9

Global symmetries

In this chapter we consider the impact of global symmetries of a field theory on its renormalization. As an example consider the theory of a charged scalar field:

$$\mathscr{L} = \partial\phi^\dagger\partial\phi - m^2\phi^\dagger\phi - g(\phi^\dagger\phi)^2/4. \tag{9.0.1}$$

This classical Lagrangian is invariant under the transformation $\phi \to e^{-i\omega\phi}$. The quantum theory is also invariant. For this particular theory, the quantum invariance is not a very deep statement. However, symmetries do not always survive quantization, as we will see in Chapter 13. Thus it is useful to examine the consequences of the symmetry in this theory. One consequence is that only invariant counterterms are needed; for example, we do not need to use non-invariant counterterms proportional to

$$\phi^2 + \phi^{\dagger 2} \quad \text{or} \quad i(\phi^2 - \phi^{\dagger 2}).$$

Other consequences are the Ward identities, which characterize the action of the symmetry at the level of Green's functions.

The main step in proving the statements is to impose an ultra-violet cut-off. If this is done by putting the theory on a lattice or by using dimensional regularization, the symmetry is preserved. The arguments given in Section 2.7 are sufficient to prove Ward identities in the bare theory. From the invariance of Green's functions follows invariance of the counterterms. As we will see in Section 9.1 we can then write renormalized Ward identities in the renormalized theory, which therefore exhibits the symmetry.

In more general cases this simple procedure fails.

One case is that the UV cut-off breaks the symmetry. For example, putting the theory on a lattice breaks Poincaré invariance. Luckily, other regulators, like dimensional continuation, preserve this invariance, and the renormalized theory with no cut-off is Poincaré invariant. Some symmetries cannot be preserved after quantization. It must be true that no regulator can preserve them. An example, to be treated in Chapter 13, is the chiral invariance of QCD.

Another case, which we will treat later in this chapter, is of spontaneous symmetry breaking, typified by the theory given by (9.0.1) with m^2 replaced by $-m^2$. This is called the Goldstone model. In this case the ground-state – the vacuum – is not invariant under the symmetry, and the field acquires a vacuum expectation value:

$$\langle 0|\phi|0\rangle = [2|m^2|/g]^{1/2} + \text{higher order.}$$

If we use an invariant regulator, like dimensional continuation, we will still be able to prove Ward identities. Hence, we will be able to prove that only symmetric counterterms are needed, so that the symmetry is preserved. From the Ward identities follows Goldstone's theorem, that there is a massless boson for each generator of a broken symmetry.

9.1 Unbroken symmetry

We first consider a totally unbroken internal symmetry. The fields carry a matrix representation of the generators. Thus:

$$\delta_\alpha \phi_i = -\mathrm{i}(t_\alpha)_i{}^j \phi_j, \tag{9.1.1}$$

in the notation of Section 2.6.

The proof that the symmetry can be preserved under quantization is elementary. We spell out the steps so that we can see what needs to be done in less trivial cases:

(1) Regulate in a way that preserves the symmetry. Lattice and dimensional regularization both do this since the symmetry commutes with all space-time transformations.
(2) Include in $\mathscr{L}$ all possible invariant counterterms up to the appropriate dimension. Thus $\delta_\alpha \mathscr{L} = 0$. For the model (9.0.1) we replace $\mathscr{L}$ by
$$\mathscr{L} = Z\partial\phi^\dagger\partial\phi - m_{\mathrm{B}}^2\phi^\dagger\phi - g_{\mathrm{B}}(\phi^\dagger\phi)^2/4. \tag{9.1.2}$$
(3) To do perturbation theory, let the free Lagrangian be invariant: $\delta_\alpha \mathscr{L}_0 = 0$. Then the interaction Lagrangian is also invariant.
(4) At each order, choose the counterterms to cancel the divergences in 1PI Green's functions. Since the free propagators and the interactions are all invariant under the symmetry, the divergences are symmetric and non-invariant counterterms are not needed.
(5) Remove the UV cut-off. The Green's functions are symmetric:
$$\delta_\alpha \langle 0|T\phi_{j_1}(y_1)\dots\phi_{j_N}(y_N)|0\rangle = 0. \tag{9.1.3}$$

