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THE AVERAGE NUMBER OF DIVISORS IN AN ARITHMETIC PROGRESSION

BY
R. A. SMITH AND M. V. SUBBARAO

1. Let l and k be positive integers. Then for each integer $n \geq 1$, define $d(n; l, k)$ to be the number of (positive) divisors of n which lie in the arithmetic progression $l \bmod k$. Note that $d(n; 1, 1) = d(n)$, the ordinary divisor function. To study the average behavior of $d(n; l, k)$, we define

$$D(x; l, k) = \sum_{1 \leq n \leq x} d(n; l, k)$$

which may be written as

$$D(x; l, k) = \sum_{\substack{1 \leq n \leq x \\ n \equiv l \bmod k}} \left[\frac{x}{n}\right] \tag{1}$$

where $[x]$ denotes the largest integer $\leq x$. If $(l, k) = 1$, the orthogonality relation for the Dirichlet characters mod k implies that (1) may be rewritten as

$$D(x; l, k) = \frac{1}{\phi(k)} \sum_{\chi \bmod k} \bar{\chi}(l) D(x, \chi) \tag{2}$$

where the sum is taken over all characters χ mod k and

$$D(x, \chi) = \sum_{1 \leq n \leq x} \left[\frac{x}{n}\right] \chi(n). \tag{3}$$

If we define

$$a_n(\chi) = \sum_{d \mid n} \chi(d),$$

then (3) can be rewritten as

$$D(x, \chi) = \sum_{1 \leq n \leq x} a_n(\chi). \tag{4}$$

Throughout this paper, the implied constants in the O-terms are independent of l, k and x. Occasionally, it will be convenient to use the Vinogradov notation $f \ll g$, which means $f = O(g)$. Finally, let $\{x\} = x - [x]$ denote the fractional part of x.

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THEOREM. *If* χ *is any non-principal character* mod k, *then*

$$D(x, \chi) = xH(x, \chi) + O((kx)^{1/3}\, d(k) \log x)$$

holds uniformly in k *and* x *tending to* ∞ *provided* $k \ll x^2$, *where*

$$H(x, \chi) = \sum_{1 \le n \le x} \frac{\chi(n)}{n}.$$

Three corollaries follow immediately from this result, the first of which has been obtained by Žogin [5] for *fixed* k with a corresponding error term of order $x^{1/2}$. Following Lehmer [3], we define Euler's constant $\gamma(l, k)$ for the arithmetic progression l mod k by

$$\gamma(l, k) = \lim_{x \to \infty} \left(H(x; l, k) - \frac{\log x}{k} \right)$$

where

$$H(x; l, k) = \sum_{\substack{1 \le n \le x \\ n \equiv l \bmod k}} \frac{1}{n}.$$

Note that $\gamma(1, 1) = \gamma$ is Euler's constant.

COROLLARY 1. *If* $(l, k) = 1$, *then*

$$D(x; l, k) = \frac{1}{k} x \log x + \left(\gamma(l, k) - \frac{1-\gamma}{k} \right) x + O((kx)^{1/3}\, d(k) \log x)$$

holds uniformly in l, k *and* x *tending to* ∞, *provided* $k \le x$.

We observe that the condition $(l, k) = 1$ is a trivial restriction since if $(l, k) = r > 1$, then $D(x; l, k) = D(x/r; l/r, k/r)$. The special case $l = k = 1$ corresponds to the well-known Divisor Problem of Dirichlet.

COROLLARY 2. *If* χ *is a non-principal character* mod k, *then*

$$\sum_{1 \le n \le x} \left\{ \frac{x}{n} \right\} \chi(n) = O((kx)^{1/3}\, d(k) \log x)$$

holds uniformly in k *and* x *tending to* ∞, *provided* $k \ll x^2$.

