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NONTRIVIAL SOLUTIONS FOR STURM–LIOUVILLE SYSTEMS VIA A LOCAL MINIMUM THEOREM FOR FUNCTIONALS

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Abstract

In this paper, employing a very recent local minimum theorem for differentiable functionals, the existence of at least one nontrivial solution for a class of systems of n second-order Sturm–Liouville equations is established.

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1. Introduction

Let $a, b \in \mathbb{R}$ with $a < b$, $p_i > 1$, ρ_i, $s_i \in L^\infty([a, b])$ with $\operatorname{essinf}_{[a,b]}\rho_i > 0$, $\operatorname{essinf}_{[a,b]} s_i > 0$, $A_i, B_i \in \mathbb{R}$, and let $\alpha_i, \beta_i, \gamma_i, \sigma_i$ be positive constants for $1 \le i \le n$.

Consider the following second-order Sturm–Liouville system on a bounded interval $[a, b]$ in $\mathbb{R}$:

$$\begin{cases} -(\rho_i \phi_{p_i}(u_i'))' + s_i \phi_{p_i}(u_i) = \lambda F_{u_i}(x, \underline{u}), \\ \alpha_i u_i'(a) - \beta_i u_i(a) = A_i, \qquad \gamma_i u_i'(b) + \sigma_i u_i(b) = B_i \end{cases} \tag{1.1}$$

for $1 \le i \le n$, where $\phi_{p_i}(t_i) = |t_i|^{p_i-2} t_i$, $\underline{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $F : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function with respect to x in $[a, b]$ for every $\underline{t} \in \mathbb{R}^n$, it is a C^1-function with respect to $\underline{t} \in \mathbb{R}^n$ for almost every x in $[a, b]$, $F(x, \underline{0}) = 0$ for almost every $x \in [a, b]$,

$$\sup_{|\underline{t}| \le s} \sum_{i=1}^{n} |F_{t_i}(x, \underline{t})| \le g_s(x)$$

for all $s > 0$ and some $g_s \in L^1$, and F_{u_i} denotes the partial derivative of F with respect to u_i.

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In this paper, using a very recent local minimum theorem for differentiable functionals due to Bonanno [1], we establish the existence of at least one nontrivial weak solution for the system (1.1).

Here, as an example, we present a special case of our main result.

THEOREM 1.1. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two continuous functions such that the differential 1-form $w := f(\xi, \eta)\, d\xi + g(\xi, \eta)\, d\eta$ is integrable and let F be a primitive of w such that $F(0, 0) = 0$. Fix $p \geq q > 1$ and assume that*

$$\lim_{(\xi,\eta)\to(0,0)} \frac{F(\xi, \eta)}{\frac{|\xi|^p}{p} + \frac{|\eta|^q}{q}} = +\infty.$$

Then there exists $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*)$ the system

$$\begin{cases} -(\rho_1\phi_p(u'))' + s_1\phi_p(u) = \lambda f(u, v), \\ -(\rho_2\phi_q(v'))' + s_2\phi_q(v) = \lambda g(u, v), \\ \alpha_1 u'(a) - \beta_1 u(a) = 0, \quad \gamma_1 u'(b) + \sigma_1 u(b) = 0 \\ \alpha_2 v'(a) - \beta_2 v(a) = 0, \quad \gamma_2 v'(b) + \sigma_2 v(b) = 0 \end{cases}$$

admits at least one nontrivial weak solution $(u_0, v_0) \in W^{1,p}([a, b]) \times W^{1,q}([a, b])$.

Problems of Sturm–Liouville type have been widely investigated by using topological degree theory, the supersolution and subsolution method, or critical point theory (see [9] and the references therein). We also refer the reader to the papers [2, 3, 5, 8, 11, 10]. Finally, we cite the papers [4, 6], in which the local minimum theorem for differentiable functionals has been successfully employed to ensure the existence of at least one nontrivial solution for differential equations.

The paper is arranged as follows. In Section 2 we give preliminaries and our main tool, that is, Theorem 2.1, while in Section 3 we present our main results.

