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Abstract

We develop the metric theory of Diophantine approximation on homogeneous varieties of semisimple algebraic groups and prove results analogous to the classical Khintchine and Jarník theorems. In full generality our results establish simultaneous Diophantine approximation with respect to several completions, and Diophantine approximation over general number fields using S-algebraic integers. In several important examples, the metric results we obtain are optimal. The proof uses quantitative equidistribution properties of suitable averaging operators, which are derived from spectral bounds in automorphic representations.

Contents

1. Introduction

In the classical theory of Diophantine approximation one studies approximations of vectors $x \in \mathbb{R}^d$ by rational vectors. Beginning with work by Khintchine in the 1920s, there has been a rich literature investigating the size of the sets of vectors in $\mathbb{R}^d$ satisfying various approximation properties [Spr79, Har98, BD99]. To set the scene we recall two fundamental results in this subject, the Khintchine and Jarník theorems.

Let us fix a nonincreasing function $\psi : \mathbb{R}^+ \to (0, 1]$ which will measure the quality of rational approximations. A vector $x \in \mathbb{R}^d$ is called *ψ-approximable* if there exist infinitely many $(p, q) \in \mathbb{Z}^d \times \mathbb{N}$ such that

$$\left\| x - \frac{p}{q} \right\| \leqslant \frac{\psi(q)}{q}.$$

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Let $\mathcal{W}(\mathbb{R}^d,\mathbb{Q}^d,\psi)$ denote the set of ψ-approximable vectors in $\mathbb{R}^d$. Since $\mathcal{W}(\mathbb{R}^d,\mathbb{Q}^d,\psi)$ is the lim sup set with respect to the balls centered at p/q and radius $\psi(q)/q$, a straightforward Borel–Cantelli argument implies that the set $\mathcal{W}(\mathbb{R}^d,\mathbb{Q}^d,\psi)$ has Lebesgue measure zero provided that

$$\sum_{(p,q)\in\mathbb{Z}^d\times\mathbb{N},p/q\in[0,1)^d}\left(\frac{\psi(q)}{q}\right)^d=\sum_{q\geqslant 1}\psi(q)^d<\infty. \tag{1.1}$$

The converse is a fundamental theorem of Khintchine [Khi24]:

Theorem 1.1 (Khintchine). *If*

$$\sum_{q\geqslant 1}\psi(q)^d=\infty,$$

then the set $\mathcal{W}(\mathbb{R}^d,\mathbb{Q}^d,\psi)$ has full Lebesgue measure.

If we strengthen (1.1) to require that

$$\sum_{(p,q)\in\mathbb{Z}^d\times\mathbb{N},p/q\in[0,1)^d}\left(\frac{\psi(q)}{q}\right)^\alpha=\sum_{q\geqslant 1}q^{d-\alpha}\psi(q)^\alpha<\infty$$

for some $\alpha\in(0,d)$, then it easily follows that the Hausdorff dimension of the set $\mathcal{W}(\mathbb{R}^d,\mathbb{Q}^d,\psi)$ is at most α. The converse was established by Jarník [Jar29]:

Theorem 1.2 (Jarník). *If for some $\alpha\in(0,d)$,*

$$\sum_{q\geqslant 1}q^{d-\alpha}\psi(q)^\alpha=\infty,$$

then the intersection of $\mathcal{W}(\mathbb{R}^d,\mathbb{Q}^d,\psi)$ with every nonempty open subset of $\mathbb{R}^d$ has Hausdorff dimension at least α.

The aim of this paper is to develop the metric theory of Diophantine approximation on homogeneous varieties of semisimple algebraic groups. In fact, we consider, more generally,

- analogues of Khintchine's theorem for homogeneous varieties (Theorems 1.5 and 1.6);
- analogues of Jarník's theorem for homogeneous varieties (Theorem 1.7).

Both theorems are developed in great generality, i.e. we consider simultaneous Diophantine approximation with respect to several completions, as well as approximation over number fields by S-algebraic integers.

Although the question of proving a Khintchine-type theorem for homogeneous varieties was raised as long as half a century ago by S. Lang [Lan65, p. 189], it has remained widely open. We are only aware of results for rational quadrics [Dru05]. See also the recent preprint [KM13]. It should be noted that this problem is very different from problems arising in the theory of Diophantine approximation with dependent quantities, also traditionally called 'Diophantine approximation on manifolds' (see, for instance, [Kle10, BD99]). In the latter subject, one studies rational approximation of points on varieties by *all* rational points in $\mathbb{R}^d$, while we are interested in approximation by rational points lying on the variety.

Let X be a nonsingular subvariety of the affine space $\mathbb{A}^n$ defined over a number field K. We denote by V_K the set of normalized absolute values $|\cdot|_v$ of K and by K_v the corresponding completions. We introduce a metric on $\mathrm{X}(K_v)$,

$$\|x-y\|_v:=\max_{1\leqslant i\leqslant n}|x_i-y_i|_v, \tag{1.2}$$

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and the height function on $\mathrm{X}(K)$,

$$\mathrm{H}(x) := \prod_{v\in V_K} \max(1, \|x\|_v). \tag{1.3}$$

For $S \subset V_K$, we set

$$X_S := \{(x_v)_{v\in S} \mid x_v \in \mathrm{X}(K_v);\ \|x_v\|_v \leqslant 1 \text{ for almost all } v\}.$$

The metric on X_S is defined as the maximum of the local metrics (1.2). We assume the existence of a nonvanishing regular differential form on X of top degree and fix such a form. This defines measures λ_v on $\mathrm{X}(K_v)$ and a measure λ_S on X_S, which is product of the measures on $\mathrm{X}(K_v)$ normalized so that the subsets $\{\|x_v\|_v \leqslant 1\}$ have measure one. Different choices of differential forms give equivalent measures.

We are interested in Diophantine approximation in X_S by elements of $\mathrm{X}(K)$, which are embedded in X_S diagonally. Let $\Psi = (\psi_v)_{v\in S}$ be a collection of nonincreasing functions $\psi_v : \mathbb{R}^+ \to (0, 1]$ such that $\psi_v = 1$ for almost all v.

Definition 1.3. We say that a point $x = (x_v)_{v\in S} \in X_S$ is Ψ-*approximable* if there are infinitely many $z \in \mathrm{X}(K)$ such that

$$\|x_v - z\|_v \leqslant \psi_v(\mathrm{H}(z)), \quad v \in S. \tag{1.4}$$

We denote by $\mathcal{W}(X_S, \mathrm{X}(K), \Psi)$ the set of Ψ-approximable points in X_S.

Let

$$\psi_S(t) := \prod_{v\in S} \psi_v(t)^{r_v},$$

with $r_v = 2$ if $K_v = \mathbb{C}$ and $r_v = 1$ otherwise. Since X is nonsingular, computing volumes with respect to the measures λ_v in local coordinates yields

$$\lambda_S(\{x \in X_S \,|\, (1.4) \text{ holds}\}) \ll_z \psi_S(\mathrm{H}(z))^{\dim(\mathrm{X})},$$

where the implied constant is uniform as z varies in compact sets. Therefore, the standard Borel–Cantelli argument implies that if

$$\sum_{z\in \mathrm{X}(K)\cap D} \psi_S(\mathrm{H}(z))^{\dim(\mathrm{X})} < \infty \tag{1.5}$$

for all bounded subsets D of X_S, then the set $\mathcal{W}(X_S, \mathrm{X}(K), \Psi)$ has measure zero.

Similarly, one can obtain an elementary estimate on the Hausdorff dimension of the set $\mathcal{W}(X_S, \mathrm{X}(K), \Psi)$. Let us assume that S is finite, and all functions ψ_v are equal. It follows from the definition of the Hausdorff dimension (see § 7 below) that if, with $\alpha > 0$,

$$\sum_{z\in \mathrm{X}(K)\cap D} \psi_S(\mathrm{H}(z))^{\alpha} < \infty \tag{1.6}$$

for all bounded subsets D of X_S, then the set $\mathcal{W}(X_S, \mathrm{X}(K), \Psi)$ has Hausdorff dimension at most α.

We prove partial converses of these statements in the setting of homogeneous varieties of simple algebraic groups. Our main results, which are optimal in many cases, are illustrated by the following example:

Example 1.4. Let
$$\mathrm{X} = \{Q(x) = a\}$$
be a rational two-dimensional ellipsoid. We fix a finite prime p, a finite set of primes $p_1, \ldots, p_s$ (possibly including ∞) different from p, and $\psi : \mathbb{R}^+ \to (0, 1]$ a nonincreasing function. We consider the problem of Diophantine approximation in $\mathrm{X}(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{X}(\mathbb{Q}_{p_s})$ by points in $\mathrm{X}(\mathbb{Z}[1/p])$. In this setting, we have the following.

(i) If there exists a bounded subset D of $\mathrm{X}(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{X}(\mathbb{Q}_{p_s})$ and $\epsilon > 0$ such that
$$\sum_{z \in \mathrm{X}(\mathbb{Z}[1/p]) \cap D} \psi(\mathrm{H}(z))^{2s+\epsilon} = \infty,$$
then, for almost every $(x_1, \ldots, x_s) \in \mathrm{X}(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{X}(\mathbb{Q}_{p_s})$, the system of inequalities
$$\|x_i - z\|_{p_i} \leqslant \psi(\mathrm{H}(z)), \quad i = 1, \ldots, s,$$
has infinitely many solutions $z \in \mathrm{X}(\mathbb{Z}[1/p])$. In view of the above discussion, this result is optimal (up to $\epsilon > 0$). Namely, as we saw in (1.5), if
$$\sum_{z \in \mathrm{X}(\mathbb{Z}[1/p]) \cap D} \psi(\mathrm{H}(z))^{2s} < \infty$$
for all bounded subsets D, then the set of such $(x_1, \ldots, x_s)$ has measure zero.