In the case of the model (9.0.1) the propagator for the charged field carries an arrow indicating the direction of flow of charge. All vertices have equal numbers of ingoing and outgoing lines. In (9.1.3) we have $\delta\phi = -\mathrm{i}\phi$ and $\delta\phi^\dagger = \mathrm{i}\phi^\dagger$, so this equation is literally a statement of charge conservation.

The current for a symmetry is defined by Noether's theorem (Section 2.6):

$$j^\mu_\alpha = \sum_i \delta_\alpha \phi_i \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi_i}. \tag{9.1.4}$$

In the case of the simple model (9.0.1) there is a single current

$$j^\mu = \mathrm{i}Z\phi^\dagger \overleftrightarrow{\partial^\mu} \phi. \tag{9.1.5}$$

We derived the Ward identities of the bare theory (2.7.6). For the theory (9.0.1) these are

$$\frac{\partial}{\partial x^\mu}\langle 0|Tj^\mu(x)\phi(y_1)\cdots\phi(y_N)\phi^\dagger(z_1)\cdots\phi^\dagger(z_N)|0\rangle$$
$$= \mathrm{i}\sum_{j=1}^{N}[\delta(x-y_j)-\delta(x-z_j)]\langle 0|T\phi(y_1)\cdots\phi(y_N)\phi^\dagger(z_1)\cdots\phi^\dagger(z_N)|0\rangle. \tag{9.1.6}$$

We showed in Section 6.6 that the current is in fact finite; no extra renormalization counterterms are needed beyond those implied by the factor Z in (9.1.5).

It is of interest to see how the divergences that are present get cancelled by the factor Z. For the two-point function of j^μ we have the 1PI graphs of Fig. 9.1.1, up to order g^2. Since $Z = 1 + O(g^2)$ in this theory, we may replace Z by 1 everywhere except in the tree graph (*a*). Graph (*b*) could be logarithmically divergent by power-counting, but is in fact zero, so no counterterm is needed at order g. Graph (*e*) is also zero. Graphs (*c*) and (*d*) are finite after their subdivergences are cancelled by a counterterm; they also cancel each other. These cancellations arise since these graphs have a

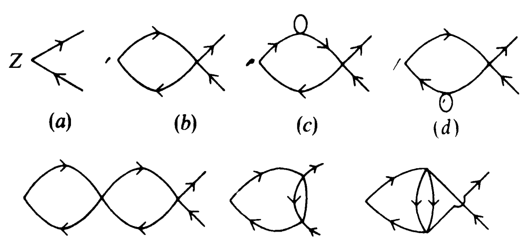

Fig. 9.1.1. Graphs up to order g^2 for the two-point function of j^μ.

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subgraph which is a graph for

$$\langle 0|Tj^\mu(x)\,\phi^\dagger\phi(y)|0\rangle.$$

In momentum space this is of the form $q^\mu f(q^2)$. The Ward identity implies that its divergence is zero:

$$\frac{\partial}{\partial x^\mu}\langle 0|Tj^\mu(x)\,\phi^\dagger\phi(y)|0\rangle = \mathrm{i}\delta(x-y)\langle 0|\delta(\phi^\dagger\phi)|0\rangle = 0,$$

so that $q^2 f(q^2) = 0$.

Graphs (f) and (g) each have a subdivergence which is cancelled by a graph of the form (b), which is zero. Their overall divergence must be cancelled by using the order g^2 term in Z in graph (a).