2. Preliminaries and basic notations

First, we recall for the reader's convenience [1, Theorem 5.1] (see also [1, Proposition 2.1]) which is our main tool. For a given nonempty set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the functions

$$\beta(r_1, r_2) = \inf_{v\in\Phi^{-1}((r_1,r_2))} \frac{\sup_{u\in\Phi^{-1}((r_1,r_2))} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) = \sup_{v\in\Phi^{-1}((r_1,r_2))} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}((-\infty,r_1))} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

THEOREM 2.1 [1, Theorem 5.1]. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in (1/\rho(r_1, r_2), 1/\beta(r_1, r_2))$ there is $u_{0,\lambda} \in \Phi^{-1}((r_1, r_2))$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}((r_1, r_2))$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Here and in the following, X will denote the Cartesian product of n Sobolev spaces $W^{1,p_i}([a, b])$ for $1 \le i \le n$, that is, $X = \prod_{i=1}^n W^{1,p_i}([a, b])$, endowed with the norm

$$\|\underline{u}\|_* = \sum_{i=1}^n \|u_i\|,$$

where

$$\|u_i\| = \Big(\int_a^b (\rho_i(x)|u_i'(x)|^{p_i} + s_i(x)|u_i(x)|^{p_i})\,dx\Big)^{1/p_i}$$

for $1 \le i \le n$. Set $\underline{p} := \min\{p_i : 1 \le i \le n\}$ and $\overline{p} := \max\{p_i : 1 \le i \le n\}$. Here, and in the sequel, we assume $\underline{p} \ge 2$.

In the sequel we need the following proposition.

PROPOSITION 2.2. *Let $T : X \to X^*$ be the operator defined by*

$$\begin{aligned} T(\underline{u})\underline{h} &= \int_a^b (\rho_i(x)\phi_{p_i}(u_i'(x))h_i'(x) + s_i(x)\phi_{p_i}(u_i(x))h_i(x))\,dx \\ &\quad + \sum_{i=1}^n \Big(\rho_i(a)\phi_{p_i}\Big(\frac{A_i + \beta_i u_i(a)}{\alpha_i}\Big)h_i(a) - \rho_i(b)\phi_{p_i}\Big(\frac{B_i - \sigma_i u_i(b)}{\gamma_i}\Big)h_i(b)\Big) \end{aligned}$$

for every $\underline{u}, \underline{h} \in X$. Then T admits a continuous inverse on X^*.

PROOF. For any $\underline{u} = (u_1, \ldots, u_n) \in X$ and $\underline{v} = (v_1, \ldots, v_n) \in X$,

$$\begin{aligned} &\langle T(\underline{u}) - T(\underline{v}), \underline{u} - \underline{v}\rangle \\ &= \int_a^b \sum_{i=1}^n (\rho_i(x)(\phi_{p_i}(u_i'(x)) - \phi_{p_i}(v_i'(x)))(u_i'(x) - v_i'(x)) \\ &\quad + s_i(x)(\phi_{p_i}(u_i(x)) - \phi_{p_i}(u_i(x)))(u_i(x) - v_i(x)))\,dx \\ &\quad + \sum_{i=1}^n \Big(\rho_i(a)\Big(\phi_{p_i}\Big(\frac{A_i + \beta_i u_i(a)}{\alpha_i}\Big) - \phi_{p_i}\Big(\frac{A_i + \beta_i v_i(a)}{\alpha_i}\Big)\Big)(u_i(a) - v_i(a))\Big) \\ &\quad - \sum_{i=1}^n \Big(\rho_i(b)\Big(\phi_{p_i}\Big(\frac{B_i - \sigma_i u_i(b)}{\gamma_i}\Big) - \phi_{p_i}\Big(\frac{B_i - \sigma_i v_i(b)}{\gamma_i}\Big)\Big)(u_i(b) - v_i(b))\Big). \end{aligned}$$