(ii) If there exists a bounded subset D of $\mathrm{X}(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{X}(\mathbb{Q}_{p_s})$ and $\alpha \in (0, 2s)$ such that
$$\sum_{z \in \mathrm{X}(\mathbb{Z}[1/p]) \cap D} \psi(\mathrm{H}(z))^{\alpha} = \infty,$$
then the set of $(x_1, \ldots, x_s) \in \mathrm{X}(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{X}(\mathbb{Q}_{p_s})$ such that the system of inequalities
$$\|x_i - z\|_{p_i} \leqslant \psi(\mathrm{H}(z)), \quad i = 1, \ldots, s,$$
has infinitely many solutions $z \in \mathrm{X}(\mathbb{Z}[1/p])$ has Hausdorff dimension at least α. This result is optimal. Indeed, according to (1.6), if
$$\sum_{z \in \mathrm{X}(\mathbb{Z}[1/p]) \cap D} \psi(\mathrm{H}(z))^{\alpha} < \infty$$
for all bounded subsets D, then the set of such $(x_1, \ldots, x_s)$ has Hausdorff dimension at most α.

These results are obtained by applying Theorem 1.5 to the adjoint group of norm 1 elements in a rational quadratic algebra which is ramified at the infinite place.

Integrability exponents

Our results depend on spherical integrability exponents which we introduce in this section. Let G_{V_K} denote the restricted direct product of G_v, $v \in V_K$. The group of K-rational points $\mathrm{G}(K)$ embeds diagonally in the locally compact group G_{V_K} as a discrete subgroup with finite covolume. A continuous unitary character χ of G_{V_K} is called automorphic if $\chi(\mathrm{G}(K)) = 1$. Then χ can be considered as an element of $L^2(G_{V_K}/\mathrm{G}(K))$. We denote by $L^2_{00}(G_{V_K}/\mathrm{G}(K))$ the subspace of $L^2(G_{V_K}/\mathrm{G}(K))$ orthogonal to all automorphic characters. The translation action of the group G_v on $G_{V_K}/\mathrm{G}(K)$ defines a unitary representation π_v of G_v on $L^2_{00}(G_{V_K}/\mathrm{G}(K))$. Fix a suitable

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maximal compact subgroup U_v of G_v (as in [GGN12, § 3]). We define the spherical *integrability exponent* of π_v with respect to U_v by

$$\mathfrak{q}_v(\mathrm{G}) := \inf \left\{ q > 0 : \begin{array}{l} \forall\, U_v\text{-inv. } w \in L^2_{00}(G_{V_K}/\mathrm{G}(K)) \\ < \pi_v(g)w, w > \in L^q(\mathrm{G}(K_v)) \end{array} \right\}. \tag{1.7}$$

We note that the Langlands program provides explicit bounds and conjectures regarding the values of the integrability exponents (see [Sar05] for a comprehensive discussion). In particular, it is known that the integrability exponents $\mathfrak{q}_v(\mathrm{G})$ are always finite (see [Clo03]). The generalized Ramanujan conjecture for SL_2 is equivalent to $\mathfrak{q}_v(\mathrm{SL}_2) = 2$ for all $v \in V_K$, and the best currently known estimates give $\mathfrak{q}_v(\mathrm{SL}_2) \leqslant 64/25$ (see [Kim03] for $\mathbb{Q}$ and [BB11] for general number fields).

There are some known cases where the Ramanujan bound is met. In the setting of Example 1.4, for instance, it is known that $\mathfrak{q}_v(\mathrm{G}) = 2$ due to the celebrated results of Deligne combined with the Jacquet–Langlands correspondence (see [Lub94, Appendix]). Other situations where $\mathfrak{q}_v(\mathrm{G}) = 2$ are provided by ([Clo02, Shi11]) (we thank P. Sarnak for bringing these to our attention).

For $S \subset V_K$, we set

$$\sigma_S := \limsup_{N\to\infty} \frac{1}{\log N} |\{v \in S : q_v \leqslant N\}|, \tag{1.8}$$

where q_v denotes the cardinality of the residue field for non-Archimedean v, and define

$$\mathfrak{q}_S(\mathrm{G}) = (1 + \sigma_S) \sup_{v\in S} \mathfrak{q}_v(\mathrm{G}). \tag{1.9}$$

Main results

Our first theorem concerns Diophantine approximation on a semisimple algebraic group G.

Theorem 1.5. *Let $\mathrm{G} \subset \mathrm{GL}_n$ be a connected simply connected almost simple algebraic group defined over a number field K, $S \subset V_K$, and $\Psi = (\psi_v)_{v\in S}$ a collection of nonincreasing functions $\psi_v : \mathbb{R}^+ \to (0, 1]$ such that $\psi_v = 1$ for almost all v. Suppose that for a bounded subset D of G_S and a constant $\alpha > ((\mathfrak{q}_{V_K\backslash S}(\mathrm{G}))/2) \dim(\mathrm{G})$, we have*

$$\sum_{z\in \mathrm{G}(K)\cap D} \psi_S(\mathrm{H}(z))^\alpha = \infty. \tag{1.10}$$

Then the set $\mathcal{W}(G_S, \mathrm{G}(K), \Psi)$ has full measure in G_S.

This result is analogous to Khintchine's theorem (Theorem 1.1). Comparing (1.10) with (1.5), we conclude that it is optimal when $\mathfrak{q}_{V_K\backslash S}(\mathrm{G}) = 2$. The assumption that G is simply connected is essential here, and Theorem 1.5 can be considered as a quantitative version of the strong approximation property [PR94, 7.4], which is only valid for simply connected semisimple groups.

More generally, we consider a quasi-affine variety $\mathrm{X} \subset \mathbb{A}^n$ defined over a number field K and equipped with a transitive action of a connected almost simple algebraic group $\mathrm{G} \subset \mathrm{GL}_n$ defined over K. For $S \subset V_K$ and a collection of functions $\Psi = (\psi_v)_{v\in S}$, we are interested in analyzing the set $\mathcal{W}(X_S, \mathrm{X}(K), \Psi)$ defined as above. Even when the set $\mathrm{X}(K)$ is not discrete in X_S, it might happen that its closure has measure zero in X_S, and in particular, the direct analogue of Theorem 1.5 fails. To describe the structure of $\mathcal{W}(X_S, \mathrm{X}(K), \Psi)$, we observe that X_S is a union of open sets

$$X_{S,S'} := \{x \in X_S : \|x_v\|_v \leqslant 1 \text{ for all } v \in S\backslash S'\},$$

where S' runs over finite subsets of S containing the Archimedean places of S. Hence, the problem reduces to analyzing the sets

$$\mathcal{W}(X_{S,S'}, X_{S,S'} \cap \mathrm{X}(K), \Psi) \subset X_{S,S'}.$$

When S' contains all places v such that $\psi_v \neq 1$, it is easy to check that

$$\mathcal{W}(X_{S,S'}, X_{S,S'} \cap \mathrm{X}(K), (\psi_v)_{v\in S}) = \mathcal{W}(X_{S'}, X_{S,S'} \cap \mathrm{X}(K), (\psi_v)_{v\in S'}) \times \left(\prod_{v\in S\setminus S'} \{x \in \mathrm{X}(K_v) : \ \|x\|_v \leqslant 1\} \right).$$

As we will see below (cf. Lemma 5.3), if $X_{S,S'} \cap \mathrm{X}(K)$ is not discrete in $X_{S'}$, then the closure of $X_{S,S'} \cap \mathrm{X}(K)$, embedded in $X_{S'}$, is open in $X_{S'}$, and we shall prove an analogue of Theorem 1.5 for the sets

$$\mathcal{W}(X_{S'}, X_{S,S'} \cap \mathrm{X}(K), (\psi_v)_{v\in S'}) \subset X_{S'}.$$

To state this result, we introduce a measure of the growth of the number of rational points in $X_{S,S'}$ defined by

$$\mathfrak{a}_{S,S'}(\mathrm{X}) := \sup_{D} \limsup_{h\to\infty} \frac{\log |\{z \in \mathrm{X}(K) \cap D : \ \mathrm{H}(z) \leqslant h\}|)}{\log h}, \tag{1.11}$$

where D runs over bounded subsets of $X_{S,S'}$. We note that $\mathfrak{a}_{S,S'}(\mathrm{X}) < \infty$. When the supremum in (1.11) is taken over all bounded subsets of X_S we use the notation $\mathfrak{a}_S(\mathrm{X})$. In the case of a group variety, if G is isotropic over $V_K\setminus S$ then $\mathfrak{a}_S(\mathrm{G}) > 0$.

Theorem 1.6. *Let $\mathrm{X} \subset A^n$ be a quasi-affine variety defined over a number field K and equipped with a transitive action of a connected almost simple algebraic group $\mathrm{G} \subset \mathrm{GL}_n$ defined over K. Let $S \subset V_K$, S' be a finite subset of S containing the Archimedean places of S, and let $\Psi = (\psi_v)_{v\in S'}$ be a collection of nonincreasing functions $\psi_v : \mathbb{R}^+ \to (0, 1]$. Suppose that for a bounded subset D of $X_{S,S'}$ and a constant*

$$\alpha > \frac{\mathfrak{a}_{S,S'}(\mathrm{X})}{\mathfrak{a}_S(\mathrm{G})} \frac{\mathfrak{q}_{V_K\setminus S}(\mathrm{G})}{2} \dim(\mathrm{X}), \tag{1.12}$$

we have

$$\sum_{z\in \mathrm{X}(K)\cap D} \psi_{S'}(\mathrm{H}(z))^{\alpha} = \infty. \tag{1.13}$$

Then $\overline{X_{S,S'} \cap \mathrm{X}(K)}$ is open in $X_{S'}$, and the set $\mathcal{W}(X_{S'}, X_{S,S'} \cap \mathrm{X}(K), \Psi)$ has full measure in $\overline{X_{S,S'} \cap \mathrm{X}(K)}$.