COROLLARY 3.

$$\sum_{\substack{1 \le n \le x \\ n \equiv l \bmod k}} \left\{ \frac{x}{n} \right\} = \frac{1-\gamma}{k} x + O((kx)^{1/3}\, d(k) \log x)$$

holds uniformly in l, k *and* x *tending to* ∞ *provided* $k \le x$; *note that* l *and* k *are not required to be coprime.*

To deduce Corollary 1 from the theorem, we separate the term in (2) arising from the principal character χ_0 from the other terms, and obtain

$$D(x; l, k) = xH(x; l, k) - F(x, \chi_0) + O((kx)^{1/3}\, d(k) \log x) \tag{5}$$

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where

$$F(x, \chi_0) = \frac{1}{\phi(k)} \sum_{1 \le n \le x} \left\{\frac{x}{n}\right\} \chi_0(n).$$

For any $x \ge k$, the Euler–Maclaurin summation formula implies that

$$H(x; l, k) = \frac{1}{k} \log x + \gamma(l, k) + O\left(\frac{1}{x}\right),$$

the constant implied in the O-term being independent of l, k and x. Since

$$F(x, \chi_0) = \frac{1}{\phi(k)} \sum_{d \mid k} \mu(d) \sum_{1 \le n \le x/d} \left\{\frac{x/d}{n}\right\},$$

it follows from (5) with $l = k = 1$ that

$$F(x, \chi_0) = \frac{1-\gamma}{k} x + O(x^{1/3}\, d(k) \log x) \tag{6}$$

which completes the proof of Corollary 1.

Corollary 2 is a trivial consequence of our theorem, as can readily be seen on replacing $[x]$ by $x - \{x\}$ in (3). Finally, corollary 3 is a trivial consequence of corollary 1 on replacing $[x]$ by $x - \{x\}$ in (1) and applying (5) and (6).

REMARKS. The main significance of these results lies in the uniformity condition in k, since corollary 2 has been obtained by Chandrasekharan and Narasimhan [2, p. 133] for *fixed* k, from which all our results can easily be deduced (k fixed). It should be pointed out that while they only considered the case of real primitive characters χ, their argument applies equally well for non-primitive $\chi \ne \chi_0$, though an extra $\log x$ factor must be inserted in their O-term. Finally, we remark that the main terms in the asymptotic formulae of Corollaries 1 and 3 dominate the error terms as x tends to ∞ whenever $k \ll x^{1/2-\epsilon}$, where $\epsilon > 0$ is arbitrary.

2. We now prove the theorem stated above. In [1], Berndt obtains a variation of Theorem 4.1 of Chandrasekharan and Narasimhan [2] which holds uniformly in $x \to \infty$ and $\lambda \to 0^+$, where $\lambda > 0$ is a parameter associated with the given Dirichlet series; our theorem is easily deduced as a special case of Berndt's theorem (with $\lambda = k^{-1/2}$ roughly).

As the first step towards establishing our theorem, we shall now introduce the following Dirichlet series for a primitive character $\chi \bmod q$ in view of the definition of $D(x, \chi)$ given by (4):

$$\varphi(s) = \varphi(s, \chi) = \sum_{n \ge 1} a_n(\chi) \lambda_n^{-s} = \lambda^{-s} \zeta(s) L(s, \chi) \tag{7}$$

where

$$\lambda_n = \lambda \cdot n \quad \text{with} \quad \lambda = \pi q^{-1/2}. \tag{8}$$

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From the well-known properties of $\zeta(s)$ and $L(s;\chi)$ [4, pp. 207–208], it follows that $\varphi(s,\chi)$ is a meromorphic function of s having a unique singularity at $s=1$ corresponding to a simple pole with residue $\lambda^{-1}L(1,\chi)\neq 0$, and satisfies the functional equation

(9) $$\Delta(s)\varphi(s)=\Delta(1-s)\psi(1-s)$$

where

$$\Delta(s)=\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+a(\chi)}{2}\right),$$

$$\psi(s)=\psi(s,\chi)=\varepsilon(\chi)\varphi(s,\bar{\chi}),$$

$$|\varepsilon(\chi)|=1,$$

and

$$a(\chi)=\begin{cases}0 & \text{if} \quad \chi(-1)=1\\ 1 & \text{if} \quad \chi(-1)=-1.\end{cases}$$