9.2 Spontaneously broken symmetry

To explain the renormalization of theories with spontaneously broken symmetry it will be sufficient to consider the case of the Goldstone model:

$$\begin{aligned}\mathscr{L} &= Z\partial_\mu\phi^\dagger\partial^\mu\phi + m^2\phi^\dagger\phi - g(\phi^\dagger\phi)^2/4 + \delta m^2\phi^\dagger\phi - \delta g(\phi^\dagger\phi)^2/4 \\ &= (\partial\phi_1)^2/2 + (\partial\phi_2)^2/2 + m^2(\phi_1^2+\phi_2^2)/2 - g(\phi_1^2+\phi_2^2)^2/16 \\ &\quad + \text{counterterms.}\end{aligned} \tag{9.2.1}$$

Here we have written the complex scalar field in terms of real fields: $\phi = (\phi_1 + \mathrm{i}\phi_2)2^{-1/2}$. The mass term is of the 'wrong sign'. This will result in spontaneous breaking of the symmetry under $\phi \to \phi\mathrm{e}^{-\mathrm{i}\omega}$. The Noether current for this symmetry is

$$j^\mu = \mathrm{i}Z\phi^\dagger\overleftrightarrow{\partial^\mu}\phi = Z(\phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1). \tag{9.2.2}$$

For small couplings the Euclidean functional integral is dominated by fields close to the minimum of the potential in (9.2.1). This is at

$$|\phi| = 2m/g^{1/2}. \tag{9.2.3}$$

The perturbation expansion amounts to a saddle point expansion about the minimum. It is set up by making the substitution

$$\phi_1 = \phi_1' + 2m/g^{1/2}, \tag{9.2.4}$$

to give

$$\begin{aligned}\mathscr{L} &= (\partial\phi_1')^2/2 + (\partial\phi_2)^2/2 - m^2\phi_1'^2 - g(\phi_1'^2+\phi_2^2)^2/16 \\ &\quad - mg^{1/2}\phi_1'(\phi_1'^2+\phi_2^2)/2 + \mathscr{L}_{\mathrm{ct}},\end{aligned} \tag{9.2.5a}$$

where

$$\begin{aligned}\mathscr{L}_{\mathrm{ct}} = &-\delta g(\phi_1'^2+\phi_2^2)^2/16 - \delta g m g^{-1/2}\phi_1'(\phi_1'^2+\phi_2^2)/2 \\ &- \phi_1'^2(3m^2\delta g/g - \delta m^2)/2 - \phi_2^2(m^2\delta g/g - \delta m^2)/2 \\ &- 2\phi_1' m g^{-1/2}(m^2\delta g/g - \delta m^2) + (Z-1)(\partial\phi_1'^2 + \partial\phi_2^2)/2.\end{aligned} \tag{9.2.5b}$$

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The idea of making this perturbation expansion is that in the functional integral we impose a boundary condition that fixes the phase of the field at ∞. By the symmetry we may make this phase real, without loss of generality. In three or more space-time dimensions, fields that have a different phase over a large region have an action so much larger that quantum fluctuations cannot destroy the boundary condition. Then ϕ_1 is forced to have a real vacuum expectation value close to $2m/g^{1/2}$.

In setting up the perturbation expansion we have tadpole graphs like Fig. 9.2.1. These generate a vacuum expectation value for ϕ_1'

$$\langle 0|\phi_1'|0\rangle = \delta v$$

that starts at order $g^{1/2}$. It means that ϕ_1 has vacuum expectation value $2mg^{-1/2} + \delta v$. There are then graphs like Fig. 9.2.2, where the tadpoles appear as subgraphs. It is possible to recast the Feynman rules by writing $\phi_1 = \phi_1'' + 2mg^{-1/2} + \delta v$ and requiring ϕ_1'' to have zero vacuum expectation value. A better practical approach is to impose $\delta v = 0$ as a renormalization condition on δm^2.

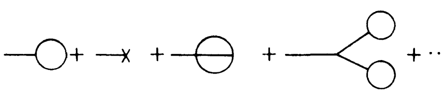

Fig. 9.2.1. Graphs for $\langle 0|\phi_1'|0\rangle$.

