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GENERALIZED SECOND FUNDAMENTAL FORM FOR LIPSCHITZIAN HYPERSURFACES BY WAY OF SECOND EPI DERIVATIVES

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ABSTRACT. Using second epi derivatives, we introduce a generalized second fundamental form for Lipschitzian hypersurfaces. In the case of a convex hypersurface, our approach leads back to the classical second fundamental form, which is usually obtained from the second fundamental forms of the outer parallel surfaces by means of a limit procedure.

1. **Introduction.** Generalized second derivatives in nonsmooth analysis have been studied from many perspectives (see *e.g.* [11], [3] or the references in [14, 15]). In [13, 14, 15], R. T. Rockafellar uses *second epi derivatives* as a tool for studying the second order behaviour of nonsmooth functions. It turns out that this is a useful concept for several classes of nonsmooth functions, including for instance convex functions, convex-concave saddle functions, lower C^2-functions (see [13]), or functions of the form $f \circ \phi$, with f convex and ϕ of class C^2. Here we address some open questions concerning second epi derivatives.

Let $f: U \longrightarrow \mathbb{R}$ be a locally Lipschitz function defined on an open set U in $\mathbb{R}^n$. Then f has second epi derivative q at $x \in U$ with respect to $y^* \in \partial f(x)$, where ∂f is the Clarke subdifferential of f (*cf.* [8]), if the second difference quotient

$$\Delta_{f,x,y^*,t}(h) = \frac{f(x+th) - f(x) - t\langle y^*, h\rangle}{t^2}, \tag{1.1}$$

considered as a locally Lipschitz function of $h \in \mathbb{R}^n$, converges to q (as $t \longrightarrow 0$) in the sense of *epi convergence* (see §2, [14, 15]). Here we mainly focus on the case of purely quadratic limit functions q defined on some linear subspace of $\mathbb{R}^n$, while Rockafellar in [14, 15] considers a greater variety of (quadratic) limit functions $q: \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$ defined on certain cones in $\mathbb{R}^n$.

A natural idea to study the second order behaviour of f is to consider pointwise convergence of $\Delta_{f,x,y^*,t}$ as $t \longrightarrow 0$, for this is equivalent to asking whether f allows for a second order Taylor's expansion at x. Consequently, with these two second order notions at our disposal, we ask for their interrelation. While some facts of a general nature concerning the interplay between both types of convergence are known, see for instance [1, 2, 10, 16], the situation for second difference quotients is more subtle and needs refined methods. For convex functions, J. Borwein and the author [4] have investigated

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the interplay between both types of convergence, including the infinite dimensional case, (see also [14, 15]). Here, in Section 3, we give conditions in a more general context that insure that one of the types of convergence entails the other. Counterexamples are given in Section 6.

A classical way to define a *generalized second fundamental form* **II** for certain points p of a convex hypersurface **F** in $\mathbb{R}^n$ is the following: Consider the usual second fundamental forms $\mathbf{II}_\epsilon$ of the outer parallel surfaces $\mathbf{F}_\epsilon$ at the corresponding points $p_\epsilon = p + \epsilon n_p \in \mathbf{F}_\epsilon$, (*cf.* [9, Chapter 3, §2]), with n_p an outer unit normal vector for **F** at p, and then take the limit $\mathbf{II} = \lim_{\epsilon\to 0} \mathbf{II}_\epsilon$. In Section 4 we show that this process is equivalent to taking the second epi derivative of the convex function f representing the surface **F** in a neighbourhood of p, ($\mathbf{F} = grph f, p = \big(x, f(x)\big)$), *i.e.* we have $\mathbf{II}(h) = q(h)$, when q is the second epi derivative of f at x. Obviously, this sheds new light on the concept of second epi derivatives, opening another field of applications. In particular, it enables us to define generalized second fundamental forms for Lipschitzian hypersurfaces in $\mathbb{R}^n$.

2. **Generalized second derivatives.** For a survey on the concept of epi convergence we refer to [1, 2, 10, 16]. Here we shall use the following formulation as our working definition (see [1], [16]):

A sequence (f_k) of proper lower semi-continuous functions on $\mathbb{R}^n$ with values in the extended reals $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is said to *epi converge* to the function $f\colon \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ if the following two conditions are satisfied:

(α) Given $x \in \mathbb{R}^n$, there exist x_k having $x_k \longrightarrow x$ such that $f_k(x_k) \longrightarrow f(x)$;

(β) For any sequence $x_k \longrightarrow x$ we have $\underline{\lim}_{k\to\infty} f_k(x_k) \geq f(x)$.

We use the notation $f_k \xrightarrow{e} f$. Notice that as a consequence of $(\alpha), (\beta)$, the limit function f is automatically lower semi-continuous. We slightly extend the concept of epi convergence by using the following notion: Given a sequence (f_k) of functions, we write $f_k(x) \xrightarrow{e} \theta$ if $(\alpha), (\beta)$ above are satisfied for a fixed x, with $f(x)$ replaced by θ.

An extended real valued function q on $\mathbb{R}^n$ is called *purely quadratic* if $N = \operatorname{dom} q$ is a linear subspace of $\mathbb{R}^n$ and q is of the form $q(h) = \frac{1}{2}\langle Th, h\rangle$ on N, with a symmetric linear $T\colon N \longrightarrow N$ (*cf.* [13]).

DEFINITION 2.1. Let $f\colon U \longrightarrow \mathbb{R}$ be a locally Lipschitz function defined on an open set U in $\mathbb{R}^n$. Let $x \in U$ and $y^* \in \partial f(x)$. Then f is said to have *generalized second derivative* q_{x,y^*} at x with respect to y^*, or equivalently, to have a *generalized second fundamental form* q_{x,y^*} at x with respect to y^* if q_{x,y^*} is a purely quadratic function such that the second order difference quotient (1.1) at x with respect to y^* epi converges to q_{x,y^*} as $t \longrightarrow 0$, *i.e.* $\Delta_{f,x,y^*,t} \xrightarrow{e} q_{x,y^*}$ as $t \longrightarrow 0$. The symmetric linear operator T_{x,y^*} associated with q_{x,y^*} is called the *generalized Hessian* at x with respect to y^*.

REMARK. We sometimes use the notation $\mathbf{II}(h) = q(h)$ for the generalized second fundamental form to indicate that $q = q_{x,y^*}$ is a quadratic form. As shown in [12], generalized second derivatives defined through epi convergence have emerged in an important role in optimization even in cases where the limit function q_{x,y^*} is not just purely quadratic. Also, it should be pointed out here that, in contrast with our present approach, the author of [12] considers one sided second order epi convergence $\Delta_{f,x,y^*,t} \xrightarrow{e} q_{x,y^*}$ as $t \searrow 0, t > 0$.

Let us now focus on pointwise convergence of $\Delta_{f,x,y^*,t}$. Let $f\colon U \longrightarrow \mathbb{R}$ be as above. Then f has a second order Taylor expansion at x if there exist $y^* \in \mathbb{R}^n$ and a symmetric linear T such that, for every $h \in \mathbb{R}^n$, f has a representation of the form

$$f(x+th) = f(x) + t\langle y^*, h\rangle + \frac{t^2}{2}\langle Th, h\rangle + o(t^2) \quad (t \longrightarrow 0). \tag{2.1}$$

Equivalently, (2.1) may be expressed by saying that $\Delta_{f,x,y^*,t}$ converges pointwise everywhere to the purely quadratic and fully defined function $q(h) = \frac{1}{2}\langle Th, h\rangle$, the symmetric operator T being the Hessian $\nabla^2 f(x)$ of f at x, and y^* being the gradient $\nabla f(x)$ of f at x.