We note that in the setting of Theorem 1.6 , $\mathfrak{a}_S(\mathrm{G}) > 0$; see § 5. The estimate (1.12) contains an interesting interplay between the arithmetic datum $\mathfrak{a}_{S,S'}(\mathrm{X})/\mathfrak{a}_S(\mathrm{G})$, which measures the growth rates of rational points, and the analytic datum $\mathfrak{q}_{V_K\setminus S}(\mathrm{G})$, which measures the spectral gap for automorphic representations. Typically, $\mathfrak{a}_{S,S'}(\mathrm{X})/\mathfrak{a}_S(\mathrm{G}) \leqslant 1$ and $\mathfrak{q}_{V_K\setminus S}(\mathrm{G})/2 \geqslant 1$, but their product must always be at least one. Indeed, if this is not the case, then

$$\frac{\mathfrak{a}_{S,S'}(\mathrm{X})}{\mathfrak{a}_S(\mathrm{G})} \frac{\mathfrak{q}_{V_K\setminus S}(\mathrm{G})}{2} \dim(\mathrm{X}) < \dim(\mathrm{X}),$$

and one can exhibit a family of approximation functions ψ_v such that both (1.5) and (1.13) hold. Since this contradicts Theorem 1.6, we conclude that

$$\mathfrak{q}_{V_K\setminus S}(\mathrm{G}) \geqslant 2 \frac{\mathfrak{a}_{S,S'}(\mathrm{X})}{\mathfrak{a}_S(\mathrm{G})},$$

which was also observed in [GGN12, Corollary 1.8]. If the equality holds (see, for instance, Example 1.4), then the result of Theorem 1.6 is optimal.

We now state an analogue of Jarník's theorem (Theorem 1.2) which estimates the Hausdorff dimension of the sets $\mathcal{W}(X_{S'}, X_{S,S'} \cap \mathrm{X}(K), \Psi)$. The problem of estimating the Hausdorff dimension in a nonconformal setting is quite subtle, and in particular, there is no version of Jarník's theorem for general rectangular regions defined by a family of functions $\psi_1, \ldots, \psi_d$. Therefore, we restrict our attention to the case where all the functions ψ_v are equal to a single function ψ and write $\mathcal{W}(X_{S'}, X_{S,S'} \cap \mathrm{X}(K), \psi)$ to denote the set of approximable points.

Theorem 1.7. *Let $\mathrm{X} \subset A^n$ be a quasi-affine variety defined over a number field K and equipped with a transitive action of a connected almost simple algebraic group $\mathrm{G} \subset \mathrm{GL}_n$ defined over K. Let $S \subset V_K$, S' be a finite subset of S containing the Archimedean places of S, and let $\psi : \mathbb{R}^+ \to (0,1]$ be a nonincreasing function. Suppose that, for some $0 < \alpha < \sum_{v \in S'} r_v \dim(\mathrm{X})$ and a bounded subset D of $X_{S,S'}$, we have*

$$\sum_{z \in \mathrm{X}(K) \cap D} \psi(\mathrm{H}(z))^\alpha = \infty. \tag{1.14}$$

Then $\overline{X_{S,S'} \cap \mathrm{X}(K)}$ is open in $X_{S'}$, and the intersection of the set $\mathcal{W}(X_{S'}, X_{S,S'} \cap \mathrm{X}(K), \Psi)$ with every nonempty open subset of $\overline{X_{S,S'} \cap \mathrm{X}(K)}$ has Hausdorff dimension at least

$$\frac{2\mathfrak{a}_S(\mathrm{G})}{\mathfrak{a}_{S,S'}(\mathrm{X})\mathfrak{q}_{V_K \setminus S}(\mathrm{G})} \cdot \alpha.$$

We note that this estimate is optimal if $\mathfrak{q}_{V_K \setminus S}(\mathrm{G}) = 2(\mathfrak{a}_{S,S'}(\mathrm{X})/\mathfrak{a}_S(\mathrm{G}))$, which holds for a number of cases (see, for instance, Example 1.4).

Comparison with [GGN12] and strategy of proof

In our earlier work [GGN12], we studied Diophantine approximation on homogeneous varieties and obtained estimates, sharp in many cases, for *Diophantine exponents*. Let X be an algebraic variety defined over a number field K and let $v \in V_K$. The Diophantine exponent of a point $x \in \mathrm{X}(K_v)$ is defined to be

$$\omega_v(x, \epsilon) := \min\{\mathrm{H}(z) : \, z \in \mathrm{X}(K), \, \mathrm{dist}_v(x, z) \leqslant \epsilon\} \tag{1.15}$$

(if no such z exists, set $\omega_v(x, \epsilon) = \infty$). This function is a generalization of the uniform irrationality exponent $\hat{\omega}(\xi)$ of a real number ξ and was introduced by M. Waldschmidt in the context of abelian varieties. It is a nonincreasing function which is bounded as $\epsilon \to 0^+$ if and only if $x \in \mathrm{X}(K)$ and is finite if and only if $x \in \overline{\mathrm{X}(K)}$. It is then a natural problem to obtain bounds for Diophantine exponents. In [GGN12], we used ergodic theorems on semisimple groups, along with a *duality principle* to obtain very general bounds which are sharp in a number of important cases for homogeneous varieties of semisimple groups. Thus [GGN12] and the present paper should be viewed as developing different facets of the metric theory of Diophantine approximation on homogeneous varieties. In the former, estimates for Diophantine exponents are developed, and in the present paper, we obtain analogues of Khintchine's and Jarník's theorems.

The broad strategy of the present paper is similar to [GGN12] and, philosophically speaking, this line of attack goes back to work by Dani [Dan85] and Kleinbock and Margulis [KM99] on the so-called *shrinking target problem*. We study Diophantine properties of points on a variety by relating them to visits to shrinking sets, of orbits of a certain group acting on a homogeneous

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space. To aid the reader, we present a short summary of the strategy of the proof in the special case of group varieties. Given a point $x \in G_S$, we associate to it the point $(e, x^{-1})G(K) \in G_{V_K}/G(K)$. We consider the action of $G_{V_K \setminus S}$ on $G_{V_K}/G(K)$. Diophantine properties of x are then connected to visits of orbits of $(e, x^{-1})G(K)$ under this action, to a shrinking sequence of neighborhoods of the identity coset. This is an example of the shrinking target property. The rate at which these neighborhoods shrink is related to asymptotic properties of the function Ψ. A *duality principle* provides the precise dictionary between the dynamics on the homogeneous space and Diophantine properties of points on varieties. A mean ergodic theorem with rate is then used to study the distribution of orbits on $G_{V_K}/G(K)$. The rate in the ergodic theorem is related to the spherical integrability exponent.

We stress that while the strategy in the present paper is similar to [GGN12], there are important technical differences. In particular, substantial analysis is necessary to study the problem after applying the ergodic theorem and the duality principle. In forthcoming work we further investigate the shrinking target property for lattice actions on homogeneous varieties obtaining best possible rates in a wide variety of examples. We refer the reader to [GGN14] for a survey of our techniques and new examples of exponents for dense lattice orbits.

2. Notation

Let K be a number field. We denote by V_K the set of normalized absolute values of K and by K_v the corresponding completions. The subsets of Archimedean and non-Archimedean absolute values are denoted by V_K^∞ and V_K^f, respectively. We use the notation $I_v = \{q_v^n\}_{n>0}$ for $v \in V_K^f$, where q_v denotes the cardinality of the residue field, and $I_v = (0,1)$ for $v \in V_K^\infty$. For non-Archimedean v, we denote by O_v the ring of integers in K_v. For a subset $T \subset V_K$, we introduce the ring O_T of T-integers:

$$O_T := \{x \in K \mid |x|_v \leqslant 1 \text{ for non-Archimedean } v \notin T\}.$$

Let $\mathrm{G} \subset GL_n$ be a connected almost simple linear algebraic group defined over K. For $v \in V_K$, we set $G_v = \mathrm{G}(K_v)$, and for $S \subset V_K$, G_S denotes the restricted direct product of G_v, $v \in S$, with respect to $\mathrm{G}(O_v)$.

When $S = S_1 \sqcup S_2$, we have $G_S = G_{S_1} \times G_{S_2}$, and in order to simplify notation we often identify a subset B of G_{S_1} with the subset $B \times \{e\}$ of G_S.

We define a height function H on G_{V_K} by

$$\mathrm{H}(g) := \prod_{v \in V_K} \max(1, \|g_v\|_v), \quad g \in G_{V_K}. \tag{2.1}$$

This extends the definition of the height function from (1.3).

For each v, we fix a good special maximal compact subgroup $U_v \subset G_v$ so that $U_v = \mathrm{G}(O_v)$ for all but finitely many v. For $S \subset V_K^f$, we set $U_S = \prod_{v \in S} U_v$. Each group G_v is equipped with an invariant measure m_v which we normalize for non-Archimedean v so that $m_v(U_v) = 1$. The groups G_S are equipped with the corresponding product measures m_S. In particular, we denote m_{V_K} by m. Since G is semisimple, the subgroup $\mathrm{G}(K)$ has finite covolume in G_{V_K}. We denote by μ the invariant probability measure on the quotient space $\Upsilon := G_{V_K}/\mathrm{G}(K)$.