Fig. 9.2.2. Graphs containing tadpoles as subgraphs.

If we start with the theory (9.0.1) without spontaneous symmetry breaking and vary m^2 until it is negative, then we should pass through a phase transition and thereby reach the Goldstone model (9.2.1). There must be an actual phase transition because $\langle 0|\phi|0\rangle$ is exactly zero in the phase with unbroken symmetry. Since this expectation value is non-zero in the Goldstone phase there must be non-analyticity of the theory as a function of m^2.

Now, we must renormalize the theory: the continuation in the renormalized mass m^2 is sensible only if the counterterms are the same functions of m^2 and g in the two phases. It is sensible to use a mass-independent renormalization prescription, for then the dependence on m^2 of the counterterms is the simplest possible. We will prove the following:

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(1) Renormalization of the Goldstone phase is accomplished by using only symmetric counterterms in the Lagrangian (9.2.1).
(2) The dimensionless counterterms $Z-1$, δg, and $\delta m^2/m^2$ can be chosen to be the same as in the phase of unbroken symmetry (the so-called Wigner phase).
(3) The current given by (9.2.2) is finite just as it is in the Wigner phase. Since the bare Lagrangian is invariant under $\phi \to \phi e^{-i\omega}$, Ward identities are valid and from them Goldstone's theorem follows, that the physical mass of ϕ_2 is exactly zero.

We must also discuss the choice of a practical renormalization prescription.

9.2.1 *Proof of invariance of counterterms*

We will do perturbation theory by choosing the free Lagrangian

$$\mathscr{L}_0 = \partial\phi_1'^2/2 + \partial\phi_2^2/2 - m^2\phi_1'^2, \tag{9.2.6}$$

and the basic interaction

$$\mathscr{L}_b = -g(\phi_1'^2 + \phi_2^2)/16 - mg^{1/2}\phi_1'(\phi_1'^2 + \phi_2^2)/2. \tag{9.2.7}$$

The counterterms are given the form (9.2.5*b*) and δg, $\delta m^2/m^2$, and Z are given the same values as in the unbroken theory with a mass-independent renormalization scheme. We will prove that these counterterms are sufficient to make the broken-symmetry theory finite.

Some of the interaction vertices are the same as in the Wigner phase. The others are obtained by substituting $2m/g^{1/2}$ for ϕ_1. Therefore graphs involving the extra vertices are obtained by erasing external ϕ_1 lines on symmetric graphs. Examples are shown in Fig. 9.2.3. The only complication is that mass terms generated from the basic interaction go into the free rather than the interaction Lagrangian. This is the sole source of complications in our proof.

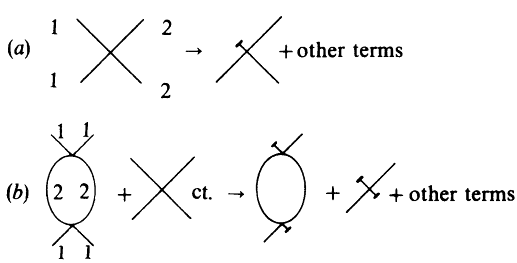

Fig. 9.2.3. Generation of graphs in theory with spontaneously broken symmetry from graphs in the symmetric theory.

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To relate the counterterms to those in the unbroken theory let us write the free propagators as follows:

$$\phi_1': \frac{\mathrm{i}}{p^2-2m^2} = \frac{\mathrm{i}}{p^2-M^2} + \frac{\mathrm{i}(2m^2-M^2)}{(p^2-M^2)^2} + \frac{\mathrm{i}(2m^2-M^2)^2}{(p^2-M^2)^2(p^2-2m^2)},$$
$$\phi_2: \frac{\mathrm{i}}{p^2} = \frac{\mathrm{i}}{p^2-M^2} + \frac{-\mathrm{i}M^2}{(p^2-M^2)^2} + \frac{\mathrm{i}M^4}{(p^2-M^2)^2 p^2}. \qquad (9.2.8)$$

Here M^2 is a arbitrary parameter. We substitute (9.2.8) for every line in a graph.