3. **Pointwise and epi convergence.** In this section we give criteria on when epi convergence of the second difference quotient implies pointwise convergence and vice versa. In a general context, conditions of this kind have been obtained by Salinetti and Wets [16], see also [10], [1, 2], [14, 15]. For second difference quotients of convex functions, a detailed study of the interplay between both types of convergence, including the situation in infinite dimensions, has been initiated by J.M. Borwein and the author [4, §6].

PROPOSITION 3.1. *Let $f\colon U \longrightarrow \mathbb{R}$ be a locally Lipschitz function, U an open set in $\mathbb{R}^n$. Let $x \in U$ and $y^* \in \partial f(x)$. Suppose the Clarke subdifferential operator ∂f has bounded difference quotient at x, i.e.*

$$\|\partial f(x+h) - y^*\| \leq C \cdot \|h\| \tag{3.1}$$

for some $C > 0$ and small $\|h\|$. Then pointwise and epi convergence of the second difference quotient $\Delta_{f,x,y^*,t}$ are equivalent.

PROOF. First assume that $\Delta_t := \Delta_{f,x,y^*,t} \xrightarrow{e} \Delta$ for a lower semi-continuous limit function Δ. Let h be fixed. Using condition (α), we find $h_t \longrightarrow h$ such that $\Delta_t(h_t) \longrightarrow \Delta(h)$. By the Lebourg Mean Value Theorem there exist vectors k_t on the segments h, h_t and subgradients $v_t^* \in \partial \Delta_t(k_t)$ such that

$$\Delta_t(h_t) - \Delta_t(h) = \langle v_t^*, h_t - h\rangle. \tag{3.2}$$

By condition (3.1), and using the fact that $\partial \Delta_t(k_t) = \frac{1}{t}\big(\partial f(x+tk_t) - y^*\big)$, we see that v_t^* is bounded for small t. Hence the right hand side of (3.2) tends to 0, giving $\Delta_t(h_t) - \Delta_t(h) \longrightarrow 0$. This proves $\Delta_t(h) \longrightarrow \Delta(h)$.

Conversely, assume $\Delta_t \longrightarrow \Delta$ pointwise. We have to check $\underline{\lim}\Delta_t(h_t) \geq \Delta(h)$ for every sequence $h_t \longrightarrow h$. Clearly this follows from (3.2) by invoking boundedness of the v_t^*. Hence $\Delta_t \xrightarrow{e} \Delta$. ∎

For convex functions, we obtain the following more flexible criterion (compare [4, §6]).

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PROPOSITION 3.2. *Let* $f: U \longrightarrow \mathbb{R}$ *be convex,* $x \in U, y^* \in \partial f(x)$*, and suppose* $\Delta_{f,x,y^*,t} \xrightarrow{e} \Delta$. *Let* $h \in \operatorname{dom}\Delta$*, and suppose there exist subgradients* $y_t^* \in \partial f(x+th)$ *such that, for some* $C > 0$,

$$\left\| \frac{1}{t}(y_t^* - y^*) \right\| \leq C \tag{3.3}$$

for small t. *Then* $\Delta_{f,x,y^*,t}(h) \longrightarrow \Delta(h)$.

PROOF. Using condition (α), find $h_t \longrightarrow h$ such that $\Delta_t(h_t) \longrightarrow \Delta(h)$. As $\frac{1}{t}(y_t^* - y^*) \in \partial \Delta_t(h)$, the subgradient inequality implies

$$\left\langle \frac{1}{t}(y_t^* - y^*), h_t - h \right\rangle \leq \Delta_t(h_t) - \Delta_t(h). \tag{3.4}$$

By (3.3), the left hand side of (3.4) tends to 0, hence $\overline{\lim}\Delta_t(h) \leq \Delta(h)$. The converse estimate $\underline{\lim}\Delta_t(h) \geq \Delta(h)$ follows from condition (β) when applied to the constant sequence h. ∎

As a consequence we obtain the following result, which was already observed in [14]. We give a direct proof using Proposition 3.1 resp. 3.2 above.

COROLLARY 3.3. *Let* f *be convex on* $\mathbb{R}^n$*, let* x *be an interior point of* $\operatorname{dom} f$*, and let* $y^* \in \partial f(x)$. *Then pointwise and epi convergence of* $\Delta_t := \Delta_{f,x,y^*,t}$ *coincide in the case where the limit function* Δ *has* $\operatorname{dom}\Delta = \mathbb{R}^n$.

PROOF. First assume $\Delta_t \longrightarrow \Delta$ pointwise with $\operatorname{dom}\Delta = \mathbb{R}^n$. Due to the convexity of Δ_t, Arzela- Ascoli gives uniform convergence on bounded sets, (see [12, Theorem 10.8]), hence $\partial\Delta_t(h)$ is uniformly bounded, $0 < |t| \leq \delta$, $\|h\| \leq 1$. In other terms, (3.1) is met, which by Proposition 3.1 implies epi convergence $\Delta_t \xrightarrow{e} \Delta$.

Conversely, let $\Delta_t \xrightarrow{e} \Delta$, with $\operatorname{dom}\Delta = \mathbb{R}^n$. Arguing as in [1, Remarque 1.12], let $h \in \mathbb{R}^n$, and let Σ be an n-simplex with vertices $h_1, \ldots, h_{n+1}$ such that, for some $\epsilon > 0$, $B(h,\epsilon) \subset B(h, 2\epsilon) \subset \Sigma$. Using condition (α) of epi convergence, find $h_{t,i} \longrightarrow h_i$, $i = 1, \ldots, n+1$ such that $\Delta_t(h_{t,i}) \longrightarrow \Delta(h_i)$. Then, for t small enough, we have $B(h,\epsilon) \subset \Sigma_t$, where Σ_t denotes the n-simplex with vertices $h_{t,1}, \ldots, h_{t,n+1}$. So by convexity, $\Delta_t(h)$ is uniformly bounded above on the ball $B(h,\epsilon)$, proving that $\partial\Delta_t(h)$ is uniformly bounded. Hence (3.3) is met, and the conclusion follows from Proposition 3.2. ∎

Notice that the assumption $\operatorname{dom}\Delta = \mathbb{R}^n$ is crucial here for both implications, as Example 1 in Section 6 shows.

We end this section with a converse to Proposition 3.2. A related result is [4, 6.2].

PROPOSITION 3.4. *Let* f *be a convex function on* $\mathbb{R}^n$. *Suppose condition (3.3) is satisfied for* $h \in \mathbb{R}^n$*, and suppose further that* $\lim_{t\to 0} \Delta_{f,x,y^*,t}(h) = \theta$. *Then* $\Delta_{f,x,y^*,t}(h) \xrightarrow{e} \theta$.