3. Group varieties

In this section we state the effective ergodic theorem and the duality principle for group varieties. An effective mean ergodic theorem involves analysis of suitable averaging operators on the space $L^2(\Upsilon)$ of square-integrable functions of Υ. For $T \subset V_K$ and a probability measure β on G_T, we introduce the averaging operator $\pi_T(\beta) : L^2(\Upsilon) \to L^2(\Upsilon)$ defined by

$$\pi_T(\beta)\phi(\varsigma) = \int_{G_T} \phi(g^{-1}\varsigma)\, d\beta(g), \quad \phi \in L^2(\Upsilon). \tag{3.1}$$

In the case where G is simply connected, the asymptotic behavior of these averaging operators is described by the following theorem.

Theorem 3.1 [GGN12, Theorem 4.2]. *Assume that* G *is simply connected, and let β be the Haar-uniform probability measure supported on a bi-$U_{V_K\backslash S}$-invariant bounded subset B of $G_{V_K\backslash S}$. Then, for every $\phi \in L^2(\Upsilon)$,*

$$\left\| \pi_{V_K\backslash S}(\beta)\phi - \int_\Upsilon \phi\, d\mu \right\|_2 \ll_\delta m_{V_K\backslash S}(B)^{-1/(\mathfrak{q}_{V_K\backslash S}(\mathrm{G}))+\delta} \|\phi\|_2$$

for every $\delta > 0$.

We refer the reader to [GN] for more number-theoretic applications of ergodic theorems. We now turn to a duality principle which connects the behavior of averaging operators and Diophantine approximation on group varieties.

Proposition 3.2 [GGN12, Proposition 5.3]. *Fix $S \subset V_K$, finite $S' \subset S$ and a bounded subset Ω of G_S. Then there exists a family of measurable subsets Φ_ϵ of Υ indexed by $\epsilon = (\epsilon_v)_{v\in S'}$, where $\epsilon_v \in I_v$, that satisfies*

$$\prod_{v\in S'} \epsilon_v^{r_v \dim(\mathrm{G})} \ll \mu(\Phi_\epsilon) \ll \prod_{v\in S'} \epsilon_v^{r_v \dim(\mathrm{G})} \tag{3.2}$$

and the following property holds. If, for a subset $B \subset G_{V_K\backslash S}$, $\epsilon = (\epsilon_v)_{v\in S'}$ as above, $x \in \Omega$ and $\varsigma := (e, x^{-1})\mathrm{G}(K) \in \Upsilon$, we have

$$B^{-1}\varsigma \cap \Phi_\epsilon \neq \emptyset,$$

then there exists $z \in \mathrm{G}(K)$ such that

$$\mathrm{H}(z) \leqslant c_0 \sup_{b\in B} \mathrm{H}(b) \tag{3.3}$$

and

$$\|x_v - z\|_v \leqslant \epsilon_v \quad \text{for all } v \in S',$$
$$\|x_v - z\|_v \leqslant 1 \quad \text{for all } v \in S\backslash S'.$$

We note that the upper bound in (3.2) was not stated explicitly in [GGN12], but it follows easily from the construction of the sets Φ_ϵ.

4. Proof of Theorem 1.5

The proof of Theorem 1.5 is quite long and involved and for the convenience of the reader, we split it up into several steps.

The approximating function

Recall that we have assumed that $\Psi = (\psi_v)_{v\in S}$ is a collection of nonincreasing functions. We first show that in fact, we may assume that $\psi_v(t) \to 0$ as $t \to \infty$ for at least one $v \in S$ and that $\mathrm{Im}(\psi_v) \subset I_v$ for all $\psi_v \neq 1$. Note that the divergence assumption (1.10) implies that the group G is isotropic over $V_K\backslash S$. Indeed, if G is anisotropic over $V_K\backslash S$, then $G_{V_K\backslash S}$ is compact (see [PR94, § 3.1]). Since $\mathrm{G}(K)$ embeds discretely in G_{V_K}, the number of $\mathrm{G}(K)$-points in $G_{V_K\backslash S} \times D$ is finite, and this contradicts (1.10).

Since G is isotropic over $V_K\backslash S$, it follows from the strong approximation property of simply connected groups (see [PR94, § 7.4]) that $\mathrm{G}(K)$ is dense in G_S. Hence, the claim of the theorem follows immediately if $\psi_v(t) \nrightarrow 0$ as $t \to \infty$ for all $v \in S$. From now on, we assume that $\psi_v(t) \to 0$ as $t \to \infty$ for at least one $v \in S$.

For $v \in V_K^f$, given a nonincreasing function $\psi_v : \mathbb{R}^+ \to (0,1]$, one can construct a nonincreasing function $\tilde{\psi}_v : \mathbb{R}^+ \to (0,1)$ such that $\tilde{\psi}_v \leqslant \psi_v \leqslant q_v\tilde{\psi}_v$ and $\mathrm{Im}(\tilde{\psi}_v) \subset I_v$. Now if we replace $\psi_v \neq 1$ in Ψ by $\tilde{\psi}_v$, then the divergence assumption (1.10) still holds. Since $\tilde{\psi}_v \leqslant \psi_v$, the theorem for $\tilde{\psi}_v$'s implies the theorem for ψ_v's. Hence, we may assume without loss of generality that $\mathrm{Im}(\psi_v) \subset I_v$ for all $\psi_v \neq 1$.

Preparations

In this section, we make more preparations in advance of applying the duality principle and the ergodic theorem. To simply notation, we set

$$d := \dim(\mathrm{G}), \quad \mathfrak{q} := \mathfrak{q}_{V_K\backslash S}(\mathrm{G}), \quad \mathfrak{a} := \mathfrak{a}_S(\mathrm{G}).$$

We consider a family of bounded bi-$U_{V_K\backslash S}$-invariant subsets of $G_{V_K\backslash S}$ defined by

$$B_h := U_{V_K\backslash S}\{g \in G_{V_K\backslash S} : \mathrm{H}(g) \leqslant h\}U_{V_K\backslash S}.$$

Let

$$U := \{g \in G_{V_K\backslash S} : \|g_v - e\| \leqslant 1 \text{ for } v \in V_K\backslash S\}.$$

Since U and $U_{V_K\backslash S}$ are compact, there exists $c_1 \geqslant 1$ such that

$$\sup_{b\in U^{-1}B_h} \mathrm{H}(b) \leqslant c_1\, h. \tag{4.1}$$

It follows from the definition of $\mathfrak{a} = \mathfrak{a}_S(\mathrm{G})$ that, for every $\delta > 0$ and sufficiently large h,

$$|\mathrm{G}(K) \cap D \cap \{h/2 < \mathrm{H} \leqslant h\}| \leqslant h^{\mathfrak{a}+\delta}. \tag{4.2}$$

Since the function ψ_S is nonincreasing, it follows from (1.10) that

$$\sum_{n=1}^{\infty} |\mathrm{G}(K) \cap D \cap \{2^{n-1} < \mathrm{H} \leqslant 2^n\}|\psi_S(2^{n-1})^{\alpha} = \infty,$$

and because of (4.2) we also get

$$\sum_{n=1}^{\infty} 2^{(\mathfrak{a}+\delta)n}\psi_S(2^{n-1})^{\alpha} = \infty. \tag{4.3}$$

Since $0 < \psi_S \leqslant 1$, there exists $\alpha_0(\delta) \in [0,\infty]$ such that this series converges for all $\alpha > \alpha_0(\delta)$ and diverges for all $\alpha < \alpha_0(\delta)$. We fix $\alpha_0 > \mathfrak{q}d/2$ such that series (1.10) diverges for $\alpha = \alpha_0$. Since divergence in (1.10) implies divergence in (4.3), we have $\alpha_0(\delta) \geqslant \alpha_0$.

Using the monotonicity of the function ψ_S, it is easy to check that (4.3) is equivalent to

$$\sum_{n=1}^{\infty} 2^{(\mathfrak{a}+\delta)n}\psi_S(c\,2^{n-1})^{\alpha} = \infty \tag{4.4}$$

with any $c > 0$. We choose $c = c_0c_1/2$, where c_0 is as in (3.3), and c_1 is as in (4.1).

In order to continue our argument, it would be convenient to know that $\alpha_0(\delta) < \infty$. This is arranged by the following construction. If

$$2^{(\mathfrak{a}+\delta)n}\psi_S(c\,2^{n-1})^{\alpha_0} \to 0 \quad \text{as } n \to \infty,$$

we set $R_n := \psi_S(c\,2^{n-1})$. Otherwise, there exists $\epsilon > 0$ such that the set

$$N := \{n :\ 2^{(\mathfrak{a}+\delta)n}\psi_S(c\,2^{n-1})^{\alpha_0} \geqslant \epsilon\}$$

is infinite. In this case, we set

$$R_n := \begin{cases} (\epsilon\,2^{-(\mathfrak{a}+\delta)n})^{1/\alpha_0} & \text{for } n \in N, \\ \psi_S(c\,2^{n-1}) & \text{otherwise.} \end{cases}$$

In both cases, we have $R_n \leqslant \psi_S(c\,2^{n-1})$, and since N is infinite,

$$\sum_{n=1}^{\infty} 2^{(\mathfrak{a}+\delta)n}R_n^{\alpha} = \infty \tag{4.5}$$

for $\alpha = \alpha_0$. There exists $\alpha_1(\delta) \in [0, \infty]$ such that this series converges for all $\alpha > \alpha_1(\delta)$ and diverges for all $\alpha < \alpha_1(\delta)$. Clearly, $\alpha_1(\delta) \geqslant \alpha_0$. It also follows from the definition of R_n that $R_n \ll 2^{-\kappa n}$ with some $\kappa > 0$. Hence, $\alpha_1(\delta) < \infty$.

Let S' be the finite subset of S on which $\psi_v \neq 1$. Recall that

$$\psi_S(t) = \prod_{v\in S'} \psi_v(t)^{r_v}.$$

Since $R_n \leqslant \psi_S(c\,2^{n-1})$, we may write

$$R_n = \prod_{v\in S'} (\epsilon_v^{(n)})^{r_v},$$

where $\epsilon_v^{(n)} \leqslant \psi_v(c\,2^{n-1})$ for all $v \in S'$.