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Hence

$$\langle T(\underline{u}) - T(\underline{v}), \underline{u} - \underline{v}\rangle \geq \int_a^b \sum_{i=1}^n (\rho_i(x)(\phi_{p_i}(u_i'(x)) - \phi_{p_i}(v_i'(x)))(u_i'(x) - v_i'(x)) \\ + s_i(x)(\phi_{p_i}(u_i(x)) - \phi_{p_i}(u_i(x)))(u_i(x) - v_i(x)))\, dx.$$

Then, by [7, Equation (2.2)],

$$\langle T(\underline{u}) - T(\underline{v}), \underline{u} - \underline{v}\rangle \geq C \sum_{i=1}^n \int_a^b (\rho_i(x)|u_i'(x) - v_i'(x)|^{p_i} + s_i(x)|u_i(x) - v_i(x)|^{p_i})\, dx$$

for some constant $C > 0$. Therefore, if $\max_{1\leq i\leq n} \|u_i - v_i\| \leq 1$ then

$$\langle T(\underline{u}) - T(\underline{v}), \underline{u} - \underline{v}\rangle \geq C \sum_{i=1}^n \|u_i - v_i\|^{p_i} \geq C \sum_{i=1}^n \|u_i - v_i\|^{\overline{p}} \\ \geq C \frac{1}{2^{(\overline{p}-1)(n-1)}} \Big(\sum_{i=1}^n \|u_i - v_i\|\Big)^{\overline{p}},$$

that is,

$$\langle T(\underline{u}) - T(\underline{v}), \underline{u} - \underline{v}\rangle \geq C \frac{1}{2^{(\overline{p}-1)(n-1)}} \|u - v\|_*^{\overline{p}}.$$

Moreover, if $\max_{1\leq i\leq n} \|u_i - v_i\| > 1$ then

$$\langle T(\underline{u}) - T(\underline{v}), \underline{u} - \underline{v}\rangle \geq C \sum_{i=1}^n \|u_i - v_i\|^{p_i} \geq C \max_{1\leq i\leq n} \|u_i - v_i\|^{p_i} \\ \geq C(\max_{1\leq i\leq n} \|u_i - v_i\|)^{\underline{p}} \geq C \frac{1}{n^{\underline{p}}} \Big(\sum_{i=1}^n \|u_i - v_i\|\Big)^{\underline{p}},$$

that is,

$$\langle T(\underline{u}) - T(\underline{v}), \underline{u} - \underline{v}\rangle \geq C \frac{1}{n^{\underline{p}}} \|u - v\|_*^{\underline{p}}.$$

It follows that

$$\langle T(\underline{u}) - T(\underline{v}), \underline{u} - \underline{v}\rangle \geq Ka(\|u - v\|_*)\|u - v\|_*$$

for all $\underline{u}, \underline{v} \in X$, where $K = C \min\{1/2^{(\overline{p}-1)(n-1)}, 1/n^{\underline{p}}\}$ and $a(t) = t^{\overline{p}-1}$ if $t \leq 1$, $a(t) = t^{\underline{p}-1}$ if $t > 1$.

Hence, T is uniformly monotone. From [12, Theorem 26.A(d)], T^{-1} exists and is continuous on X^*. This completes the proof. □

Put

$$m_i := \sup\Big\{\frac{\max_{x\in[a,b]} |u_i(x)|}{\|u_i\|} : u_i \in W^{1,p_i}([a, b]) \setminus \{0\}\Big\}$$

for $1 \le i \le n$. One has $m_i < +\infty$. For our goal it is enough to know an explicit upper bound for the constant m_i. In this context [3, Proposition 2.1], setting

$$k_i = 2^{(p_i-1)/p_i} \frac{1}{(b-a)^{1/p_i}} \Big(\max\Big\{\frac{1}{\operatorname{essinf} s_i}, \frac{(b-a)^{p_i}}{\operatorname{essinf} \rho_i}\Big\}\Big)^{1/p_i}$$

for $1 \le i \le n$, one has $m_i \le k_i$. Hence,

$$\|u_i\|_\infty \le m_i \|u_i\|$$

for every $u_i \in W^{1,p_i}([a,b])$. Further, we also put

$$M = \max\Big\{ \sup_{u_i \in W^{1,p_i}([a,b])\setminus\{0\}} \frac{\max_{x\in[a,b]} |u_i(x)|^{p_i}}{\|u_i\|^{p_i}} : 1 \le i \le n\Big\}. \tag{2.1}$$