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PROOF. We have to show that, given $h_t \longrightarrow h$, the relation $\underline{\lim}\Delta_t(h_t) \geq \theta$ is valid. Now let $y_t^* \in \partial f(x + th)$ be chosen in accordance with (3.3). Then the subgradient inequality gives us

$$\langle v_t^*, h_t - h\rangle \leq \Delta_t(h_t) - \Delta_t(h), \tag{3.5}$$

where $v_t^* = \frac{1}{t}(y_t^* - y^*)$. By condition (3.3), the v_t^* are bounded, whence the left hand side in (3.5) tends to 0. From this we deduce $\underline{\lim}\Delta_t(h_t) \geq \lim\Delta_t(h) = \theta$, which proves the statement. ∎

4. **Parallel surface.** In this section we give a geometric characterization of the generalized second fundamental form of a convex function in terms of the outer parallel surface of its graph.

Let $f: \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be a proper lower semi continuous convex function. We consider the graph of f as a convex hypersurface $\mathbf{F}$ in $\mathbb{R}^{n+1}$. The outer parallel surface $\mathbf{F}_\epsilon$ of distance $\epsilon > 0$ is obtained by rolling a ball of diameter ϵ on $\mathbf{F}$. In other terms, $\mathbf{F}_\epsilon$ is the convex hypersurface represented as the graph of the convex function f_ϵ whose epigraph is the set

$$\operatorname{epi} f_\epsilon = \left\{(x, \xi) \in \mathbb{R}^n \times \mathbb{R} : \operatorname{dist}\big((x, \xi), \operatorname{epi} f\big) \leq \epsilon\right\} = \operatorname{epi} f + \epsilon B_{n+1}, \tag{4.1}$$

B_{n+1} the unit ball in $\mathbb{R}^{n+1}$. It is known that f_ϵ is of class $C^{1,1}$ on the interior of its domain $\operatorname{dom} f_\epsilon = \operatorname{dom} f + \epsilon B_n$, B_n the unit ball in $\mathbb{R}^n$. We proceed to give an analytic representation of f_ϵ using Young - Fenchel duality.

Let $p \in \mathbf{F}$ be represented as $\big(y, f(y)\big)$, with $y \in \operatorname{dom} f$, and let $v^* \in \partial f(y)$. Then $\mathbf{F}$ has an outer unit normal vector

$$n_p = \frac{(v^*, -1)}{\sqrt{1 + \|v^*\|^2}} \in \mathbb{R}^n \times \mathbb{R} \tag{4.2}$$

at p (*cf.* [12, p.15]), and $p_\epsilon = p + \epsilon n_p$ is a point on the outer parallel surface $\mathbf{F}_\epsilon$. Suppose p_ϵ is represented as $(x, f_\epsilon(x))$, *i.e.*

$$x - y = \frac{\epsilon v^*}{\sqrt{1 + \|v^*\|^2}}. \tag{4.3}$$

Let h, h_ϵ be the support functionals of $\operatorname{epi} f, \operatorname{epi} f_\epsilon$ respectively (*cf.* [12, §13]). By (4.1), we have

$$h_\epsilon = h + \sigma_\epsilon, \tag{4.4}$$

where $\sigma_\epsilon(x, \xi) = \epsilon\sqrt{\|x\|^2 + \xi^2}$ is the support functional of the ball ϵB_{n+1}, (*cf.* [12, p. 115]). Now the supporting hyperplanes of $\mathbf{F}, \mathbf{F}_\epsilon$ at p, p_ϵ respectively have the equations

$$\xi(x) = \langle x, v^*\rangle - h(v^*, -1), \quad \xi_\epsilon(x) = \langle x, v^*\rangle - h_\epsilon(v^*, -1). \tag{4.5}$$

Here we use the fact that $(v^*, -1)$ is an outer normal vector for both $\mathbf{F}$ at p and $\mathbf{F}_\epsilon$ at p_ϵ. By the definition of the Young Fenchel conjugate (*cf.* [12, p. 104]), we have $f^*(v^*) = -\xi(0)$, $f_\epsilon^*(v^*) = -\xi_\epsilon(0)$. Combining these with (4.3) and (4.4) gives

$$f_\epsilon^*(v^*) = f^*(v^*) + \epsilon\sqrt{1 + \|v^*\|^2}. \tag{4.6}$$

As $v^* \in \partial f(y)$ was arbitrary, we see that (4.6) holds for $v^* \in \text{range}(\partial f) = \text{dom}(\partial f^*)$. Using the closedness of f^*, f_ϵ^*, we deduce that (4.6) holds on $\text{dom} f^* = \text{dom} f_\epsilon^*$, since $\text{dom}(\partial f^*)$ is dense in $\text{dom} f^*$.

Let ϕ be the convex function $\phi(x) = -\sqrt{1 - \|x\|^2}$ defined on the unit ball. The conjugate function ϕ^* is $\phi^*(v) = \sqrt{1 + \|v\|^2}$, (see [12, p. 106]), so (4.6) reads as $f_\epsilon^* = f^* + \epsilon\phi^*$. Dualizing (4.6), we obtain the following representation for f_ϵ (see [12, Theorem 16.4]):

$$
\begin{aligned}
f_\epsilon(x) &= (f^* + \epsilon\phi^*)^*(x) = \big(f \,\square\, \epsilon\phi(\epsilon^{-1}\cdot)\big)(x) \\
&:= \inf_{y \in \mathbb{R}^n} \Big(f(y) - \sqrt{\epsilon^2 - \|x - y\|^2} \Big)
\end{aligned}
\tag{4.7}
$$

where $\square$ denotes infimal convolution (see [12, p. 34]). It remains to observe that the infimum (4.7) is attained for a unique $y = H_\epsilon(x)$, which is given by (4.3). Indeed, using (4.6), (4.7), $v^* \in \partial f(y)$, and (4.3), we find

$$
f_\epsilon(x) + f_\epsilon^*(v^*) \le f(y) + (\epsilon\phi^*)^*(x - y) + f^*(v^*) + (\epsilon\phi^*)(v^*) \le \langle x, v^* \rangle,
$$

which shows that $v^* \in \partial f_\epsilon(x)$ (see [12, Theorem 23.5]). Hence we have equality here, proving that the infimum (4.7) is attained at y when v^*, x, y are as in (4.3). Uniqueness of y now follows from the fact that f_ϵ is differentiable at x, *i.e.*, $\partial f_\epsilon(x) = \{v^*\}$, the latter being a consequence of the fact that f_ϵ^* is strictly convex (*cf.* [12, Theorem 26.3]). Notice that the operator H_ϵ so defined may be written in the form

$$
H_\epsilon = \text{proj}_x \circ P_{\text{epi} f} \circ \text{id} \otimes f_\epsilon,
\tag{4.8}
$$

where $\text{proj}_x : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ is the projection along the ξ_{n+1}-axis, $P_{\text{epi} f}$ is the orthogonal projection onto the closed convex set $\text{epi} f$, and where $\text{id} \otimes f_\epsilon$ denotes the mapping $x \longrightarrow \big(x, f_\epsilon(x)\big)$. In particular, (4.8) shows that H_ϵ is a locally Lipschitz operator. Now using (4.2) and (4.3), we find

$$
\nabla f_\epsilon(x) = v^* = \frac{x - H_\epsilon(x)}{\sqrt{\epsilon^2 - \|x - H_\epsilon(x)\|^2}},
\tag{4.9}
$$

which in particular shows that ∇f_ϵ is a locally Lipschitz operator, so f_ϵ is of class $C^{1,1}$ on the interior of its domain.