Applying the ergodic theorem in conjunction with duality

We are now in a position to apply the ergodic theorem coupled with the duality principle. Let $\alpha < \alpha_1(\delta)$. We denote by Φ_n the collection of measurable subsets of Υ defined by Proposition 3.2 with $\epsilon_v = \epsilon_v^{(n)}$, $v \in S'$, and consider the sequence of functions on Υ defined by

$$\phi_n := c_n 1_{\Phi_n} \quad \text{with } c_n = 2^{(\mathfrak{a}+\delta)n}R_n^{\alpha-d}.$$

According to (3.2), we have $\mu(\Phi_n) \gg R_n^d$ and $\alpha < \alpha_1(\delta)$, and so it follows that

$$\sum_{n\geqslant 1} \int_{\Upsilon} \phi_n\, d\mu = \sum_{n\geqslant 1} c_n\mu(\Phi_n) = \infty. \tag{4.6}$$

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Proposition 4.1. *Let β_n denote the Haar-uniform probability measure supported on B_{2^n} and*

$$F_k := \sum_{n\geqslant k}\left|\pi_{V_K\backslash S}(\beta_n)\phi_n - \int_\Upsilon \phi_n\,d\mu\right|.$$

Then F_k is L^2-integrable for sufficiently small $\delta > 0$ and $\alpha < \alpha_1(\delta)$ sufficiently close to $\alpha_1(\delta)$.

Proof. Observe that by Theorem 3.1 and Proposition (3.2), for every $\delta > 0$,

$$\begin{aligned}\|F_k\|_2 &\ll_\delta \sum_{n\geqslant k} m_{V_K\backslash S}(B_{2^n})^{-(1/\mathfrak{q})+\delta}\|\phi_n\|_2\\ &= \sum_{n\geqslant k} m_{V_K\backslash S}(B_{2^n})^{-(1/\mathfrak{q})+\delta} c_n\mu(\Phi_n)^{1/2}\\ &\leqslant \sum_{n\geqslant k} m_{V_K\backslash S}(B_{2^n})^{-(1/\mathfrak{q})+\delta} 2^{(\mathfrak{a}+\delta)n}R_n^{\alpha-d/2}.\end{aligned}$$

By [GGN12, Lemma 6.1], for every $\delta > 0$ and sufficiently large n,

$$m_{V_K\backslash S}(B_{2^n}) \gg 2^{(\mathfrak{a}-\delta)n}.$$

Hence, for sufficiently large k,

$$\begin{aligned}\|F_k\|_2 &\ll_\delta \sum_{n\geqslant k} 2^{(a(1-1/\mathfrak{q})+\theta)n}R_n^{\alpha-d/2}\\ &= \sum_{n\geqslant k} 2^{(\mathfrak{a}(1-1/\mathfrak{q})+2\theta)n}R_n^{\alpha-d/2}\cdot 2^{-\theta n},\end{aligned}$$

where $\theta = \theta(\delta)$ satisfies $\theta(\delta)\to 0^+$ as $\delta\to 0^+$. We apply Hölder's inequality to the above sum with the exponents

$$r = \mathfrak{a}/(\mathfrak{a}(1-1/\mathfrak{q})+2\theta) \quad \text{and} \quad \bar{r} = (1-1/r)^{-1}.$$

Note that when δ is sufficiently small, we have $r > 1$. Then we obtain

$$\|F_k\|_2 \ll_\delta \left(\sum_{n\geqslant k} 2^{\mathfrak{a}n}R_n^{r(\alpha-d/2)}\right)^{1/r}\cdot\left(\sum_{n\geqslant k} 2^{-\theta\bar{r}n}\right)^{1/\bar{r}}. \tag{4.7}$$

Since $\alpha_0 > \mathfrak{q}d/2$,

$$\frac{\alpha_0 - d/2}{1-1/\mathfrak{q}} > \alpha_0.$$

This implies that

$$\frac{\mathfrak{a}(\alpha-d/2)}{\mathfrak{a}(1-1/\mathfrak{q})+2\theta} > \alpha$$

for every sufficiently small $\delta > 0$ and every $\alpha \geqslant \alpha_0$. Since $\alpha_1(\delta)\geqslant\alpha_0$, it follows that for $\alpha < \alpha_1(\delta)$ sufficiently close to $\alpha_1(\delta)$ and for sufficiently small $\delta > 0$, we also have

$$r(\alpha-d/2) = \frac{\mathfrak{a}(\alpha-d/2)}{\mathfrak{a}(1-1/\mathfrak{q})+2\theta} > \alpha_1(\delta).$$

This implies that the series in (4.7) converges, and we conclude that F_k is L^2-integrable. □

Now let $\delta > 0$ and $\alpha < \alpha_1(\delta)$ be such that F_k is L^2-integrable. We consider a sequence of subsets

$$\Upsilon_n := \{\varsigma \in \Upsilon : B_{2^n}^{-1}\varsigma \cap \Phi_n = \emptyset\}$$

of Υ. Note that by (4.6), on the set $\bigcap_{n\geqslant k} \Upsilon_n$,

$$F_k = \sum_{n\geqslant k} \int_\Upsilon \phi_n \, d\mu = \infty.$$

Since F_k is L^2-integrable, it follows that $\mu(\bigcap_{n\geqslant k} \Upsilon_n) = 0$ and the set $\Upsilon_\infty := \liminf(\Upsilon_n)$ also has measure zero. We denote by $\widetilde{\Upsilon}_\infty$ the preimage of Υ_∞ in G_{V_K}. Then $m(\widetilde{\Upsilon}_\infty) = 0$.

Let Ω be a compact subset of G_S, and

$$\Omega' = \{x \in \Omega : \exists\, y \in U : (y, x^{-1}) \notin \widetilde{\Upsilon}_\infty\}.$$

Since

$$(U \times (\Omega\backslash\Omega')^{-1}) \subset \widetilde{\Upsilon}_\infty,$$

and U has positive measure, it follows that the set $\Omega\backslash\Omega'$ has measure zero. Let us take an exhaustion $G_S = \cup_{j\geqslant 1}\Omega_j$ of G_S by compact sets. Then $\cup_{j\geqslant 1}\Omega'_j$ is a subset of G_S of full measure. Hence, it suffices to show that given a compact subset Ω of G_S, almost every element of Ω' is contained in $\mathcal{W}(G_S, \mathrm{G}(K), \Psi)$.

For $x \in \Omega'$, there exists $y \in U$ such that $\tilde{\varsigma} := (y, x^{-1}) \notin \widetilde{\Upsilon}_\infty$. Then

$$\tilde{\varsigma}\mathrm{G}(K) \notin \Upsilon_\infty = \liminf(\Upsilon_n),$$

and $\tilde{\varsigma}\mathrm{G}(K) \notin \Upsilon_n$ for infinitely many n. This means that for infinitely many n, we have

$$B_{2^n}^{-1}\tilde{\varsigma}\mathrm{G}(K) \cap \Phi_n \neq \emptyset,$$

and

$$(U^{-1}B_{2^n})^{-1}(e, x)\mathrm{G}(K) \cap \Phi_n \neq \emptyset.$$

Now we are in position to apply Proposition 3.2. It follows that for infinitely many n, there exists $z_n \in \mathrm{G}(K)$ such that

$$\mathrm{H}(z_n) \leqslant c_0 \sup_{b\in U^{-1}B_{2^n}} \mathrm{H}(b) \leqslant c_0 c_1\, 2^n = c\, 2^{n-1},$$

and

$$\|x_v - z_n\|_v \leqslant \epsilon_v^{(n)} \leqslant \psi_v(c\, 2^{n-1}) \leqslant \psi_v(\mathrm{H}(z_n)) \quad \text{for all } v \in S',$$
$$\|x_v - z_n\|_v \leqslant 1 \quad \text{for all } v \in S\backslash S'.$$

Recall that $\psi_v(t) \to 0$ as $t \to \infty$ for at least one v. Then since

$$\|x_v - z_n\|_v \leqslant \psi_v(c\, 2^{n-1}) \to 0,$$

we conclude that if $x_v \notin \mathrm{G}(K)$, then the set $\{z_n\}$ must be infinite. This proves that almost every element in Ω' belongs to $\mathcal{W}(G_S, \mathrm{G}(K), \Psi)$, and finishes the proof of Theorem 1.5.

5. Homogeneous varieties

In this section, we state the ergodic theorem and duality principle for homogeneous varieties. Let $\mathrm{X} \subset \mathbb{A}^n$ be a quasi-affine algebraic variety defined over a number field K and equipped with a transitive action of a connected almost simple algebraic group $\mathrm{G} \subset \mathrm{GL}_n$ defined over K. Because G is not assumed to be simply connected, it usually has nontrivial automorphic characters, and the behavior of the averaging operators is more subtle than in Theorem 3.1.

Let $\mathcal{X}_{\mathrm{aut}}(G_{V_K})$ be the set of automorphic characters of G_{V_K}, namely the set consisting of continuous unitary characters χ of G_{V_K} such that $\chi(\mathrm{G}(K)) = 1$. Let S' be a finite subset of V_K. For an open subgroup U of $G_{V_K^f \setminus S'}$, we denote by $\mathcal{X}_{\mathrm{aut}}(G_{V_K})^U$ the subset of $\mathcal{X}_{\mathrm{aut}}(G_{V_K})$ consisting of U-invariant characters. By [GGN12, Lemma 4.4], the set $\mathcal{X}_{\mathrm{aut}}(G_{V_K})^U$ is finite. We denote by G^U the kernel of $\mathcal{X}_{\mathrm{aut}}(G_{V_K})^U$ in G_{V_K}. Then G^U is a finite index subgroup in G_{V_K} (see [GGN12, Lemma 4.4]), which clearly contains $G(K)$.