From (2.1),

$$\|u_i\|_\infty \le M^{1/p_i}\|u_i\| \quad \text{for } i = 1, \dots, n, \ \forall\, \underline{u} \in X. \tag{2.2}$$

For all $\vartheta > 0$ we denote by $Q(\vartheta)$ the set

$$\Big\{\underline{t} = (t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |t_i| \le \vartheta\Big\}.$$

Moreover, we set

$$\mathcal{M} = \sum_{i=1}^{n} (M p_i)^{1/p_i}.$$

Now, put

$$\mathcal{F}(\underline{\tau}) = \sum_{i=1}^{n}\Big(\Big(\int_a^b s_i(x)\,dx\Big)\frac{|\tau_i|^{p_i}}{p_i} + \frac{\gamma_i \rho_i(b)}{\sigma_i p_i}\Big|\frac{B_i - \sigma_i \tau}{\gamma_i}\Big|^{p_i} + \frac{\alpha_i \rho_i(a)}{\beta_i p_i}\Big|\frac{A_i + \beta_i \tau}{\alpha_i}\Big|^{p_i}\Big)$$

for all $\underline{\tau} \equiv (\tau_1, \tau_2, \dots, \tau_n) \in \mathbb{R}^n$ and

$$a_{\tau,q}(\nu) := \frac{\int_a^b \max_{\underline{t}\in Q(\nu)} F(x, \underline{t})\,dx - \int_a^b F(x, \underline{\tau})\,dx}{(\frac{\nu}{\mathcal{M}})^q - \mathcal{F}(\underline{\tau})}$$

for all $\underline{\tau} \in \mathbb{R}^n$, $q > 1$ and $\nu > 0$, with $(\nu/\mathcal{M})^q \ne \mathcal{F}(\underline{\tau})$.

3. Main results

Our main result is the following theorem.

THEOREM 3.1. *Assume that there exist* $\nu_1, \nu_2 > 0$, $\underline{\tau} \in \mathbb{R}^n$, *with* $\nu_1 < \mathcal{M} \le \nu_2$ *and* $(\nu_1/\mathcal{M})^{\overline{p}} < \mathcal{F}(\underline{\tau}) < (\nu_2/\mathcal{M})^{\underline{p}}$ *such that*

$$a_{\underline{\tau},\underline{p}}(\nu_2) < a_{\underline{\tau},\overline{p}}(\nu_1).$$

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Then, for each $\lambda \in (1/a_{\underline{\tau},\overline{p}}(\nu_1), 1/a_{\underline{\tau},\underline{p}}(\nu_2))$ *the system* (1.1) *admits at least one nontrivial weak solution* $\underline{u_0} = (u_{01}, u_{02}, \ldots, u_{0n}) \in X$ *such that*

$$0 < \sum_{i=1}^{n} \frac{\|u_{0i}\|^{p_i}}{p_i} < \frac{\nu_2^{\underline{p}}}{\mathcal{M}^{\underline{p}}}.$$

Proof. Our aim is to apply Theorem 2.1. To this end, fix λ as in the conclusion and define $\Phi, \Psi : X \to \mathbb{R}$ as

$$\Phi(\underline{u}) = \sum_{i=1}^{n}\bigg(\frac{\|u_i\|^{p_i}}{p_i} + \frac{\gamma_i\rho_i(b)}{\sigma_i p_i}\bigg|\frac{B_i - \sigma_i u_i(b)}{\gamma_i}\bigg|^{p_i} + \frac{\alpha_i\rho_i(a)}{\beta_i p_i}\bigg|\frac{A_i + \beta_i u_i(a)}{\alpha_i}\bigg|^{p_i}\bigg)$$