Using the above representation of the outer parallel surface $\mathbf{F}_\epsilon$, we may now proceed to give a geometric characterization of the points of generalized second order smoothness of a convex surface. Here a point p on a convex surface $\mathbf{F}$ in $\mathbb{R}^{n+1}$ is called a *point of second order smoothness* (resp. a *point of generalized second order smoothness* with respect to the outer normal vector n_p at p) if, in a neighbourhood of p, $\mathbf{F}$ admits a representation as the graph of a convex function $f: U \longrightarrow \mathbb{R}$ such that $p = \big(x, f(x)\big)$, and f is second order differentiable at x (resp. has a generalized second derivative at x with respect to $v^* \in \partial f(x)$ such that $(v^*, -1)$ is parallel to n_p).

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THEOREM 4.1. *Let* $\mathbf{F}$ *be a convex surface in* $\mathbb{R}^{n+1}$, *and* $\mathbf{F}_\epsilon$ *its outer parallel surface at distance* ϵ. *Let* $p \in \mathbf{F}$ *with an outer unit normal* n_p *at* p, *and let* p_ϵ *be the corresponding point* $p_\epsilon = p + \epsilon n_p$ *on the surface* $\mathbf{F}_\epsilon$. *Then* p *is a point of generalized second order smoothness for* $\mathbf{F}$ *with respect to* n_p *if and only if* p_ϵ *is a point of second order smoothness for* $\mathbf{F}_\epsilon$.

PROOF. Let $\mathbf{F}$ be represented as the graph of $f: U \longrightarrow \mathbb{R}$ in a neighbourhood of p, and assume $p = \big(0, f(0)\big) = (0,0)$ and n_p parallel to $(0,-1)$, *i.e.* $0 \in \partial f(0)$. Hence $\mathbf{F}_\epsilon$ is represented by f_ϵ in a neighbourhood of $p_\epsilon = (0,-\epsilon) = \big(0, f_\epsilon(0)\big)$. The statement of the Theorem now reduces to showing that f has a generalized second derivative q at 0 with respect to $0 \in \partial f(0)$ if and only if f_ϵ is second order differentiable at 0 in the usual sense.

The first statement means $\Delta_{f,0,0,t} \xrightarrow{e} q$ for a purely quadratic convex function q. Using the fact that epi convergence is invariant under Young-Fenchel conjugation (*cf.* [1], [16]), this is equivalent to $(\Delta_{f,0,0,t})^* \xrightarrow{e} q^*$. Observe that q^* is again a purely quadratic convex function (see [13]). Now we need the following general equality: $(\Delta_{f,x,y^*,t})^* = \Delta_{f^*,y^*,x,t}$, which in our case gives the new equivalent statement

$$\Delta_{f^*,0,0,t} \xrightarrow{e} q^*. \tag{4.10}$$

Next observe that the function $\epsilon\phi^*(y) = \epsilon\sqrt{1+\|y\|^2}$ is of class C^∞, hence its second difference quotient at 0 converges pointwise everywhere, and by convexity therefore converges uniformly on bounded sets. So $\Delta_{\epsilon\phi^*,0,0,t} \longrightarrow q_\epsilon$, where $\nabla(\epsilon\phi^*)(0) = 0$ and $\nabla^2(\epsilon\phi^*)(0) = \epsilon\,\mathrm{id}$, whence $q_\epsilon(h) = \frac{\epsilon}{2}\|h\|^2$. Now observe the following fact: Given functions f_n, f, g_n, g such that $f_n \xrightarrow{e} f$ in the epi sense and $g_n \longrightarrow g$ uniformly on compact sets, we have $f_n + g_n \xrightarrow{e} f + g$. This may be checked using the defining conditions (α), (β) for epi convergence. Applying this observation, we see that (4.10) is equivalent to

$$\Delta_{f^*+\epsilon\phi^*,0,0,t} = \Delta_{f^*,0,0,t} + \Delta_{\epsilon\phi^*,0,0,t} \xrightarrow{e} q^* + q_\epsilon. \tag{4.11}$$

Dualizing (4.11) again, using (4.7) and $\big(\Delta_{f^*+\epsilon\phi^*,0,0,t}\big)^* = \Delta_{f\Box(\epsilon\phi^*)^*,0,0,t} = \Delta_{f_\epsilon,0,0,t}$, we obtain the new equivalent statement

$$\Delta_{f_\epsilon,0,0,t} \xrightarrow{e} (q^* + q_\epsilon)^* = q \Box q_{\frac{1}{\epsilon}}, \tag{4.12}$$

which uses $q_\epsilon^* = q_{\frac{1}{\epsilon}} = \frac{1}{2\epsilon}\|\cdot\|^2$. But now we apply the fact that f_ϵ is of class $C^{1,1}$, or rather, that ∇f_ϵ is a locally Lipschitz operator. For then Proposition 3.1 or 3.2 implies that, in (4.12), epi convergence is equivalent to pointwise convergence. So we end up with the equivalent statement $\Delta_{f_\epsilon,0,0,t} \longrightarrow q \Box q_{\frac{1}{\epsilon}}$ pointwise, which by convexity means that f_ϵ has second derivative $q \Box \frac{1}{2\epsilon}\|\cdot\|^2$ at 0.

For the converse argument, observe that $\Delta_{f_\epsilon,0,0,t} \longrightarrow \tilde{q}$ pointwise for some purely quadratic and fully defined function $\tilde{q}$ implies $\Delta_{f_\epsilon,0,0,t} \xrightarrow{e} \tilde{q}$ by 3.1, hence $\Delta_{f_\epsilon^*,0,0,t} \xrightarrow{e} \tilde{q}^*$, hence $\Delta_{f^*,0,0,t} \xrightarrow{e} \tilde{q}^* - q_\epsilon =: \tilde{q}_0$ by (4.11) resp. (4.6). In particular, $\tilde{q}_0$ is a purely quadratic convex function. Dualizing finally gives the equivalent statement $\Delta_{f,0,0,t} \xrightarrow{e} \tilde{q}_0^* = (\tilde{q}^* - q_\epsilon)^*$. ∎

REMARKS. 1) The reasoning above relies on the fact that, for a convex function, pointwise convergence of the second order difference quotient, *i.e.* the existence of a second order Taylor expansion, is equivalent to second order differentiability. See [6, §2] for a proof of Jessen's classical result for functions on the line, and [4, §3] for a general proof using modern terminology.

2) Theorem 4.1 shows that the generalized second fundamental form $\mathbf{II}(h) = q(h)$ at the point $p \in \mathbf{F}$ can be obtained as the limit of the second fundamental forms $\mathbf{II}_\epsilon$ of the $\mathbf{F}_\epsilon$ at p_ϵ when $\epsilon \longrightarrow 0$. Indeed, with respect to the local coordinates chosen in the proof, the second fundamental form of $\mathbf{F}_\epsilon$ at p_ϵ is $\mathbf{II}_\epsilon = q \square \frac{1}{2\epsilon}\|\cdot\|^2$, which converges pointwise to $\mathbf{II} = q$ as $\epsilon \longrightarrow 0$ (*cf.* [4, Lemma 5.1]). This shows that our concept of a generalized second fundamental form for Lipschitzian hypersurfaces, as introduced in Definition 2.1, extends the known concept of a generalized second fundamental form in convexity as the limit $\epsilon \longrightarrow 0$ of the second fundamental forms of the parallel surfaces $\mathbf{F}_\epsilon$.

