cdots \times M_{p^a}(K_{\mathrm{sep}})$, $C_A(B) \otimes_K K_{\mathrm{sep}} \simeq M_{p^b}(K_{\mathrm{sep}}) \times \cdots \times M_{p^b}(K_{\mathrm{sep}})$ (both p^{n-a-b} times) and*

$$\operatorname{cdim}_p \mathbf{Aut}_K(A,B) = \dim \mathbf{Aut}_K(A,B) = p^{n+a-b} + p^{n-a+b} - p^{n-a-b} - 1.$$

(iv) *Let $n = 2^r$ for some $r \geqslant 1$. Then there exist a central simple algebra A of degree n over some field extension K of F and an involution σ of orthogonal (respectively symplectic) type on A such that*

$$\operatorname{cdim}_2 \mathbf{Aut}_K(A,\sigma) = \dim \mathbf{Aut}_K(A,\sigma) = \begin{cases} \dfrac{n(n-1)}{2} & \text{if } \sigma \text{ is orthogonal,} \\ \dfrac{n(n+1)}{2} & \text{if } \sigma \text{ is symplectic.} \end{cases}$$

(v) *Let $n = 2^r$ for some $r \geqslant 0$ and let K/F be a separable quadratic extension. Then there exist a field extension L/F linearly disjoint from K/F, a central simple $M := L \otimes_F K$-algebra A of degree n and a unitary L-linear involution σ on A such that*

$$\operatorname{cdim}_2 \mathbf{Aut}_M(A,\sigma) = \dim \mathbf{Aut}_M(A,\sigma) = n^2 - 1.$$

Remark 5.4. The split forms of the groups appearing in Corollary 5.3 usually have clearly lower canonical p-dimension, e.g. for the special orthogonal groups

$$\operatorname{cdim}_2 \mathbf{O}^+_{2n+1} = \operatorname{cdim}_2 \mathbf{O}^+_{2n+2} = \frac{n(n+1)}{2},$$

which was conjectured in [BR05] and proven independently in [Kar05] and [Vis05]. This value should be compared with the values $n(2n+1) - 1$ (respectively $(n+1)(2n+1) - 1$) for the quadratic forms q of dimension $2n+1$ and $2n+2$, respectively, from part (i) of the corollary.

Another example is the group $\mathbf{Aut}_K(A,B)$ from part (iii) of the corollary, where A is a central division K-algebra of degree $d = p^n$, B is a maximal étale subalgebra of A and $\operatorname{cdim}_p \mathbf{Aut}_K(A,B) = \dim \mathbf{Aut}_K(A,B) = d - 1$. Here the split form $\mathbf{Aut}_K(M_d(K), K^d) \simeq \mathbf{G}_m^d/\mathbf{G}_m \rtimes S_d$ has canonical p-dimension equal to 0 (this follows from [KM06, Remark 3.7], since the maps $H^1(-, \mathbf{G}_m^d/\mathbf{G}_m \rtimes S_d) \to H^1(-, S_d)$ of pointed sets have a trivial kernel and since $\operatorname{cdim}_p S_d = 0$).

Now we turn our attention to the case of groups of multiplicative type.

COROLLARY 5.5. *Let G be a group scheme of multiplicative type which splits over a Galois extension of p-power degree. Let C be any subgroup of G containing $C(G)$ and set $H = G/C$. Then, for every $p \in \mathbb{P} \cup \{0\}$,*

$$\operatorname{cdim}_p H \geqslant \dim H - \operatorname{ed}_p C.$$

Proof. As recorded in Example 4.3(i), the subgroup $C(G)$ of G is p-exhaustive. Hence, by Lemma 4.2, C is p-exhaustive as well. The claim now follows from Theorem 5.1. This proves Corollary 5.5 and hence Theorem 1.2 from the introduction. □