The following theorem describes the asymptotic behavior of the averaging operators acting in $L^2(\Upsilon)$ for G which is not necessarily simply connected.

Theorem 5.1 [GGN12, Theorem 4.5]. *Let S be a subset of V_K and S' a finite subset of S. Let U^0 be a finite index subgroup of $U_{V_K^f \cap (S \setminus S')}$ and $U = U_{V_K^f \setminus S} U^0$. Let B be a bounded measurable subset of $G_{V_K \setminus S} \cap G^U$ which is bi-$U_{V_K^f \setminus S}$-invariant and β the Haar-uniform probability measure supported on the subset $U^0 B$ of $G_{(V_K \setminus S) \cup (V_K^f \setminus S')}$. Then for every $\phi \in L^2(\Upsilon)$ such that $\mathrm{supp}(\phi) \subset G^U/\mathrm{G}(K)$, we have*

$$\left\| \pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)\phi - \left(\int_\Upsilon \phi \, d\mu \right) \xi_U \right\|_2 \ll_\delta m_{V_K \setminus S}(B)^{-1/(\mathfrak{q}_{V_K \setminus S}(\mathrm{G})) + \delta} \|\phi\|_2$$

for every $\delta > 0$, where ξ_U is the function on Υ such that $\xi_U = |G_{V_K} : G^U|$ on the open set $G^U/\mathrm{G}(K) \subset \Upsilon$ and $\xi_U = 0$ otherwise.

The version of Proposition 3.2 for general homogeneous varieties, i.e. the duality principle, is as follows:

Proposition 5.2 [GGN12, Proposition 5.5]. *Fix $S \subset V_K$, finite $S' \subset S$, $x^0 \in X_{S'}$, and a bounded subset Ω of $G_{S'}$. Then there exist $\epsilon_0 \in (0, 1)$ and a family of measurable subsets Φ_ϵ of Υ indexed by $\epsilon = (\epsilon_v)_{v \in S'}$, where $\epsilon_v \in I_v \cap (0, \epsilon_0)$, that satisfy*

$$\prod_{v \in S'} \epsilon_v^{r_v \dim(\mathrm{X})} \ll \mu(\Phi_\epsilon) \ll \prod_{v \in S'} \epsilon_v^{r_v \dim(\mathrm{X})} \tag{5.1}$$

and the following property holds. If, for $B \subset G_{V_K \setminus S} \times \prod_{v \in V_K^f \cap (S \setminus S')} \mathrm{G}(O_v)$, $\epsilon = (\epsilon_v)_{v \in S'}$ as above, and $\varsigma := (e, g^{-1})\mathrm{G}(K) \in \Upsilon$ with $g \in \Omega$, we have

$$B^{-1}\varsigma \cap \Phi_\epsilon \neq \emptyset,$$

then there exists $\gamma \in \mathrm{G}(O_{(V_K \setminus S) \cup S'})$ such that

$$\mathrm{H}(\gamma) \leqslant c_0 \sup_{b \in B} \mathrm{H}(b) \tag{5.2}$$

and, for $x = gx^0 \in X_{S'}$,

$$\|x_v - \gamma x_v^0\| \leqslant \epsilon_v \quad \text{for all } v \in S'.$$

The upper bound in (5.1) was not stated in [GGN12], but it follows from the explicit construction of the sets Φ_ϵ. We will also need the following lemma about the structure of rational points on homogeneous varieties.

Lemma 5.3. *Let S be a subset of V_K and S' a finite subset of S containing the Archimedean places of S. Suppose that the set $X_{S,S'} \cap \mathrm{X}(K)$, embedded in $X_{S'}$, is not discrete. Then the group* G *is isotropic over $V_K \backslash S$, and $\overline{X_{S,S'} \cap \mathrm{X}(K)}$ is open in $X_{S'}$.*

Proof. We first show that $X_{V_K\backslash S}$ must be noncompact. Indeed, since $X_{S,S'} \cap \mathrm{X}(K)$ is not discrete in $X_{S'}$, there exists a bounded subset D of $X_{S,S'}$ which contains infinitely many elements of $X_{S,S'} \cap \mathrm{X}(K)$. On the other hand, $\mathrm{X}(K)$ is discrete in X_{V_K}, and if $X_{V_K\backslash S}$ were compact, then there would have been only finitely many elements of $\mathrm{X}(K)$ contained in $X_{V_K\backslash S} \times D$, which gives a contradiction.

If G is anisotropic over $V_K \backslash S$, then $V_K \backslash S$ is finite by [PR94, Theorem 6.7], and $G_{V_K\backslash S}$ is compact by [PR94, Theorem 3.1]. It follows from finiteness of Galois cohomology over local fields [PR94, Theorem 6.14] that $X_{V_K\backslash S}$ consists of finitely many orbits of $G_{V_K\backslash S}$. Then $X_{V_K\backslash S}$ is compact, contradicting the previous paragraph. This shows that G must be isotropic over $V_K \backslash S$.

To describe the structure of $\overline{X_{S,S'} \cap \mathrm{X}(K)}$ in $X_{S'}$, we note that

$$X_{S,S'} \cap \mathrm{X}(K) = \mathrm{X}(O_{(V_K\backslash S)\cup S'}).$$

Let $p : \tilde{\mathrm{G}} \to \mathrm{G}$ denote the simply connected cover of G. By [GGN12, Lemma 6.3], the closure $\overline{\mathrm{X}(O_{(V_K\backslash S)\cup S'})}$ in $X_{S'}$ is a union of finitely many open orbits of $p(\tilde{G}_{S'})$. This proves the lemma. □

6. Proof of Theorem 1.6

The proof of Theorem 1.6 follows the same outline as the proof of Theorem 1.5 but with a few more technicalities as necessitated by the more general setup.

Preliminaries and the approximating function

The assumption (1.13) implies that $\mathrm{X}(K) \cap D$ is infinite. In particular, it follows that the set $X_{S,S'} \cap \mathrm{X}(K)$, embedded in $X_{S'}$, is not discrete, and by Lemma 5.3,

$$\overline{X_{S,S'} \cap \mathrm{X}(K)} = \overline{\mathrm{X}(O_{(V_K\backslash S)\cup S'})},$$

is open in $X_{S'}$. Moreover, the closure $\overline{\mathrm{X}(O_{(V_K\backslash S)\cup S'})}$ in $X_{S'}$ is a union of finitely many open orbits of $p(\tilde{G}_{S'})$. Therefore, it suffices to show that, for $x^0 \in \mathrm{X}(O_{(V_K\backslash S)\cup S'})$, almost all points in $p(\tilde{G}_{S'})x^0$ are approximable. Moreover, it suffices to show that for every compact subset Ω of $p(\tilde{G}_{S'})$, almost all points in Ωx^0 are approximable. From now on we fix such Ω and x^0.

If $\psi_v(t) \nrightarrow 0$ as $t \to \infty$ for all $v \in S'$, then the claim of the theorem follows from density. Hence, we assume that $\psi_v(t) \to 0$ as $t \to \infty$ for at least one $v \in S'$.

As in the proof of Theorem 1.5, we may assume, without loss of generality, that $\mathrm{Im}(\psi_v) \subset I_v \cap (0, \epsilon_0)$ with notation as in Proposition 5.2.

We set

$$U^0 = \prod_{v \in V_K^f \cap (S\backslash S')} (U_v \cap \mathrm{G}(O_v)) \quad \text{and} \quad U = U_{V_K^f\backslash S} U^0.$$

Since both U_v and $\mathrm{G}(O_v)$ are open and compact in G_v, it follows that the subgroup $U_v \cap \mathrm{G}(O_v)$ has finite index in U_v. Hence, since $U_v = \mathrm{G}(O_v)$ for almost all v, U^0 is a finite index subgroup in $U_{V_K^f \cap (S\backslash S')}$. Then U is a finite index subgroup in $U_{V_K^f\backslash S'}$. Recall that G^U denotes the kernel of U-invariant automorphic characters of G_{V_K}. It contains $\mathrm{G}(K)$ and has finite index in G_{V_K} (see [GGN12, Lemma 4.4]). We note that $p(\tilde{G}_{S'}) \subset G^U$ because $\tilde{G}$ has no nontrivial automorphic characters (see [GGN12, Lemma 4.1]). We also fix a compact neighborhood U' of identity in $G_{V_K^\infty \backslash S'}$ contained in G^U. Then UU' is a neighborhood of identity in $G_{V_K \backslash S'}$. Let

$$B_h := U_{V_K \backslash S}\{g \in G_{V_K \backslash S} : \mathrm{H}(g) \leqslant h\}U_{V_K \backslash S}$$

and

$$B'_h := U^0(B_h \cap G^U).$$

Since U, U', and $U_{V_K \backslash S}$ are compact, there exists $c_1 \geqslant 1$ such that

$$\sup_{b \in (UU')^{-1}B'_h} \mathrm{H}(b) \leqslant c_1\, h. \tag{6.1}$$

There exists $c_2 = c_2(x^0) \geqslant 1$ such that

$$\mathrm{H}(\gamma x^0) \leqslant c_2\, \mathrm{H}(\gamma). \tag{6.2}$$

To simply notation, we set

$$d := \dim(\mathrm{G}), \quad \mathfrak{q} := \mathfrak{q}_{V_K \backslash S}(\mathrm{G}), \quad \mathfrak{a} := \mathfrak{a}_{S,S'}(\mathrm{X}), \quad \mathfrak{a}_0 := \mathfrak{a}_S(\mathrm{G}). \tag{6.3}$$

Since the function ψ_S is nonincreasing, we deduce from (1.13) that

$$\sum_{n=1}^{\infty} |\mathrm{X}(K) \cap D \cap \{2^{n-1} < \mathrm{H} \leqslant 2^n\}| \cdot \psi_S(2^{n-1})^\alpha = \infty,$$

and from the definition of $\mathfrak{a}$, we also get

$$\sum_{n=1}^{\infty} 2^{(\mathfrak{a}+\delta)n} \psi_S(2^{n-1})^\alpha = \infty \tag{6.4}$$

for every $\delta > 0$. Since $0 < \psi_S \leqslant 1$, there exists $\alpha_0(\delta) \in [0, \infty]$ such that series (6.4) converges for all $\alpha > \alpha_0(\delta)$ and diverges for all $\alpha < \alpha_0(\delta)$. We fix $\alpha_0 > \mathfrak{a}\mathfrak{a}_0^{-1}\mathfrak{q}d/2$ such that series (1.13) diverges. Since divergence in (1.13) implies divergence in (6.4), we have $\alpha_0(\delta) \geqslant \alpha_0$.