Let us extract some additional information from the proof of Theorem 4.1. With the same notations let $\operatorname{dom} q = \operatorname{Ker} q \oplus R$, $\dim(\operatorname{Ker} q) = k$, $\dim R = r$, so $\dim(\operatorname{dom} q) = k + r$. Then the conjugate q^* has domain $\operatorname{dom} q^* = \operatorname{Ker} q^\perp$ with $\dim(\operatorname{dom} q^*) = n - k$. Let $\lambda_1, \ldots, \lambda_{k+r}$ be the eigenvalues of q resp. its corresponding generalized Hessian, with $\lambda_1 = \cdots = \lambda_k = 0$, say. Then q^* has eigenvalues $0(n-k-r$ times$)$, $\lambda_{k+1}^{-1}, \ldots, \lambda_{k+r}^{-1}$. Hence $q^* + q_\epsilon$ has eigenvalues $\epsilon(n-k-r$ times$)$, $\lambda_{k+1}^{-1} + \epsilon, \ldots, \lambda_{k+r}^{-1} + \epsilon$. Consequently, the form $q \square q_{\epsilon^{-1}}$, which is fully defined, has eigenvalues $\mu_1, \ldots, \mu_n$, where $\mu_1 = \cdots = \mu_k = 0$, $\mu_i = (\lambda_i^{-1} + \epsilon)^{-1}$, $(i = k+1, \ldots, k+r)$, $\mu_{k+r+1} = \cdots = \mu_n = \epsilon^{-1}$. In other terms, the eigenvalues λ_i of q may be obtained from the eigenvalues μ_i of $q \square q_{\epsilon^{-1}}$ by the formula

$$\lambda_i = \frac{\mu_i}{1 - \epsilon\mu_i}, \quad i = 1, \ldots, n, \tag{4.13}$$

with the convention $\lambda_i = \infty$ in case $\mu_i = \epsilon^{-1}$. Here the space $\operatorname{dom} q^\perp$ may be interpreted as eigenspace with corresponding eigenvalue ∞.

As a consequence of this observation, we have the following

COROLLARY 4.2. *Let* $\mathbf{F}$ *be a convex hypersurface in* $\mathbb{R}^{n+1}$. *Let* $p \in \mathbf{F}$, *and let* n_p *be an outer unit normal vector at p. Let* $p_\epsilon = p + \epsilon n_p$ *be the corresponding point on* $\mathbf{F}_\epsilon$. *Then p is a point of second order smoothness for* $\mathbf{F}$ *if and only if* p_ϵ *is a point of second order smoothness for* $\mathbf{F}_\epsilon$ *such that the principal curvatures at* p_ϵ *are* $< \epsilon^{-1}$. *The latter is equivalent to saying that the largest ball tangent to* $\mathbf{F}_\epsilon$ *at* p_ϵ *and lying locally below the surface has radius* $> \epsilon$.

Indeed, with the same notation as above, the principal curvatures of $\mathbf{F}_\epsilon$ at p_ϵ being the eigenvalues μ_i of the Hessian at p_ϵ, we see that $\mu_i < \epsilon^{-1}$ for all i implies $\lambda_i \in \mathbb{R}$ for all i, which is to say that q is fully defined. By Corollary 3.3, this means that q is a second derivative in the classical sense, giving the statement.

Let K be a convex body in $\mathbb{R}^n$ having C^2-boundary. Let $\epsilon > 0$ be chosen so that a ball of radius ϵ can be rolled inside K, whence the inner parallel body $K_{-\epsilon}$ of K exists. Then $K_{-\epsilon}$ has outer parallel body $(K_{-\epsilon})_\epsilon = K$. By Corollary 4.2 above, $K_{-\epsilon}$ has boundary of class C^2 if and only if, for some $\epsilon' > \epsilon$, a ball of radius ϵ' can be rolled inside K, which is to say that $(K_{-\epsilon'})_{\epsilon'} = K$.

5. Theorem of Meusnier. In the previous section we showed that, in the case of a convex hypersurface $\mathbf{F}$, the generalized second fundamental form at a point p of generalized second order smoothness of $\mathbf{F}$ may be obtained from the classical second fundamental forms of the outer parallel surfaces $\mathbf{F}_\epsilon$ at the corresponding points p_ϵ. Now we proceed to show that, in the general case of a Lipschitzian hypersurface, the generalized second fundamental form, when it exists, has an equally nice geometric interpretation.

Let $\mathbf{F}$ be a Lipschitzian hypersurface in $\mathbb{R}^{n+1}$, represented as the graph of a Lipschitz function $f: U \longrightarrow \mathbb{R}$ in a neighbourhood of a point $p \in \mathbf{F}$ (*cf.* [13]). Assume $p = (0,0) = \big(0, f(0)\big)$. Suppose further that p is a strictly regular point, which means that the Clarke tangent cone $T_{\mathbf{F}}(p)$ of $\mathbf{F}$ at p (*cf.* [13]) is a linear subspace of dimension n. Equivalently, this means that the Clarke subdifferential $\partial f(0)$ is singleton. We may assume that $\nabla f(0) = 0$, or rather $T_{\mathbf{F}}(p) = \mathbb{R}^n \times \{0\}$.

Suppose now that the second difference quotient at p converges for a fixed direction h, $\|h\| = 1$, *i.e.* $\Delta_{f,0,0,t}(h) \longrightarrow \theta$. This means that the plane surface curve $\kappa_h: t \longrightarrow f(th)$ lying in the normal section spanned by the tangent $t_p = p + \mathbb{R}(h,0)$ at p, and the normal n_p, which in our local coordinates is the ξ_{n+1}-axis, has curvature 2θ at p. We proceed to prove a nonsmooth version of the Theorem of Meusnier (*cf.* [7, 5]) which tells that we can deduce some additional information concerning the curvature of surface curves κ other than κ_h but having the same tangent line t_p at $p = (0,0)$.

Let us first recall the notion of the curvature of κ at p. Let $r = \big(x, f(x)\big)$ be a point on the curve κ. Let $\mathbb{R}^n = \mathbb{R}h \oplus \mathbb{R}h^\perp$, and let $x = \xi h + y$ be the corresponding decomposition. Let r_h be the footpoint of the orthogonal projection of r onto the tangent t_p. So $r_h = (\xi h, 0)$ in our local coordinates. Let $\sigma_r = \operatorname{dist}(r, r_h)$ and $\tau_r = \operatorname{dist}(r_h, p)$. Then $\lim_{r\longrightarrow p} \sigma_r / \tau_r = 0$, since κ has tangent $t_p = (0,0) + \mathbb{R}(h,0)$ at $p = (0,0)$. Now we say that κ has *curvature* $1/\rho$ at p if the limit

$$\frac{1}{\rho} = \lim_{r\longrightarrow p} \frac{2\operatorname{dist}(r, r_h)}{\operatorname{dist}(r,p)^2} = \lim_{r\longrightarrow p} \frac{2\sigma_r}{\sigma_r^2 + \tau_r^2}$$

exists (*cf.* [7]). The corresponding limit inferior is called the *lower curvature* of κ at p.