Example 5.6. Let L/F be a field extension such that the normal closure of L has p-power degree over F. Let K be an intermediate field of the extension L/F. Let $T := R_{L/F}(\mathbf{G}_m)/R_{K/F}(\mathbf{G}_m)$. Then

$$\operatorname{cdim}_p T = \dim T = [L:F] - [K:F].$$

Proof. Apply Corollary 5.5 to $G = R_{L/F}(\mathbf{G}_m)$ and $C = R_{K/F}(\mathbf{G}_m)$, and note that $\operatorname{ed}_p C = 0$. □

A famous and frequently applied result of Karpenko says that Severi–Brauer varieties $\operatorname{SB}(A)$ of p-power degree central division algebras A are p-incompressible, i.e., have $\operatorname{cdim}_p \operatorname{SB}(A) = \dim \operatorname{SB}(A)$, [Kar00, Theorem 2.1] (see also [Kar10, Proposition 2.2]). Karpenko more recently proved that Weil transfers $R_{K/F}(\operatorname{SB}(A))$ of 'suitably generic' central simple K-algebras A of 2-power degree are 2-incompressible, when K/F is a quadratic separable extension [Kar12]. The following Corollary 5.7 basically tells us that the same happens for Weil restrictions with respect to separable field extensions of higher degree.

Corollary 5.7. *Under the assumptions of Example 5.6, there exist a field extension M/F and an Azumaya $M \otimes_F K$-algebra A of degree $[L:K]$ over M and split over $M \otimes_F L$ such that the Weil restriction $R_{M\otimes_F K/M}(\operatorname{SB}(A))$ of the Severi–Brauer variety $\operatorname{SB}(A)$ is p-incompressible.*

Proof. There is a natural isomorphism

$$H^1(M,T) \simeq \ker(\operatorname{Br}(K \otimes_F M) \to \operatorname{Br}(L \otimes_F M)).$$

Let a be a T-torsor over some field extension M/F with maximal canonical p-dimension. Let A be an Azumaya $K \otimes_F M$-algebra (split by $L \otimes_F M$) of degree $[L:K]$ corresponding to a. Then the splitting fields of t are precisely the splitting fields of $R_{M\otimes_F K/M}(\operatorname{SB}(A))$. Therefore

$$\begin{aligned}\operatorname{cdim}_p R_{M\otimes_F K/M}(\operatorname{SB}(A)) &= \operatorname{cdim}_p a = [L:F] - [K:F] = [K:F]([L:K]-1)\\ &= \dim R_{M\otimes_F K/M}(\operatorname{SB}(A)),\end{aligned}$$

which proves the claim. □

Our goal is now to find a condition on an algebraic torus T which ensures that the lower bound from Corollary 5.5 is an equality.

Corollary 5.8. *Let T be an algebraic torus over a field F, p a prime, and T_d the largest split subtorus of T. Let K/F be a splitting field of T. Make the following two assumptions:*

(i) *$[K:F]$ is a power of p;*

(ii) *the Tate cohomology group $\hat{H}^{-1}(\operatorname{Gal}(K/F), X(T))$ is trivial.*

Then, for every diagonalizable subgroup C of T containing the p-torsion of T_d,

$$\operatorname{cdim} T/C = \operatorname{cdim}_p T/C = \dim T/T_d = \dim T/C - \operatorname{ed}_p C.$$

Proof. Assumption (ii) implies $C \subseteq T_d$. Moreover, T_d/C is a split torus and therefore special. Thus $\operatorname{cdim} T/C \leqslant \operatorname{cdim} T/T_d$ by [KM06, Lemma 6.5]. The inequalities $\operatorname{cdim}_p T/C \leqslant \operatorname{cdim} T/C \leqslant \dim T/T_d$ follow.