Since ψ_S is monotone, (6.4) is equivalent to

$$\sum_{n=1}^{\infty} 2^{(\mathfrak{a}+\delta)n} \psi_S(c2^{n-1})^\alpha = \infty \tag{6.5}$$

with any $c > 0$. We choose $c = c_0c_1c_2/2$ where c_0 is as in (5.2) and c_1, c_2 as in (6.1), (6.2).

As in the proof of Theorem 1.5, we make a reduction to the case when $\alpha_0(\delta) < \infty$. Let $\alpha_0 > \mathfrak{a}\mathfrak{a}_0^{-1}\mathfrak{q}d/2$ be such that series (6.5) diverges. We define R_n as in the proof of Theorem 1.5. Then $R_n \leqslant \psi_S(c\, 2^{n-1})$, $R_n \ll 2^{-\kappa n}$ with some $\kappa > 0$, and

$$\sum_{n=1}^{\infty} 2^{(\mathfrak{a}+\delta)n} R_n^\alpha = \infty \tag{6.6}$$

for $\alpha = \alpha_0$. Since $R_n \ll 2^{-\kappa n}$, series (6.6) converges for sufficiently large α. There exists $\alpha_1(\delta) \in [\alpha_0, \infty)$ such that series (6.6) converges for all $\alpha > \alpha_1(\delta)$ and diverges for all $\alpha < \alpha_1(\delta)$. Since $\psi_S(t) = \prod_{v \in S'} \psi_v(t)^{r_v}$ and $R_n \leqslant \psi_S(c\, 2^{n-1})$, we may write

$$R_n = \prod_{v \in S'} (\epsilon_v^{(n)})^{r_v},$$

where $\epsilon_v^{(n)} \leqslant \psi_v(c\, 2^{n-1})$ for $v \in S'$.

Let β'_n be the Haar-uniform probability measure supported on B'_n. By Theorem 5.1, the averages along β'_n satisfy the mean ergodic theorem: for every $\phi \in L^2(\Upsilon)$ such that $\mathrm{supp}(\phi) \subset G^U/\mathrm{G}(K)$,

$$\left\| \pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta'_n)\phi - \left(\int_\Upsilon \phi\, d\mu\right)\xi_U \right\|_2 \ll_\delta \; m_{V_K\backslash S}(B_{2^n}\cap G^U)^{-1/\mathfrak{q}+\delta}\|\phi\|_2 \tag{6.7}$$

for every $\delta > 0$, where ξ_U is the function on Υ such that $\xi_U = |G_{V_K} : G^U|$ on $G^U/\mathrm{G}(K) \subset \Upsilon$ and $\xi_U = 0$ otherwise.

Applying the mean ergodic theorem in conjunction with duality

Let Φ_n be a family of measurable subsets of Υ defined by Proposition 5.2 with $\epsilon_v = \epsilon_v^{(n)}$, $v \in S'$. We set

$$\phi_n := c_n 1_{\Phi_n} \quad \text{with } c_n = 2^{(\mathfrak{a}+\delta)n} R_n^{\alpha-d}.$$

By construction of the sets Φ_n in Proposition 5.2, Φ_n is a neighborhood of the identity coset in Υ with size determined by $\epsilon_v^{(n)} \leqslant \psi_v(c\, 2^{n-1})$ (see the proof of [GGN12, Proposition 5.5]). Since the divergence condition (1.13) is stable under rescaling of the functions ψ_v, we may arrange that

$$\mathrm{supp}(\phi_n) = \Phi_n \subset G^U/\mathrm{G}(K).$$

By (5.1),

$$\|\phi_n\|_2 = c_n \mu(\Phi_n)^{1/2} \ll 2^{(\mathfrak{a}+\delta)n} R_n^{\alpha-d/2}. \tag{6.8}$$

As in the case of group varieties, we now consider integrability of the functions F_k.

Proposition 6.1. *Let*

$$F_k := \sum_{n\geqslant k} \left| \pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta'_n)\phi_k - \left(\int_\Upsilon \phi_k\, d\mu\right)\xi_U \right|.$$

Then F_k is L^2-integrable for sufficiently small $\delta > 0$ and $\alpha < \alpha_1(\delta)$ sufficiently close to $\alpha_1(\delta)$.

Proof. By (6.7) and (6.8), for every $\delta > 0$,

$$\begin{aligned}\|F_k\|_2 &\ll_\delta \sum_{n\geqslant k} m_{V_K\backslash S}(B_{2^n}\cap G^U)^{-(1/\mathfrak{q})+\delta}\|\phi_n\|_2 \\ &\ll \sum_{n\geqslant k} m_{V_K\backslash S}(B_{2^n}\cap G^U)^{-(1/\mathfrak{q})+\delta} 2^{(\mathfrak{a}+\delta)n} R_n^{\alpha-d/2}.\end{aligned}$$

Moreover, by [GGN12, Lemmas 6.1–6.2], for every $\delta > 0$ and sufficiently large n,

$$m_{V_K\backslash S}(B_{2^n}\cap G^U) \gg m_{V_K\backslash S}(B_{2^n}) \gg_\delta 2^{(\mathfrak{a}_0-\delta)n}.$$

Therefore, for sufficiently large k,

$$\begin{aligned}\|F_k\|_2 &\ll_\delta \sum_{n\geqslant k} 2^{(\mathfrak{a}-\mathfrak{a}_0/\mathfrak{q}+\theta)n} R_n^{\alpha-d/2} \\ &= \sum_{n\geqslant k} 2^{(\mathfrak{a}-\mathfrak{a}_0/\mathfrak{q}+2\theta)n} R_n^{\alpha-d/2}\cdot 2^{-\theta n},\end{aligned}$$

where $\theta = \theta(\delta)$ satisfies $\theta(\delta) \to 0^+$ as $\delta \to 0^+$. Now we apply Hölder's inequality with the exponents

$$r = \mathfrak{a}/(\mathfrak{a} - \mathfrak{a}_0/\mathfrak{q} + 2\theta) \quad \text{and} \quad \bar{r} = (1 - 1/r)^{-1}.$$

When δ is sufficiently small, $r > 1$. This gives

$$\|F_k\|_2 \ll_\delta \left(\sum_{n \geqslant k} 2^{\mathfrak{a}n} R_n^{r(\alpha - d/2)}\right)^{1/r} \cdot \left(\sum_{n \geqslant k} 2^{-\theta \bar{r} n}\right)^{1/\bar{r}}. \tag{6.9}$$

Since $\alpha_0 > \mathfrak{a}\mathfrak{a}_0^{-1}\mathfrak{q}d/2$, it is easy to check that

$$\frac{\mathfrak{a}(\alpha_0 - d/2)}{\mathfrak{a} - \mathfrak{a}_0/\mathfrak{q}} > \alpha_0.$$

Moreover, it follows that for all sufficiently small $\delta > 0$ and all $\alpha \geqslant \alpha_0$,

$$\frac{\mathfrak{a}(\alpha - d/2)}{\mathfrak{a} - \mathfrak{a}_0/\mathfrak{q} + 2\theta} > \alpha.$$

Since $\alpha_1(\delta) \geqslant \alpha_0$, it follows that for $\alpha < \alpha_1(\delta)$ sufficiently close to $\alpha_1(\delta)$ and for sufficiently small $\delta > 0$,

$$r(\alpha - d/2) = \frac{\mathfrak{a}(\alpha - d/2)}{\mathfrak{a} - \mathfrak{a}_0/\mathfrak{q} + 2\theta} > \alpha_1(\delta).$$

Hence, by the definition of $\alpha_1(\delta)$, the series in (6.9) converges, which completes the proof that F_k is L^2-integrable. □

We fix $\delta > 0$ and $\alpha < \alpha_1(\delta)$ such that F_k is L^2-integrable. Let

$$\Upsilon_n := \{\varsigma \in G^U/\mathrm{G}(K) : (B'_{2^n})^{-1}\varsigma \cap \Phi_n = \emptyset\}.$$

By the definition of F_k, on the set $\cap_{n \geqslant k}\Upsilon_n$,

$$F_k = |G_{V_K} : G^U| \sum_{n \geqslant k} \int_\Upsilon \phi_n \, d\mu = |G_{V_K} : G^U| \sum_{n \geqslant k} c_n \mu(\Phi_n).$$

Since by Proposition 5.2, $\mu(\Phi_n) \gg R_n^d$ and $\alpha < \alpha_1(\delta)$, we conclude that

$$F_k \gg \sum_{n \geqslant k} 2^{(\mathfrak{a}+\delta)n} \psi_S(c 2^{n-1})^\alpha = \infty$$

on the set $\cap_{n \geqslant k}\Upsilon_n$. In particular, this shows that $\mu(\cap_{n \geqslant k}\Upsilon_n) = 0$ and $\Upsilon_\infty := \liminf(\Upsilon_n)$ also has measure zero. Let $\widetilde{\Upsilon}_\infty$ be the preimage of Υ_∞ in G^U. Then $m(\widetilde{\Upsilon}_\infty) = 0$. Let

$$\Omega' = \{g \in \Omega : \; \exists y \in UU' : (y, g^{-1}) \notin \widetilde{\Upsilon}_\infty\}.$$

Then since

$$(UU' \times (\Omega \backslash \Omega')^{-1}) \subset \widetilde{\Upsilon}_\infty,$$

and UU' has positive measure in $G_{V_K \backslash S'}$, the set $\Omega \backslash \Omega'$ has measure zero. Then $\Omega' x^0$ has full measure in Ωx^0.