Let ϵ_r be the plane spanned by the point r and the tangent line $p + \mathbb{R}(h,0)$. Then the angle α_r between ϵ_r and the normal n_p at p, *i.e.* the ξ_{n+1}-axis, satisfies $f(x) = \|y\| / \tan\alpha_r$. Here, in order to obtain the usual Meusnier formula for the curvature of a surface curve in Proposition 5.1 below, we define the angle α_r between ϵ_r and n_p by $\alpha_r = \frac{\pi}{2} - \beta_r$, where β_r is the angle between n_p and the normal of ϵ_r. We say that κ has *osculating plane* ϵ at p if ϵ is the limit of the planes ϵ_r (as $r \longrightarrow p$, $r \in \kappa$), or equivalently, if α_r converges (as $r \longrightarrow p$, $r \in \kappa$), (*cf.* [7]).

PROPOSITION 5.1. *In the above situation, suppose the surface curve κ has osculating plane ϵ not contained in the tangent hyperplane, and let $\alpha \neq \frac{\pi}{2}$ be the angle between ϵ and the normal n_p at p. Then κ has curvature $\frac{2\theta}{\cos\alpha}$ at p.*

PROOF. With the above notations, for a point $r = \big(x, f(x)\big)$ on κ, consider the corresponding point $r' = \big(\xi h, f(\xi h)\big)$ on the normal section spanned by n_p and t_p (where $x = \xi h + y$). Then, by assumption,

$$\lim_{\xi \to 0} \frac{f(\xi h)}{\xi^2} = \lim_{\xi \to 0} \Delta_{f,0,0,\xi}(h) = \theta. \tag{5.1}$$

But now observe that, due to the fact that f is locally Lipschitz and strictly differentiable at 0 with $\nabla f(0) = 0$, we have

$$\lim_{\xi \to 0} \frac{f(\xi h + y) - f(\xi h)}{\|y\|} = 0. \tag{5.2}$$

Now (5.2) implies $f(x)/f(\xi h) \longrightarrow 1$, $(x = \xi h + y;\ r \longrightarrow p)$, hence (5.1) gives us $f(x)/\xi^2 \longrightarrow \theta$ $(x = \xi h + y)$. Let r_h be the orthogonal projection of $r \in \kappa$ onto the tangent line t_p. So we have $\sigma_r = \operatorname{dist}(r, r_h) = f(x)/\cos\alpha_r$ and $\tau_r = \operatorname{dist}(r_h, p) = \xi$, whence the curvature of κ at p is

$$\lim_{r \to p} \frac{2\sigma_r}{\sigma_r^2 + \tau_r^2} = \lim_{r \to p} \frac{2\frac{f(x)}{\cos\alpha_r}}{\left(\frac{f(x)}{\cos\alpha_r}\right)^2 + \xi^2} = \frac{2\theta}{\cos\alpha}. \tag{5.3}$$

Here we use $f(x)/\xi \longrightarrow 0$, $f(x)/\xi^2 \longrightarrow \theta$ and $\alpha_r \longrightarrow \alpha$. This proves the result. ∎

While pointwise convergence $\Delta_{f,0,0,t}(h) \longrightarrow \theta$ controls the curvature of certain surface curves κ having tangent line $t_p = p + \mathbb{R}(h,0)$ at $p = (0,0)$, namely of those κ having osculating plane not contained in the tangent hyperplane, we shall see next that epi convergence $\Delta_{f,0,0,t}(h) \xrightarrow{e} \theta$ of the second difference quotient in a fixed direction h, $\|h\| = 1$, has a similar interpretation in terms of the curvature of surface curves κ having tangent t_p at p. Notice that the following result shows in particular that epi convergence of $\Delta_{f,t}$ at a point p where $\mathbf{F}$ is locally represented as the graph of f is invariant under change of coordinates.

PROPOSITION 5.2. *With the same notations as above, epi convergence $\Delta_{f,0,0,t}(h) \xrightarrow{e} \theta$ of the second difference quotient is equivalent to the following: Given any surface curve κ having tangent t_p at p, the orthogonal projection κ' of κ onto the normal section spanned by t_p and n_p has lower curvature $\geq 2\theta$ at p and, moreover, there is at least one such curve κ such that κ' has curvature 2θ at p.*

PROOF. For a point $r = \big(x, f(x)\big) \in \kappa$, the corresponding projected point is $r' = \big(\xi h, f(x)\big) \in \kappa'$, for $t_p = \mathbb{R}(h,0)$ and n_p is the ξ_{n+1} axis in our local coordinates. Now let $h_k \longrightarrow h$ and $t_k \longrightarrow 0$ be fixed. Consider any surface curve κ containing the $\big(t_k h_k, f(t_k h_k)\big)$. Then condition (β) of epi convergence is

$$\varliminf_{k \to \infty} \Delta_{f,0,0,t_k}(h_k) = \varliminf_{k \to \infty} \frac{f(t_k h_k)}{t_k^2} \geq \theta. \tag{5.4}$$

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On the other hand, writing $t_kh_k = \xi_k h + y_k$ with $y_k \in \mathbb{R}h^{\perp}$, we have

$$(5.5)\qquad \frac{f(t_kh_k)}{t_k^2} = \frac{f(t_kh_k)}{\xi_k^2 + \|y_k\|^2}\cdot\|h_k\|^2 = \frac{\frac{f(t_kh_k)}{\xi_k^2}}{1+\left(\frac{\|y_k\|}{\xi_k}\right)^2}\cdot\|h_k\|^2.$$

As $\|h_k\| \longrightarrow \|h\| = 1$ and $\|y_k\|/\xi_k \longrightarrow 0$ (κ has tangent $t_p = \mathbb{R}(h,0)$ at $p = (0,0)$), the term (5.5) behaves like $f(t_kh_k)/\xi_k^2$ hence (5.4) is equivalent to

$$(5.6)\qquad \underline{\lim}_{k\to\infty} \frac{f(t_kh_k)}{\xi_k^2} \geq \theta,$$

which is the same as saying that κ' has lower curvature $\geq 2\theta$ at p. Concerning statement (α) of epi convergence, suppose κ is a surface curve having tangent t_p at p such that κ' has curvature 2θ at p. Then we may select any $h_k \longrightarrow h$ and $t_k \longrightarrow 0$ having $\big(t_kh_k, f(t_kh_k)\big) \in \kappa$. Then $\Delta_{f,0,0,t_k}(h_k) \longrightarrow \theta$. Conversely, suppose $h_t \longrightarrow h$ is given such that $\Delta_{f,0,0,t}(h_t) \longrightarrow \theta$. Selecting sequences $h_k \longrightarrow h$ and $t_k \longrightarrow 0$, we interpolate the t_kh_k linearly to obtain a surface curve κ containing the $\big(t_kh_k, f(t_kh_k)\big)$, such that the corresponding projected curve κ' has curvature 2θ at p. This completes the argument. ∎

There are two natural candidates for a generalized second fundamental form at a point $p \in \mathbf{F}$. Indeed, with the same local coordinates as above, suppose $\Delta_{f,0,0,t} \xrightarrow{e} q$ in the epi sense, and $\Delta_{f,0,0,t} \longrightarrow q^\sharp$ pointwise, with $q, q^\sharp$ both purely quadratic functions. Then, in the case of a convex $\mathbf{F}$, $q(h) = \mathbf{II}(h)$ arises naturally as the limit of the second fundamental forms of the outer parallel surfaces of $\mathbf{F}$ (Theorem 4.1), while by Proposition 5.1 above, $q^\sharp(h) = \mathbf{II}^\sharp(h)$ is a natural choice as well, since it is what we would call the *surface curvature* of $\mathbf{F}$ at p in direction h. Consequently, the situation is most satisfactory when $q, q^\sharp$ exist and do coincide. However, as our examples in Section 6 show, this need not be the case for convex hypersurfaces.