Suppose we substitute the first term on the right of (9.2.8) for every line of a basic graph which has only four-point basic vertices. Then we obtain a graph in the symmetric theory with mass M.

The difference between these symmetric graphs and the true theory is given by:

(1) graphs with one or more three-point vertices,
(2) graphs with the second or third term on the right of (9.2.8) substituted for one or more propagators.

In either case the degree of divergence is reduced. Now the maximum degree of divergence is two. So substitution of the third term in (9.2.8) always makes a graph overall convergent. We are allowed at most one substitution of the second term.

Let us now suppose that all graphs with fewer than N loops are successfully renormalized by our symmetric counterterms. We will prove inductively that all N-loop graphs are renormalized. The induction starts because tree graphs need no renormalization. We decompose the mass counterterm in $\mathscr{L}_{\mathrm{ct}}$ as

$$-\tfrac{1}{2}\phi_1'^2[3m^2\delta g/g + (Z_m-1)(-m^2-M^2) + (Z_m-1)M^2]$$
$$-\tfrac{1}{2}\phi_2^2[m^2\delta g/g + (Z_m-1)(-m^2-M^2) + (Z_m-1)M^2]. \qquad (9.2.9)$$

Here $Z_m = (m^2+\delta m^2)/m^2$ is the mass renormalization factor.

After substitution of (9.2.8) for each propagator in a basic 1PI graph with N-loop all subdivergences are cancelled by counterterms of lower order, according to the inductive hypothesis. We are left with the following overall divergences:

(1) Logarithmically divergent graphs for the four-point function with all propagators set to $\mathrm{i}/(p^2-M^2)$ and with only four-point vertices. Such graphs have an overall divergence independent of M which is removed by counterterms in δg for the symmetric theory. No other 1PI graph for the four-point function has an overall divergence.

(2) Self-energy graphs with four-point vertices only and with propagators $i/(p^2 - M^2)$. Field renormalization and the $(Z_m - 1)M^2$ terms in (9.2.9) renormalize these, again exactly as in the symmetric theory.
(3) Self-energy graphs as in (2) but with one propagator replaced by a second term in (9.2.8). In the numerators of (9.2.8) we write

$$2m^2 - M^2 = 3m^2 - (m^2 + M^2) = (2m/g^{1/2})^2(3g/4) - (m^2 + M^2),$$
$$-M^2 = m^2 - (m^2 + M^2) = (2m/g^{1/2})^2(g/4) - (m^2 + M^2). \qquad (9.2.10)$$

The terms with $-(m^2 + M^2)$ are renormalized by the $(Z_m - 1)(-m^2 - M^2)$ parts of the mass counterterms. They correspond to the effect of differentiating the self-energy graphs with respect to M^2. The other terms in (9.2.10) we will regard as an insertion of a four-point vertex on a line when two ϕ_1 fields are replaced by $2m/g^{1/2}$. These terms are considered under (4).
(4) Graphs of classes (1) and (2) in which one or more external ϕ_1 fields are deleted and replaced by $2m/g^{1/2}$. Examples are Fig. 9.2.3(*b*) and Fig. 9.2.4. The same replacement generates the counterterm Lagrangian (9.2.5*b*) from the symmetric theory, so we have counterterms for them.

This completes the proof.

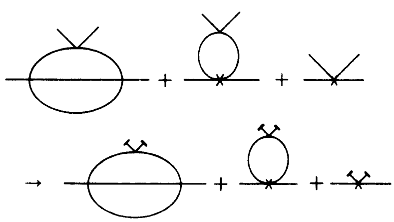

Fig. 9.2.4. Generation of graphs with loops in theory with spontaneously broken symmetry from graphs in the symmetric theory.