and

$$\Psi(\underline{u}) = \int_a^b F(x, \underline{u}(x))\, dx$$

for all $\underline{u} \in X$. Let us prove that the functionals Φ and Ψ satisfy the conditions required in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $\underline{u} \in X$ is

$$\Psi'(\underline{u})(\underline{v}) = \int_a^b \sum_{i=1}^{n} F_{u_i}(x, \underline{u}(x))v_i(x)\, dx$$

for every $\underline{v} \in X$, and it is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Moreover, it is well known that Φ is a continuously differentiable functional whose differential at the point $\underline{u} \in X$ is

$$\begin{aligned}\Phi'(\underline{u})(\underline{v}) = &\int_a^b \sum_{i=1}^{n}(\rho_i(x)\phi_{p_i}(u_i'(x))v_i'(x) + s_i(x)\phi_{p_i}(u_i(x))v_i(x))\, dx \\ &+ \sum_{i=1}^{n}\bigg(\rho_i(a)\phi_{p_i}\Big(\frac{A_i + \beta_i u_i(a)}{\alpha_i}\Big)v_i(a) - \rho_i(b)\phi_{p_i}\Big(\frac{B_i - \sigma_i u_i(b)}{\gamma_i}\Big)v_i(b)\bigg)\end{aligned}$$

for every $\underline{v} \in X$, and since Φ is convex, from [8, Proposition 25.20(i)] we deduce that Φ is sequentially weakly lower semicontinuous, while Proposition 2.2 gives that Φ' admits a continuous inverse on X^*. Now, put $\underline{w}(x) = \underline{\tau} \equiv (\tau_1, \tau_2, \ldots, \tau_n)$, $r_1 = (\nu_1/\mathcal{M})^{\overline{p}}$ and $r_2 = (\nu_2/\mathcal{M})^{\underline{p}}$. Clearly, $\underline{w} \in X$ and since $\nu_1 < \mathcal{M} \le \nu_2$, one has $r_1 < 1 \le r_2$. Moreover, taking into account that $\Phi(\underline{w}) = \mathcal{F}(\underline{\tau})$, from $(\nu_1/\mathcal{M})^{\overline{p}} < \mathcal{F}(\underline{\tau}) < (\nu_2/\mathcal{M})^{\underline{p}}$ one has $r_1 < \Phi(\underline{w}) < r_2$. Finally,

$$\sup_{\Phi(\underline{u})<r_2} \Psi(\underline{u}) = \sup_{\Phi(\underline{u})<r_2} \int_a^b F(x, \underline{u}(x))\, dx \le \int_a^b \max_{\underline{t}\in Q(\nu_2)} F(x, \underline{t})\, dx. \tag{3.1}$$

In fact, from (2.2),

$$\sum_{i=1}^{n} \frac{\|u_i\|_\infty^{p_i}}{p_i} \le M \sum_{i=1}^{n} \frac{\|u_i\|^{p_i}}{p_i}.$$

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So, in particular,

$$\|u_i\|_\infty \le (p_i M)^{1/p_i}\Big(\sum_{i=1}^{n}\frac{\|u_i\|^{p_i}}{p_i}\Big)^{1/p_i}.$$

Hence, for all $\underline{u} \in X$ such that $\Phi(\underline{u}) < r_2$ (and hence, in particular, $\sum_{i=1}^n \|u_i\|^{p_i}/p_i < r_2$) one has $\|u_i\|_\infty \le (p_i M)^{1/p_i} r_2^{1/p_i}$, and taking into account that $r_2 \ge 1$, one has $\sum_{i=1}^n \|u_i\|_\infty < r_2^{1/\underline{p}}\mathcal{M} = \nu_2$. It follows that (3.1) holds.