6. **Examples.** In this section we construct limiting examples showing that pointwise resp. epi convergence of second difference quotients may be strikingly different. In other terms, we exhibit points p on a convex surface $\mathbf{F}$ such that the generalized second fundamental form $\mathbf{II}$ at p exists, but for at least one direction h, the value $\mathbf{II}(h)$ differs from the surface curvature in direction h. Here the results from Section 5 provide the main idea. Observe that for functions $f: \mathbb{R} \longrightarrow \mathbb{R}$, pointwise and epi convergence of the second difference quotient do coincide, so we are led to study functions on $\mathbb{R}^2$.

We ask for a convex function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ which is differentiable at $(0,0)$ with $f(0,0) = 0$ and $\nabla f(0,0) = (0,0)$ such that $\Delta_{f,0,0,t} \longrightarrow q^\sharp$ pointwise everywhere, $\Delta_{f,0,0,t} \xrightarrow{e} q$ in the epi sense, with $q, q^\sharp$ purely quadratic convex functions having $q^\sharp \neq q$. Clearly, by Corollary 3.3, this is only possible when $\operatorname{dom} q$ and $\operatorname{dom} q^\sharp$ are proper subspaces of $\mathbb{R}^2$. By the definition of epi convergence, we know that $q \leq q^\sharp$ in general, so $\operatorname{dom} q^\sharp \subset \operatorname{dom} q$.

EXAMPLE 1. Here we construct f as above with $\operatorname{dom} q^\sharp = \operatorname{dom} q = \{0\} \times \mathbb{R}$, the y-axis, such that $q(0,1) < q^\sharp(0,1) < \infty$. The idea is as follows. Let us prescribe the values of the function f along the y-axis, say $f(0,y) = a\cdot y^2$ for some $a > 0$. Then $q^\sharp(0,1) = a$. Now suppose we can, in addition, fix the values of f along the parabola $x = y^\alpha$ (with $1 < \alpha < 2$ fixed) by $f(y^\alpha, y) = b\cdot y^2$ for some $0 \leq b < a$. Then we obtain a surface curve $\kappa\colon y \longrightarrow (y^\alpha, y, b\cdot y^2)$ having tangent the y-axis at $(0,0,0)$, whose projection $\kappa'\colon y \longrightarrow (0, y, b\cdot y^2)$ onto the yz-plane has curvature $2b$ at $(0,0,0)$. By Proposition 5.2, this implies $q(0,1) \leq b < a = q^\sharp(0,1)$.

For $0 < |t| \leq 1$, let ϵ_t be the plane spanned by the points $(0,t,t^2)$, $(|t|^\alpha, t, \frac{1}{2}t^2)$ and $(0, \frac{1}{2}t, \frac{1}{4}t^2)$, with equation $z_t = -\frac{1}{2}t^2 - \frac{1}{2}|t|^{2-\alpha}x + \frac{3}{2}ty$, and define $f_1\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f_1(x,y) = \sup_{|t|\leq 1} z_t(x,y), \tag{6.1}$$

where $z_0 = 0$. Then f_1 is a convex function which is globally defined, since the angles between the normal vectors of the planes ϵ_t and the xy-plane are bounded away from 0. We list the relevant properties of f_1:

1. $f_1(0,0) = 0$ and $\nabla f_1(0,0) = (0,0)$, so $f_1 \geq 0$;
2. On the y-axis, $f_1(0,y) = \frac{9}{8}y^2$;
3. $f_1(x,0) = 0$ on the positive x-axis $x \geq 0$;
4. On the negative x-axis, $f_1(x,0) = C_\alpha\cdot|x|^{\frac{2}{\alpha}}$, $x \leq 0$, with $C_\alpha = \alpha\cdot(2-\alpha)^{\frac{2-\alpha}{\alpha}}\cdot 2^{-\frac{2+\alpha}{\alpha}}$;
5. There exists a critical parabola $y = \pm K_\alpha\cdot x^{-\alpha}$ in the half plane $x \geq 0$ such that $f_1(x,y) = 0$ for the points (x,y) inside the parabola.

These items may be obtained by calculating explicitly, for the special (x,y) needed, the value $t = t(x,y)$ where the supremum (6.1) is attained. Next consider the planes with equations $\tilde{z}_s = -\frac{1}{2}s^2 + \frac{1}{2}|s|^{2-\alpha}x + sy$ where $|s| \leq 1$, and define $f_2\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f_2(x,y) = \sup_{|s|\leq 1} \tilde{z}_s(x,y).$$

The relevant properties of f_2 are the following:

1. $f_2(0,0) = 0$, $\nabla f_2(0,0) = (0,0)$, so $f_2 \geq 0$;
2. On the y-axis $f_2(0,y) = \frac{1}{2}y^2$;
3. $f_2(x,0) = 0$ on the negative x-axis $x \leq 0$;
4. $f_2(x,0) = C_\alpha\cdot x^{\frac{2}{\alpha}}$, $x \geq 0$, where C_α is as above.

Define $f(x,y) = \max\big(f_1(x,y), f_2(x,y)\big)$, then f is a convex function with the desired properties. Indeed, it is easy to see that $q^\sharp(0,\pm 1) = \mathbf{II}^\sharp(0,\pm 1) = \frac{9}{8}$, while $q^\sharp(h) = \mathbf{II}^\sharp(h) = \infty$ for all $h \notin \mathbb{R}(0,1)$. Concerning $q = \mathbf{II}$, we first check that $\mathbf{II}(0,1) < \frac{9}{8}$. Consider f_1, f_2 along the parabola $(|y|^\alpha, y)$. The second difference quotient of f_1 is

$$\frac{f_1(y^\alpha,y)}{y^{2\alpha}+y^2} \doteq \frac{f_1(y^\alpha,y)}{y^2} = \frac{z_{t(y)}(y^\alpha,y)}{y^2} =: \theta_1, \tag{6.2}$$

where $t(y)$ denotes the unique value where the supremum $f_1(y^\alpha, y)$ is attained. Setting $A = t(y)/y$, we see that A must satisfy the equation $A(1 + \frac{2-\alpha}{2}A^{-\alpha}) = \frac{3}{2}$, whence θ_1 has constant value $\theta_1 = -\frac{1}{2}A^2 - \frac{1}{2}A^{2-\alpha} + \frac{3}{2}A$. A similar reasoning for f_2 shows that

$$\frac{f_2(y^\alpha,y)}{y^{2\alpha}+y^2} \doteq \frac{\tilde{z}_{s(y)}(y^\alpha,y)}{y^2} = \frac{1}{2}B^{2-\alpha} + B - \frac{1}{2}B^2 =: \theta_2, \tag{6.3}$$