We have $C(T) = T_d[p]$ and therefore $\operatorname{ed}_p C = \dim T_d/C$. Applying Corollary 5.5 we immediately get the inequality $\operatorname{cdim}_p T/C \geqslant \dim T/C - \operatorname{ed}_p C = \dim T/T_d$. This completes the proof. □

Definition 5.9. Let $r \in \mathbb{N}_0$, p a prime. Define $\mathcal{C}_p^{(r)}$ to be the class of all F-tori of the form T/C, where:

(i) T is a torus admitting a Galois-splitting field K/F of p-power degree such that

$$\hat{H}^{-1}(\mathrm{Gal}(K/F), X(T)) = 0;$$

(ii) C is a diagonalizable subgroup of T which contains $T_d[p]$ and which has $\mathrm{ed}_p\, C = r$.

Some properties of this construction are listed in the following lemma.

LEMMA 5.10. (i) *For* S *in* $\mathcal{C}_p^{(r)}$*:* $\mathrm{cdim}_p\, S = \dim S - r$.

(ii) *If* $S_1 \in \mathcal{C}_p^{(r_1)}$, $S_2 \in \mathcal{C}_p^{(r_2)}$ *then* $S_1 \times S_2 \in \mathcal{C}_p^{(r_1+r_2)}$.

(iii) *Assume* $S \in \mathcal{C}_p^{(r)}$ *and* $S' \subseteq S$ *is a subtorus with* S/S' *anisotropic. Then* $S' \in \mathcal{C}_p^{(r)}$.

Proof. (i) This is a reformulation of Corollary 5.8.

(ii) The simple proof is left to the reader.

(iii) Write $S = T/C$, with T, C (and K) as in the definition of $\mathcal{C}_p^{(r)}$. Let T' be the preimage of S' under the canonical projection $T \to S$. Clearly T' is split over K as well and C contains $T'_d[p] \subseteq T_d[p]$. Moreover, $S' \simeq T'/C$. Note that S' contains the image of T_d, since S/S' is anisotropic. Hence T' contains T_d, which in turn contains C. Thus T'/T_d is an epimorphic image of S', and hence a torus. It follows that T' is a torus as well.

It remains to verify the condition $\hat{H}^{-1}(\mathrm{Gal}(K/F), X(T')) = 0$. We have a short exact sequence $1 \to X(S/S') \to X(T) \to X(T') \to 1$. Since S/S' is anisotropic, $X(S/S')$ has trivial fixed point set under $\mathrm{Gal}(K/F)$. In particular, $\hat{H}^0(\mathrm{Gal}(K/F), X(S/S')) = 0$. We also have $\hat{H}^{-1}(\mathrm{Gal}(K/F), X(T)) = 0$, and hence the claim follows from the (standard) long exact sequence in Tate cohomology. □

Example 5.11. Let $L_1, \ldots, L_n$ be separable field extensions whose normal closures $K_1, \ldots, K_n$ have p-power degree over F. Then any subtorus T of the product $\prod_{i=1}^n (R_{L_i/F}(\mathbf{G}_m)/\mathbf{G}_m)$ belongs to $\mathcal{C}_p^{(0)}$, and hence has $\mathrm{cdim}\, T = \mathrm{cdim}_p\, T = \dim T$.

Proof. For every i, the torus $T_i = R_{L_i/F}(\mathbf{G}_m)$ is split by K_i and satisfies the condition

$$\hat{H}^{-1}(\mathrm{Gal}(K_i/F), X(T_i)) = 0.$$

The subgroup $C_i = \mathbf{G}_m$ coincides with $(T_i)_d$, hence contains $(T_i)_d[p]$, and has $\mathrm{ed}_p\, C_i = 0$. Therefore $T_i/C_i \in \mathcal{C}_p^{(0)}$. Lemma 5.10(ii) implies that the torus $S := \prod_{i=1}^n (R_{L_i/F}(\mathbf{G}_m)/\mathbf{G}_m)$ lies in $\mathcal{C}_p^{(0)}$, too. Since S is anisotropic (and therefore S/T as well), Lemma 5.10(iii) implies that T also belongs to $\mathcal{C}_p^{(0)}$. □