Arguing in a similar way,

$$\sup_{\Phi(\underline{u})<r_1} \Psi(\underline{u}) \le \int_a^b \max_{\underline{t}\in Q(\nu_1)} F(x,\underline{t})\,dx. \tag{3.2}$$

Therefore, using (3.1) and (3.2),

$$\begin{aligned}\beta(r_1,r_2) &\le \frac{\sup_{\underline{u}\in\Phi^{-1}((-\infty,r_2))}\Psi(\underline{u}) - \Psi(\underline{w})}{r_2-\Phi(\underline{w})}\\ &\le \frac{\int_a^b \sup_{\underline{t}\in Q(\nu_2)} F(x,\underline{t})\,dx - \int_a^b F(x,\underline{\tau})\,dx}{(\frac{\nu_2}{\mathcal{M}})^{\underline{p}} - \mathcal{F}(\underline{\tau})}\\ &\le a_{\underline{\tau},\underline{p}}(\nu_2)\end{aligned}$$

and

$$\begin{aligned}\rho(r_1,r_2) &\ge \frac{\Psi(\underline{w}) - \sup_{u\in\Phi^{-1}((-\infty,r_1))}\Psi(\underline{u})}{\Phi(\underline{w})-r_1}\\ &\ge \frac{\int_a^b F(x,\underline{\tau})\,dx - \int_a^b \sup_{\underline{t}\in Q(\nu_1)} F(x,\underline{t})\,dx}{\mathcal{F}(\underline{\tau}) - (\frac{\nu_1}{\mathcal{M}})^{\overline{p}}}\\ &\ge a_{\underline{\tau},\overline{p}}(\nu_1),\end{aligned}$$

respectively. Hence, taking [8, Lemma 2.1] into account, the weak solutions of the system (1.1) are exactly the solutions of the equation $\Phi'(\underline{u}) - \lambda\Psi'(\underline{u}) = 0$, and from Theorem 2.1 the conclusion follows. □

Now we point out the following consequence of Theorem 3.1.

THEOREM 3.2. *Assume that there exist $\nu \ge \mathcal{M}$, $\underline{\tau} \in \mathbb{R}^n$, with $\mathcal{F}(\underline{\tau}) < (\nu/\mathcal{M})^{\underline{p}}$, such that*

$$\frac{\int_a^b \max_{\underline{t}\in Q(\nu)} F(x,\underline{t})\,dx}{\nu^{\underline{p}}} < \frac{\int_a^b F(x,\underline{\tau})\,dx}{\mathcal{M}^{\underline{p}}\mathcal{F}(\underline{\tau})}.$$

Then, for each

$$\lambda \in \Bigg(\frac{\mathcal{F}(\underline{\tau})}{\int_a^b F(x,\underline{\tau})\,dx}, \frac{(\frac{\nu}{\mathcal{M}})^{\underline{p}}}{\int_a^b \max_{\underline{t}\in Q(\nu)} F(x,\underline{t})\,dx}\Bigg)$$

the system (1.1) admits at least one nontrivial weak solution $\underline{u}_0 = (u_{01},\dots,u_{0n}) \in X$.

PROOF. Applying Theorem 3.1, we get the conclusion by picking $\nu_1 = 0$ and $\nu_2 = \nu$. Indeed, owing to our assumptions, one has

$$\begin{aligned} a_{\underline{\tau},\underline{p}}(\nu) &< \frac{\Big(1 - \frac{\mathcal{F}(\underline{\tau})\mathcal{M}^{\underline{p}}}{\nu^{\underline{p}}}\Big) \int_a^b \max_{\underline{t}\in Q(\nu)} F(x,\underline{t})\,dx}{\frac{\nu^{\underline{p}}}{\mathcal{M}^{\underline{p}}} - F(\underline{\tau})} \\ &= \frac{\mathcal{M}^{\underline{p}}}{\nu^{\underline{p}}} \int_a^b \max_{\underline{t}\in Q(\nu)} F(x,\underline{t})\,dx \\ &< \frac{\int_a^b F(x,\underline{\tau})\,dx}{\mathcal{F}(\underline{\tau})} \\ &= a_{\underline{\tau},\overline{p}}(0). \end{aligned}$$

In particular,

$$a_{\underline{\tau},\underline{p}}(\nu) < \frac{\mathcal{M}^{\underline{p}}}{\nu^{\underline{p}}} \int_a^b \max_{\underline{t}\in Q(\nu)} F(x,\underline{t})\,dx.$$

Hence, Theorem 3.1 ensures the result. □

Here, we give a special case of the main result.