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where $B = s(y)/y$ is a root of the equation $B(1 - \frac{2-\alpha}{2}B^{-\alpha}) - 1 = 0$. It follows that $f_1 \leq f_2$ along the parabola $(|y|^\alpha, y)$, whence the second difference quotient of f along the parabola tends to θ_2. As the curve $\big(|y|^\alpha, y, f(|y|^\alpha, y)\big)$ has tangent the y-axis at 0, Proposition 5.2 implies

$$q(0, \pm 1) = \mathbf{II}(0, \pm 1) \leq \theta_2 < \frac{9}{8}.$$

The latter may be tested *e.g.* by specializing α. For instance, $\alpha \searrow 1$ gives $\theta_2 \longrightarrow \frac{7}{9}$, while $\alpha \nearrow 2$ gives $\theta_2 \longrightarrow 1$. Finally, for the directions $h \notin \mathbb{R}(0, 1)$, we obtain $\mathbf{II}(h) = q(h) = \infty$, hence $\operatorname{dom} q = \operatorname{dom} q^\sharp = \{0\} \times \mathbb{R}$. Notice that f is of class C^∞ away from the curve $\big(x(y), y\big)$ where $f_1\big(x(y), y\big) = f_2\big(x(y), y\big)$, but is not differentiable at the points on this curve.

EXAMPLE 2. We modify Example 1 by fixing $1 < \alpha < \beta < 2$ and taking the planes ϵ_t through the points $(0, t, |t|^\beta)$, $(|t|^\alpha, t, \frac{1}{2}|t|^\beta)$, $\big(0, \frac{1}{2}t, (\frac{1}{2}|t|)^\beta\big)$ with equation $z = z_t(x, y)$, $0 < |t| \leq 1$, setting $f_1(x, y) = \sup_t z_t(x, y)$ as above. We define f_2 accordingly. Let $f = \max(f_1, f_2)$. Here, $f_1(0, y) = \text{const} \cdot y^\beta > f_2(0, y)$ along the y-axis, which shows $q^\sharp(0, 1) = \infty$, so $\operatorname{dom} q^\sharp = \{(0, 0)\}$, while as in Example 1, $\operatorname{dom} q = \{0\} \times \mathbb{R}$ with $q(0, 1) < \infty$. Again $f(x, 0) = C \cdot |x|^{\frac{2}{\alpha}}$ along the x-axis, with C now depending on α and β.

EXAMPLE 3. We may obtain another variant of the above examples where we prescribe the values $f(0, y)$ along the y-axis in such a way that $f(0, \cdot)$ has no second derivative at $y = 0$, while the rest of the construction is similar. The effect is that $\Delta_{f,0,0,t}$ does not have any pointwise limit at all, while its epi limit q exists as above with $\operatorname{dom} q$ the y-axis.

REMARKS. 1) In [14, 15], Rockafellar discusses one sided generalized second derivatives, *i.e.* $\Delta_{f,0,0,t} \xrightarrow{e} q$ for $t > 0$, $t \longrightarrow 0$. Consider for instance the function f_1 defined in Example 1. Here we have $\Delta_{f_1,0,0,t} \xrightarrow{e} q$, $t \downarrow 0$ with $\operatorname{dom} q = \{(\xi, \eta) : \xi \geq 0\}$, $q(\xi, \eta) = 0$ on $\operatorname{dom} q$, using item (5) for f_1 above, while $\Delta_{f_1,0,0,t} \longrightarrow q^\sharp$, $t \downarrow 0$ pointwise with $\operatorname{dom} q^\sharp = \operatorname{dom} q$, $q^\sharp(\xi, \eta) = 0$ for $\xi > 0$, $q^\sharp(0, 1) = q^\sharp(0, -1) = \frac{9}{8}$. Se here we have an even wider variety of situations where epi- and pointwise limits $q, q^\sharp$ are different.

2) We sketch how Example 1 may be modified to get a convex function f which is smooth on a pointed neighbourhood of (0,0). Starting with f as in Example 1, we observe that the graph of f consists of two C^∞ surfaces pasted together along the curve $f_1 = f_2$ as their common boundary. Using standard methods, f may be replaced by a smoothed convex function $\tilde{f}$ on any domain $R(\epsilon, \delta) = \{v \in \mathbb{R}^2 : \epsilon < \|v\| < \delta\}$ so that f differs from $\tilde{f}$ by a small amount in a small neighbourhood of the curve $f_1 = f_2$. Repeating this procedure successively on the overlapping domains domains $R_n = R(\frac{1}{n+2}, \frac{1}{n})$ in such a way that the corresponding smoothed functions $\tilde{f}_n$ may be pasted together to give a convex function, while taking care that $\tilde{f}_n$ agrees with f outside smaller and smaller neighbourhoods of the curve $f_1 = f_2$ on R_n by smaller and smaller amounts, we obtain a convex function $\tilde{f}$ which is of class C^∞ on a pointed neighbourhood of (0,0). Clearly we may arrange $\tilde{f}(0, y) = \frac{9}{8}y^2$ and $\tilde{f}(y^\alpha, y) = f(y^\alpha, y)$, which gives $\mathbf{II}(0, 1) < \mathbf{II}^\sharp(0, 1)$ as in Example 1.

References

1. H. Attouch, *Familles d'opérateurs maximaux monotones et mesurabilité*, Ann. Mat. Pura Appl. **120**(1979), 35–111.
2. H. Attouch and R.J.-B. Wets, *A convergence theory for saddle functions*, Trans. Amer. Math. Soc. **280**(1983), 1–41.
3. A. Ben-Tal and J. Zowe, *Directional derivatives in nonsmooth optimization*, J. Optimiz. Theory Appl.**47**(1985), 483–490.
4. J. M. Borwein and D. Noll, *Second order differentiability of convex functions in Banach space*, Trans. Amer. Math. Soc., to appear.
5. G. Bouligand, *Géometrie infinitésimale directe*, Paris, 1932.
6. H. Busemann, *Convex Surfaces*, Interscience Publishers, New York, 1955.
7. H. Busemann and W. Feller, *Krümmungseigenschaften konvexer Flächen*, Acta Math. **66**(1936), 1–47.
8. F. Clarke, *Generalized gradients and applications*, Trans. Amer. Math. Soc. **205**(1975), 247–262.
9. M. DoCarmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976.
10. S. Dolecki, G. Salinetti and R. J.-B. Wets, *Convergence of functions: equi-semi continuity*, Trans. Amer. Math. Soc. **276**(1983), 409ff.
11. J.-B. Hiriart-Urruty and A. Seeger, *Calculus rules on a new set-valued second derivative for convex functions*, Nonlin. Anal., Theory, Methods, Appl. **13**(1989), 721–738.
12. R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, NJ, 1970.
13. ———, *Maximal monotone relations and the second derivatives of nonsmooth functions*, Ann. Inst. H. Poincaré. Anal. Non Lin. **2**(1985), 167–184.
14. ———, *Generalized second derivatives of convex functions and saddle functions*, Trans. Amer. Math. Soc. **322**(1990), 51–77.
15. ———, *Second order optimality conditions in non- linear programming obtained by way of epi derivatives*, Math. Oper. Res. **14**(1989), 462–484.
16. G. Salinetti and R. J.-B. Wets, *On the relation between two types of convergence for convex functions*, J. Math. Anal. Appl. **60**(1977), 211–226.

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