9.2.2 *Renormalization of the current*

The same procedure shows that the current

$$j^\mu = Z(\phi_1'\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1') + 2Zmg^{-1/2}\partial^\mu\phi_2 \qquad (9.2.11)$$

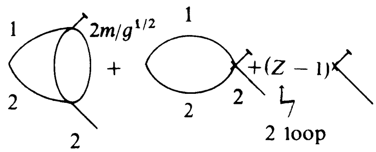

Fig. 9.2.5. Renormalization of current in spontaneously broken theory.

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has finite Green's functions. Note that the term $2Zmg^{-1/2}\partial^\mu\phi_2$ contains the counterterms that renormalize graphs like Fig. 9.2.5.

The Ward identities are then true and involve finite quantities. A typical case is

$$\begin{aligned}\partial_\mu\langle 0|Tj^\mu(x)\phi_2(y)|0\rangle &= \mathrm{i}\langle 0|\delta\phi_2(y)|0\rangle \\ &= -\mathrm{i}\langle 0|\phi_1(y)|0\rangle \\ &= -\mathrm{i}(2m/g^{1/2}+\delta v). \end{aligned} \tag{9.2.12}$$

By multiplying by the inverse propagator for ϕ_2 and going to momentum space, we find

$$p_\mu\Gamma^\mu_{j,2}(p) = (2m/g^{1/2}+\delta v)(\mathrm{i}/G_{22}(p^2)), \tag{9.2.13}$$

where $\Gamma^\mu_{j,2}$ is the set of graphs for $\langle 0|Tj^\mu\phi_2|0\rangle$ that are 1PI in ϕ_2. This is illustrated in Fig. 9.2.6. Since $\Gamma^\mu \propto p^\mu$ as $p^2\to 0$, (9.2.13) implies that G_{22}^{-1} has a zero at $p^2=0$, in other words that ϕ_2 is massless to all orders of perturbation theory. This is the Goldstone theorem (Goldstone, Salam & Weinberg (1962)).

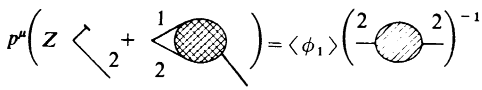

Fig. 9.2.6. The Ward identity that implies Goldstone's theorem.

9.2.3 *Infra-red divergences*

Individual graphs with a self-energy insertion on a ϕ_2 line have infra-red divergences. Such a graph is illustrated in Fig. 9.2.7, and the divergence comes from the region where the momentum k on the ϕ_2 line is close to zero:

$$\int_{k\sim 0} \mathrm{d}^4k\frac{1}{(k^2)^2}.$$

If uncancelled, this divergence indicates that the self-energy shifts the mass to a value other than zero. But the Goldstone theorem tells us that the self-energy is zero at $k=0$. So the infra-red divergence cancels against divergences in other graphs of the same order.

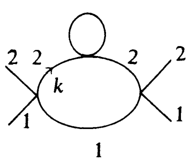

Fig. 9.2.7. Graph with infra-red divergence.

9.3 Renormalization methods

One of the practical problems that arises in making calculations in a theory with spontaneous symmetry breaking is to find the most convenient renormalization prescription. Fundamentally, there is no problem, for all renormalization prescriptions are related by renormalization-group transformations, and are therefore equally good. But, in practice, choice of one prescription over another can save some labor. The problems become particularly acute in gauge theories of weak interactions (Beg & Sirlin (1982)).