THEOREM 3.3. *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, n$ be continuous functions such that the differential 1-form $w := \sum_{i=1}^n f_i(\underline{\xi})\, d\xi_i$ is integrable and let F be a primitive of w such that $F(\underline{0}) = 0$. Assume that*

$$\lim_{|\underline{\xi}|\to 0^+} \frac{F(\underline{\xi})}{\sum_{i=1}^n \frac{|\xi_i|^{p_i}}{p_i}} = +\infty.$$

Then, for each $\lambda \in (0, (1/\mathcal{M}^{\underline{p}}(b-a)) \sup_{\nu>0}(\nu^{\underline{p}}/\max_{\underline{t}\in Q(\nu)} F(\underline{t})))$ the system

$$\begin{cases} -(\rho_i\phi_{p_i}(u_i'))' + s_i\phi_{p_i}(u_i) = \lambda f_i(x,\underline{u}), \\ \alpha_i u_i'(a) - \beta_i u_i(a) = 0, \quad \gamma_i u_i'(b) + \sigma_i u_i(b) = 0 \end{cases}$$

for $1 \le i \le n$, admits at least one nontrivial weak solution $\underline{u}_0 \in X$.

PROOF. For fixed λ as in the conclusion, there exists a positive constant ν such that $\lambda < (1/\mathcal{M}^{\underline{p}}(b-a))(\nu^{\underline{p}}/\max_{\underline{t}\in Q(\nu)} F(\underline{t}))$, that is,

$$\frac{\int_a^b \max_{\underline{t}\in Q(\nu)} F(\underline{t})\,dx}{(\frac{\nu}{\mathcal{M}})^{\underline{p}}} < \frac{1}{\lambda}.$$

Taking into account that $\lim_{|\underline{\xi}|\to 0^+} F(\underline{\xi})/(\sum_{i=1}^n |\xi_i|^{p_i}/p_i) = +\infty$, and, hence,

$$\lim_{|\underline{\tau}|\to 0^+} \frac{F(\underline{\tau})}{\mathcal{F}(\underline{\tau})} = +\infty,$$

we can choose $\underline{\tau}$ satisfying

$$\mathcal{F}(\underline{\tau}) < \Big(\frac{\nu}{\mathcal{M}}\Big)^{\underline{p}}$$

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and such that

$$\frac{F(\underline{\tau})}{\mathcal{F}(\underline{\tau})} > \frac{1}{\lambda}\frac{1}{(b-a)}.$$

Hence, one has

$$\frac{\int_a^b \max_{\underline{t}\in Q(\nu)} F(\underline{t})\,dx}{(\frac{\nu}{M})^{\underline{p}}} < \frac{1}{\lambda} < \frac{\int_a^b F(\underline{\tau})\,dx}{\mathcal{F}(\underline{\tau})},$$

and from Theorem 3.2 the conclusion follows. □

Remark 3.4. Theorem 1.1 in the Introduction is an immediate consequence of Theorem 3.3 when $n = 2$.

Now, we present the following example to illustrate the result.

Example 3.5. Let $p = q = 4$. Consider the system

$$\begin{cases} -(|u'|^2u')' + |u|^2u = \lambda\dfrac{u^3}{\sqrt{u^4+v^4}}, \\ -(|v'|^2v')' + |v|^2v = \lambda\dfrac{v^3}{\sqrt{u^4+v^4}}, \\ u'(0) - u(0) = 0, \quad u'(1) + u(1) = 0, \\ v'(0) - v(0) = 0, \quad v'(1) + v(1) = 0. \end{cases} \tag{3.3}$$