We observe that $b_n=b_n(\chi)=\varepsilon(\chi)a_n(\bar{\chi})$ in the notation of [1], and furthermore, we note that (9) implies

$$\varphi(0,\chi)=-\tfrac{1}{2}a(\chi)\varepsilon(\chi)L(1,\bar{\chi}).$$

Therefore, in the notation of [1], we have

$$\begin{aligned}Q^*(x,\lambda)&=(\operatorname*{res}_{s=1}\varphi(s,\chi))\lambda x+\varphi(0,\chi)\\ &=L(1,\chi)x-\tfrac{1}{2}a(\chi)\varepsilon(\chi)L(1,\bar{\chi}).\end{aligned}$$

Since $L(1,\chi)\ll\log q$ (this follows by partial summation), then $Q^*(x,\lambda)\ll x\log q$, whence it follows that $\rho=-\frac{2}{3}$ upon noting that $|a_n|=|b_n|\leq d(n)$. Choosing $\eta=\frac{1}{6}$, we see that $E(x,\lambda)=c(qx)^{1/3}\log x$ for some absolute constant $c>0$. Since the remaining hypotheses of Berndt's theorem are readily verified (noting that $f(\lambda)=0$ in our case), we therefore can conclude that

$$\begin{aligned}D(x,\chi)-xL(1,\chi)&\ll\sum_{x<n\leq x+y}|a_n(\chi)|+(qx)^{1/3}\log q\\ &\ll\sum_{x<n\leq x+y}d(n)+(qx)^{1/3}\log q\end{aligned}$$

where $y\ll(qx)^{1/3}$. Therefore, the asymptotic formula for $\sum_{n\leq x}d(n)$ given by, say, Chandrasekharan and Narasimhan (cf. [2, equation (10.3)]) implies that

(10) $$D(x,\chi)=xL(1,\chi)+O((qx)^{1/3}\log qx)$$

holds for all primitive characters $\chi \bmod q$. Since $L(1,\chi)=H(x,\chi)+0(x^{-1}q^{1/2}\log q)$ for primitive χ (again by partial summation), it follows that

(10) may be rewritten as

$$D(x, \chi) = xH(x, \chi) + O((qx)^{1/3} \log x) \tag{11}$$

since $q \ll x^2$ implies $q^{1/2} \ll (qx)^{1/3}$.

To complete the proof of the theorem, let χ be a non-principal character mod k. Then there exists a unique primitive character χ^* mod q which induces χ, where $q > 1$ and q divides k [4, p. 126]. Since $\chi(n) = \chi^*(n)$ whenever $(n, k) = 1$, then (3) implies that

$$D(x, \chi) = \sum_{d \mid k} \mu(d)\chi^*(d)D\left(\frac{x}{d}, \chi^*\right).$$

By (11), this may be rewritten as

$$\begin{aligned} D(x, \chi) &= \sum_{d \mid k} \mu(d)\chi^*(d)\left(\frac{x}{d}H\left(\frac{x}{d}, \chi^*\right) + O((qx)^{1/3} \log x)\right) \\ &= xH(x, \chi) + O((kx)^{1/3}\, d(k) \log x) \end{aligned}$$

since $q \le k$ and

$$H(x, \chi) = \sum_{d \mid k} \frac{\mu(d)\chi^*(d)}{d} H\left(\frac{x}{d}, \chi^*\right).$$

This completes the proof of the theorem.

References

1. B. Berndt, *On the average order of ideal functions and other arithmetic functions*, Bull. Amer. Math. Soc. **76** (1970), 1270–1274.

2. K. Chandrasekharan and R. Narasimhan, *Functional equations with multiple gamma factors and the average order of arithmetic functions*, Annals of Math. (2) **76** (1962), 93–136.

3. D. H. Lehmer, *Euler's constant for arithmetic progressions*, Acta Arith. **27** (1975), 125–142.

4. K. Prachar, *Primzahlverteilung*, Springer-Verlag, Berlin (1957).

5. I. I. Žogin, *Certain asymptotic equations connected with the problem of Dirichlet on divisors. A generalization of the Dirichlet Theorem*. Sverdlovsk. Gos. Ped. Inst. Učen. Zap. **31** (1965), 87–96.

Department of Mathematics,
University of Toronto
Toronto M5S 1A1, Canada.

Department of Mathematics,
University of Alberta,
Edmonton T6G 2G1, Canada.