Among the issues to be considered in choosing a renormalization prescription are:

(1) If we ignore higher-order corrections, then some parameters are equal to quantities, like particle masses, that are easily measurable. It is often convenient to impose exact equality as a renormalization condition.
(2) One must treat tadpole graphs. Their effect is to provide an additional shift δv in the vacuum expectation value of the field. Leaving these graphs as they are considerably increases the number of graphs contributing to a given Green's function. Shifting the field by δv gives many extra terms in the formulae for the coefficients in (9.2.5*a*) and (9.2.5*b*). One can impose $\delta v = 0$ as a renormalization condition, at the expense of removing the simple connection between the phases of broken and unbroken symmetry (as was exploited in Section 9.2).
(3) It is necessary to relate calculations done by different people. Direct comparisons can be made only if the same renormalizations are used. It is evidently useful to agree on a standard.
(4) If the coupling is not very small or if there occur very large ratios of masses and momenta, then one must choose a renormalization prescription with the ability to remove the large logarithms.

One approach is to use dimensional regularization with minimal subtraction. Graphs can be renormalized by the forest formula. At one-loop order this amounts to subtraction of the pole part from each 1PI graph. We can do this without regard to the symmetry relations between counterterms for different Green's functions. Since the counterterms have the pure-pole form, these relations are automatically satisfied.

Another approach is to compute Z, δg, and δm^2 by three renormalization conditions imposed on some of the 1PI Green's functions in the broken-symmetry phase. Then the values of counterterms for other Green's functions are computed from (9.2.5*b*). It is convenient to determine $m^2\delta g/g - \delta m^2$ by requiring $\delta v = 0$, i.e., $\langle 0|\phi_1|0\rangle = 2mg^{-1/2}$ exactly. Then

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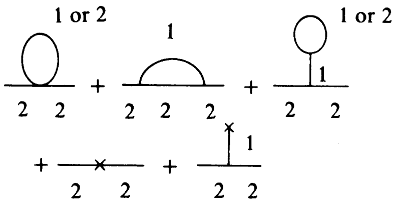

Fig. 9.3.1. Renormalization of self-energy at one-loop order.

the mass counterterm for ϕ_2 forces the 1PI self-energy of ϕ_2 to be exactly zero when $k^2 = 0$. The one-loop graphs are shown in Fig. 9.3.1.

Both of these approaches require explicit computation of the values of Z, δg, and δm^2 to find the values of counterterms for the various Green's functions. It is also possible (Symanzik (1970a)) to use the Ward identities to generate renormalization conditions for all divergent 1PI Green's functions from the three basic conditions. These conditions are simple if the three basic conditions are imposed at zero external momentum.

9.3.1 *Generation of renormalization conditions by Ward identities*

The general Ward identity is

$$\frac{\partial}{\partial x^\mu}\langle 0|Tj^\mu(x)\prod_{j=1}^{N}\phi_{n_j}(y_j)|0\rangle$$
$$= -\mathrm{i}\prod_{j=1}^{N}\delta(x-y_j)\langle 0|T\delta\phi_{n_j}(x)\prod_{i\neq j}\phi_{n_i}(y_i)|0\rangle. \qquad (9.3.1)$$

In the Goldstone model, the labels n_i take the values 1′ or 2, and we have $\delta\phi_1' = \phi_2$ and $\delta\phi_2 = -\phi_1 = -(\phi_1' + 2m/g^{1/2})$. For simplicity we impose the condition $\langle 0|\phi_1'|0\rangle = 0$. Suppose we have obtained renormalization conditions valid up to $l-1$ loops. We will now find the appropriate conditions for l-loop graphs.

The case $N = 1$ was given in (9.2.12) and (9.2.13), and in Fig. 9.2.6. We saw that one renormalization condition on the self-energy of ϕ_2 is that it is zero at $p^2 = 0$. Another condition on the derivative can be chosen arbitrarily, corresponding to the freedom to multiply Z by a finite factor. Suppose we choose to make the residue of the Goldstone pole equal to unity, and we choose to make $\delta v = 0$. Then we also obtain the renormalization condition on the Green's function $\langle 0|Tj^\mu(x)\phi_2(y)|0\rangle$ of the current with ϕ_2. The condition is that it is equal to its lowest-order value at $p = 0$. This condition is equivalent to making the counterterm equal to

$$2mg^{-1/2}(Z-1)\partial^\mu\phi_2,$$

as required if j^μ is to be the Noether current.