Example 5.12. Let p be a prime and S an anisotropic algebraic torus over F whose minimal Galois splitting field is cyclic of p-power degree over F. Then S belongs to the class $\mathcal{C}_p^{(0)}$. In particular,

$$\mathrm{cdim}_p\, S = \mathrm{cdim}\, S = \dim S.$$

Proof. Let K/F be the minimal Galois splitting field of S. Embed S in $R_{K/F}(\mathbf{G}_m)^N$ for some $N \gg 0$. Since S is anisotropic, it lies in the subtorus $(R^{(1)}_{K/F}(\mathbf{G}_m))^N$, which is isomorphic to $S' := (R_{K/F}(\mathbf{G}_m)/\mathbf{G}_m)^N$ since K/F is cyclic. Thus we are in the situation of Example 5.11 and the claim follows. □

Example 5.13. Let L/F be a cyclic Galois extension of degree $p^r > 1$ and let $T = R^{(1)}_{L/F}(\mathbf{G}_m)$ be the corresponding norm 1 torus. Then $\operatorname{ed} T = \operatorname{ed}_p T = 1$, but $\operatorname{cdim} T = \operatorname{cdim}_p T = \dim T = p^r - 1$ can be arbitrarily large.

Example 5.14. Let T be an algebraic torus over F. Assume that there exists an element τ of $\operatorname{Gal}(F_{\mathrm{sep}}/F)$ which acts as -1 on $X(T)$. Then $\operatorname{cdim} T = \operatorname{cdim}_2 T = \dim T$.

Proof. Let F' be the fixed field F^{τ}_{sep}. Then $T_{F'}$ is of the form $(T')^{\dim T}$, where T' is a non-split 1-dimensional torus. By Example 5.12, $T_{F'}$ has canonical 2-dimension equal to $\dim T$. Since $\dim T \geqslant \operatorname{cdim}_2 T \geqslant \operatorname{cdim} T \geqslant \operatorname{cdim} T_{F'}$, the claim follows. □

Recall that a torus T over F is said to be quasi-split if its character $\operatorname{Gal}(F_{\mathrm{sep}}/F)$-lattice $X(T)$ is permutation.

Example 5.15. Let T be a 2-dimensional algebraic torus. Then we have:

(i) $\operatorname{cdim} T = 0$ if and only if T is quasi-split;

(ii) $\operatorname{cdim} T = 1$ if and only if $T \simeq \mathbf{G}_m \times T'$ where T' is a non-split one-dimensional torus;

(iii) $\operatorname{cdim} T = 2$, otherwise.

Proof. In every case it is clear that $\operatorname{cdim} T$ cannot be larger than the claimed value. Moreover, the equality $\operatorname{cdim} \mathbf{G}_m \times T' = 1$ for a non-split one-dimensional torus T' is contained in Example 5.14. It remains to show that if T is neither quasi-split, nor of the form $\mathbf{G}_m \times T'$ with T' non-split, then $\operatorname{cdim} T \geqslant 2$. Let L/F be the minimal Galois splitting field of T. Then $\operatorname{Gal}(L/F)$ is a finite group embedding in $\mathbf{GL}_2(\mathbb{Z})$.

First assume that there exists an element σ of order 3 in $\operatorname{Gal}(L/F)$. Let $F' = L^{\sigma}$. Then $T_{F'}$ is isomorphic to $R_{L/F'}(\mathbf{G}_m)/\mathbf{G}_m$, which has canonical dimension 2 by Example 5.12. Hence T has canonical dimension 2 as well.

Now assume that $\operatorname{Gal}(L/F)$ does not contain elements of order 3. Then $\operatorname{Gal}(L/F)$ embeds in the (unique up to conjugacy) maximal 2-subgroup D_8 of $\mathbf{GL}_2(\mathbb{Z})$. Since T is neither quasi-split, nor of the form $\mathbf{G}_m \times T'$ with T' one-dimensional, one easily sees that $\operatorname{Gal}(L/F)$ contains an element which acts as -1 on $X(T)$. Now the claim follows from Example 5.14. □

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