Finally, we show that almost every element of $\Omega' x^0$ belongs to the set $\mathcal{W}(X_{S'}, X_{S,S'} \cap \mathrm{X}(K), \Psi)$. For $g \in \Omega'$, we set $\varsigma := (e, g^{-1})\mathrm{G}(K)$. There exists $y \in UU'$ such that

$$y\varsigma \notin \Upsilon_\infty.$$

This implies that for infinitely many n, $y\varsigma \in \Upsilon_n$, i.e.

$$(y^{-1}B'_{2^n})^{-1}\varsigma \cap \Phi_n \neq \emptyset.$$

Then it follows from Proposition 5.2 that for infinitely many n, there exists $\gamma_n \in \mathrm{G}(O_{(V_K\setminus S)\cup S'})$ such that

$$\mathrm{H}(\gamma_n) \leqslant c_0 \sup_{b\in(UU')^{-1}B'_{2^n}} \mathrm{H}(b) \leqslant c_0 c_1 2^n,$$

and for $x = gx^0$ and $z_n = \gamma_n x^0$,

$$\|x_v - z_n\|_v \leqslant \epsilon_v^{(n)} \leqslant \psi_v(c2^{n-1}) \quad \text{for all } v \in S'.$$

We have $z_n \in \mathrm{X}(O_{(V_K\setminus S)\cup S'})$ and $\mathrm{H}(z_n) \leqslant c_2\mathrm{H}(\gamma_n) \leqslant c2^{n-1}$. Hence, since ψ_v is monotone, we conclude that

$$\|x_v - z_n\|_v \leqslant \psi_v(\mathrm{H}(z_n)) \quad \text{for all } v \in S'.$$

Recall that $\psi_v(t) \to 0$ as $t \to \infty$ for at least one $v \in S'$. If $x_v \notin \mathrm{X}(O_{(V_K\setminus S)\cup S'})$, then it follows that the set $\{z_n\}$ is infinite. Therefore, almost every element of $\Omega' x^0$ is in $\mathcal{W}(X_{S'}, X_{S,S'} \cap \mathrm{X}(K), \Psi)$. This completes the proof of the theorem.

7. Hausdorff dimension

We start by recalling the notion of Hausdorff measure and dimension. Let (M, dist) be a locally compact separable metric space. The s-Hausdorff measure $\mathcal{H}^s$ is a Borel measure on M defined by

$$\mathcal{H}^s(E) := \lim_{\rho\to 0^+} \mathcal{H}^s_\rho(E),$$

where

$$\mathcal{H}^s_\rho(E) := \inf \sum_i r(B_i)^s,$$

and the infimum is taken over all countable covers of E by closed balls B_i such that each B_i has radius at most ρ, and $r(B_i)$ denotes the radius of B_i. The Hausdorff dimension of the set E is defined by

$$\dim(E) := \sup\{s :\ \mathcal{H}^s(E) = \infty\} = \inf\{s :\ \mathcal{H}^s(E) < \infty\}.$$

We assume that for some $d, r_0 > 0$,

$$r^d \ll \mathcal{H}^d(B(x,r)) \ll r^d \tag{7.1}$$

uniformly over all closed balls $B(x,r)$ with $r \leqslant r_0$. Then for every nontrivial closed ball B and $s < d$, we have $\mathcal{H}^s(B) = \infty$.

The following mass transfer principle was proved in [BV06]:

Theorem 7.1 [BV06, Theorem 3]. *Let $\{B(x_i, r_i)\}_{i\in\mathbb{N}}$ be a sequence of closed balls in M with $r_i \to 0$ as $i \to \infty$. Suppose that for some $s \in (0,d)$ and every closed ball C in M,*

$$\mathcal{H}^d(C \cap \limsup B(x_i, r_i^{s/d})) = \mathcal{H}^d(C). \tag{7.2}$$

Then for any closed ball B in M,

$$\mathcal{H}^s(B \cap \limsup B(x_i, r_i)) = \mathcal{H}^s(B). \tag{7.3}$$

The proof of Theorem 7.1 is based on construction of a Cantor-like set in $B \cap \limsup_{i\in\mathbb{N}} B(x_i, r_i)$ and a suitable measure supported on this set with large dimension. The same proof still applies provided that:

(i) (7.1) holds for all closed balls $B(x, r)$ with $r \leqslant r_0$ such that $B \cap B(x, r) \neq \emptyset$;

(ii) (7.2) holds for all closed balls C contained in the ball B.

Proof of Theorem 1.7

In the proof we use notation as in (6.3). We remind the reader that we restrict our attention to the case where all the functions ψ_v are equal to a single function ψ, and we may assume without loss of generality that $\psi(t) \to 0$ as $t \to \infty$.

The assumption (1.13) implies that the set $X_{S,S'} \cap \mathrm{X}(K)$, embedded in $X_{S'}$, is not discrete, and, by Lemma 5.3, its closure $\overline{X_{S,S'} \cap \mathrm{X}(K)}$ is open in $X_{S'}$.

We consider the space $X_{S'}$ with the metric which is the product of local metrics (1.2). We cover X with a collection of Zariski open subsets U such that each U supports a nonvanishing regular differential form of top degree. Since the sets $U_{S'}$ form an open cover of $X_{S'}$, it is sufficient to show that

$$\dim(B_0 \cap \mathcal{W}(X_{S,S'}, X_{S,S'} \cap \mathrm{X}(K), \psi)) \geqslant \frac{2\mathfrak{a}_0}{\mathfrak{a}\mathfrak{q}}\alpha$$

for all nontrivial closed balls B_0 in $X_{S'}$ such that $B_0 \subset U_{S'} \cap \overline{X_{S,S'} \cap \mathrm{X}(K)}$ for some $U_{S'}$.

Let

$$\rho := \mathrm{dist}(B_0, (U_{S'} \cap \overline{X_{S,S'} \cap \mathrm{X}(K)})^c) > 0$$

and

$$\tilde{B}_0 := \{x \in X_{S'} : \mathrm{dist}(x, B_0) \leqslant \rho/2\} \subset U_{S'} \cap \overline{X_{S,S'} \cap \mathrm{X}(K)}.$$

Then every closed ball $B(x, r)$ such that $r \leqslant \rho/4$ and $B(x, r) \cap B_0 \neq \emptyset$ satisfies $B(x, r) \subset \tilde{B}_0$.

Let $\lambda_{S'}$ be the measure on $U_{S'}$ defined by the nowhere-zero differential form. Different choices of differential forms lead to equivalent measures. Since X is a homogeneous variety, it is nonsingular, and, computing in local coordinates, we obtain that, for all closed balls $B(x, r) \subset U_{S'}$,

$$r^d \ll \lambda_{S'}(B(x, r)) \ll r^d,$$

where $d := (\sum_{v\in S'} r_v) \dim(\mathrm{X})$. This estimate is uniform over all closed balls $B(x, r)$ with $x \in \tilde{B}_0$ and bounded r. Therefore, we conclude that

$$\mathcal{H}^d \ll \lambda_{S'} \ll \mathcal{H}^d \tag{7.4}$$

uniformly over Borel subsets of $\tilde{B}_0$. In particular, it follows that property (i) (stated after Theorem 7.1) holds.

We apply Theorem 7.1 to the collection of closed balls $B(z, \psi(\mathrm{H}(z)))$ with $z \in X_{S,S'} \cap \mathrm{X}(K)$. Choose any positive $s < (2\mathfrak{a}_0/\mathfrak{a}\mathfrak{q})\alpha$ and set $\tilde{\psi} := \psi^{s/d}$. We have

$$\mathcal{W}(X_{S,S'}, X_{S,S'} \cap \mathrm{X}(K), \tilde{\psi}) = \limsup B(z, \tilde{\psi}(\mathrm{H}(z))).$$

According to our assumption, for $\beta := (d/s)\alpha > (\mathfrak{a}\mathfrak{q}/2\mathfrak{a}_0)d$,

$$\sum_{z\in \mathrm{X}(K)\cap D} \tilde{\psi}_{S'}(\mathrm{H}(z))^\beta = \infty.$$

Hence, by Theorem 1.6, the set $\mathcal{W}(X_{S,S'}, X_{S,S'} \cap \mathrm{X}(K), \tilde{\psi})$ has full measure in $X_{S'}$, and in particular, it follows from (7.4) that for every closed ball C contained in B_0,

$$\mathcal{H}^d(C \cap \limsup B(z, \tilde{\psi}(\mathrm{H}(z)))) = \mathcal{H}^d(C).$$

This verifies property (ii) (stated after Theorem 7.1), and Theorem 7.1 now implies that

$$\mathcal{H}^s(B_0 \cap \limsup B(z, \psi(\mathrm{H}(z)))) = \mathcal{H}^s(B_0) = \infty$$

for every $s < (2\mathfrak{a}_0/\mathfrak{a}\mathfrak{q})\alpha$. This proves that

$$\dim(B_0 \cap \mathcal{W}(X_{S,S'}, X_{S,S'} \cap \mathrm{X}(K), \psi)) \geqslant \frac{2\mathfrak{a}_0}{\mathfrak{a}\mathfrak{q}}\alpha,$$

as required.

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