Taking into account that the differential 1-form $(u^3/\sqrt{u^4+v^4})\,du + (v^3/\sqrt{u^4+v^4})\,dv$ is integrable and its primitive is $F(u, v) = (1/2)\sqrt{u^4+v^4}$, one has

$$\lim_{(u,v)\to(0,0)} \frac{F(u,v)}{\frac{u^4}{4}+\frac{v^4}{4}} = +\infty.$$

Hence, owing to Theorem 3.3, by choosing $\rho_1 = \rho_2 = s_1 = s_2 = \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, and by picking $\nu = 1$, for each $\lambda \in (0, 1/8(1 + 2^{1/4})^2)$, the system (3.3) has at least one nontrivial weak solution $(u_0, v_0) \in W^{1,4}([0, 1]) \times W^{1,4}([0, 1])$.

Here we want to point out the following consequence of Theorem 3.1 when $n = 1$.

Let $\rho_1 = \rho$, $s_1 = s$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\sigma_1 = \sigma$, $A_1 = A$, $B_1 = B$ and $p_1 = p$. Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Let F be the function defined by $F(x, t) = \int_0^t f(x, s)\,ds$ for each $(x, t) \in [a, b] \times \mathbb{R}$. Put

$$m := \sup_{u\in W^{1,p}([a,b])\setminus\{0\}} \frac{\max_{x\in[a,b]} |u(x)|}{\left(\int_a^b (\rho(x)|u'(x)|^p + s(x)|u(x)|^p)\,dx\right)^{1/p}}.$$

Now, put

$$\mathcal{F}(\tau) = \left(\left(\int_a^b s(x)\,dx\right)\frac{|\tau|^p}{p} + \frac{\gamma\rho(b)}{\sigma p}\left|\frac{B-\sigma\tau}{\gamma}\right|^p + \frac{\alpha\rho(a)}{\beta p}\left|\frac{A+\beta\tau}{\alpha}\right|^p\right)$$

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for all $\tau \in \mathbb{R}$ and

$$b_\tau(\nu) := \frac{\int_a^b \max_{|t|\le\nu} F(x,t)\,dx - \int_a^b F(x,\tau)\,dx}{\frac{1}{p}(\frac{\nu}{m})^p - \mathcal{F}(\tau)}$$

for all $\tau \in \mathbb{R}$ and $\nu > 0$, with $(1/p)(\nu/m)^p \neq \mathcal{F}(\tau)$. Then, we have the following result.

THEOREM 3.6. *Assume that there exist $\nu_1, \nu_2 > 0$, $\tau \in \mathbb{R}$, with $\nu_1 < p^{1/p}m \le \nu_2$ and $(1/p)(\nu_1/m)^p < \mathcal{F}(\tau) < (1/p)(\nu_2/m)^p$ such that*

$$b_\tau(\nu_2) < b_\tau(\nu_1).$$

Then, for each $\lambda \in (1/b_\tau(\nu_1), 1/b_\tau(\nu_2))$ the problem

$$\begin{cases} -(\rho\phi_p(u'))' + s\phi_p(u) = \lambda f(x,u), \\ \alpha u'(a) - \beta u(a) = A, \quad \gamma u'(b) + \sigma u(b) = B \end{cases}$$

admits at least one nontrivial weak solution $u_0 \in W^{1,p}([a,b])$ such that

$$0 < \int_a^b (\rho(x)|u'(x)|^p + s(x)|u(x)|^p)\,dx < \left(\frac{\nu_2}{m}\right)^p.$$

Finally, as a special case of Theorem 3.3, we point out the following result.

THEOREM 3.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative and continuous function such that*

$$\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi^{p-1}} = +\infty.$$

Fix $\nu > 0$. Then, for each $\lambda \in (0, (1/pm^p)(\nu^p/\int_0^\nu f(t)\,dt))$, the problem

$$\begin{cases} -u'' + u = \lambda f(u), \\ \alpha u'(a) - \beta u(a) = 0, \quad \gamma u'(b) + \sigma u(b) = 0 \end{cases}$$

admits at least one nontrivial classical solution u_0 such that $\|u_0\|_\infty < \nu$.

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