Similarly we may treat the case $N = 2$:

$$\begin{aligned}\partial_\mu \langle 0|Tj^\mu(x)\phi_1'(y)\phi_2(z)|0\rangle \\ = -\mathrm{i}\delta(x-y)\langle 0|T\phi_2(y)\phi_2(z)|0\rangle \\ +\mathrm{i}\delta(x-z)\langle 0|T\phi_1'(y)\phi_1'(z)|0\rangle,\end{aligned} \tag{9.3.2}$$

where we used $\langle 0|\phi_1'|0\rangle = 0$. In terms of 1PI graphs in momentum space this gives Fig. 9.3.2. After use of Fig. 9.2.6 we find Fig. 9.3.3.

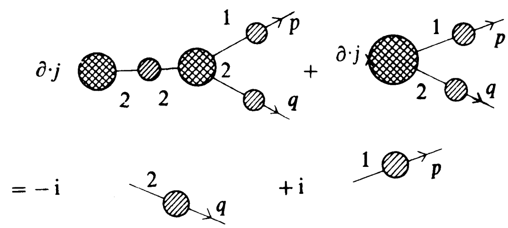

Fig. 9.3.2. Ward identity for two-point function of j^μ.

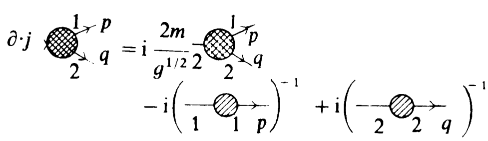

Fig. 9.3.3. Result of multiplying Fig. 9.3.2 by two inverse propagators.

We now set $p + q = 0$ to eliminate the left-hand side. This gives

$$\begin{aligned}0 &= \frac{2m}{g^{1/2}}\mathrm{i}\Gamma_{212}(0,p,-p) - [p^2 - 2m^2 - \Sigma_1(p^2)] + [p^2 - \Sigma_2(p^2)] \\ &= \frac{2m}{g^{1/2}}\mathrm{i}\Gamma_{212}(0,p,-p) + 2m^2 - \Sigma_1(p^2) - \Sigma_2(p^2).\end{aligned} \tag{9.3.3}$$

Here Σ_1 and Σ_2 are self-energies and Γ_{212} is the 1PI Green's function for two ϕ_2 fields and one ϕ_1'. We choose a mass renormalization condition for Σ_1, say $\Sigma_1(0) = 0$. Since we already know that $\Sigma_2(0) = 0$, this tells us that the renormalization condition on Γ_{212} is

$$\Gamma_{212}(0,0,0) = -imq^{-1/2} = \text{lowest-order value.} \tag{9.3.4}$$

Since graphs for Γ_{212} are at worst logarithmically divergent we know $\Gamma_{212}(p,q,r)$ completely at this order. From (9.3.3) we can now determine $\Sigma_1(p^2)$. But the calculation of $\Sigma_1(p^2)$ from its graphs is already fixed except for a renormalization condition that determines the value of the field-

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strength counterterm. So (9.3.3) gives us the renormalization in such a way that the counterterm is $-(Z-1)p^2$.

Similar arguments may be applied to give renormalization conditions for all the remaining 1PI Green's functions that have overall divergences. They are easiest to express in terms of Ward identities for 1PI Green's functions. (See Lee (1976), and references therein.)

The structure of these arguments generalizes what would be done in the unbroken phase. For example, (9.3.2) integrated over x would give $\Sigma_1 = \Sigma_2$ in this phase. This condition would say that the counterterms for Σ_1 and Σ_2 are equal. But in the Goldstone phase this is not so. Integrating over x is equivalent to setting the momentum at the vertex for the current to zero. The derivative with respect to x gives a factor of this momentum, but since there is a pole $1/p^2$ the right-hand side of (9.3.2) is not zero. The argument that we had to use is more complicated.

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