ON WELL-BOUNDED OPERATORS OF TYPE (B)

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1. Introduction

The notion of a well-bounded operator was introduced by Smart (**9**). The properties of well-bounded operators were further investigated by Ringrose (**6**, **7**), Sills (**8**) and Berkson and Dowson (**2**). Berkson and Dowson have developed a more complete theory for the type (A) and type (B) well-bounded operators than is possible for the general well-bounded operator. Their work relies heavily on Sills' treatment of the Banach algebra structure of the second dual of the Banach algebra of absolutely continuous functions on a compact interval.

The main result of this paper (Theorem 5) is the characterisation of a type (B) operator by means of the weak compactness of its $\mathscr{A}_J$-operational calculus (as in Theorem 4.2 of (**2**)) and the description of the operational calculus using Stieltjes integrals of a kind suggested by Krabbe (**5**). Our results are also stronger than those of Berkson and Dowson in that we show the $\mathscr{B}_J$-operational calculus for a type (B) operator to be continuous on pointwise convergent nets of uniformly bounded variation.

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2. Preliminaries

Let X be a complex Banach space with dual space X'. We write $\langle x, x' \rangle$ for the value of the functional x' in X' at x in X. When Y is a subset of X we write Y^w for the weak closure of Y, and Y_1 for $\{y \in Y: \| y \| \leqq 1\}$.

Let $L(X)$ be the Banach algebra of bounded linear operators on X. When T is in $L(X)$, let T' in $L(X')$ be its adjoint.

We shall abbreviate " weak/strong operator topology " to " weak/strong topology ". When $\mathscr{T}$ is a subset of $L(X)$, we write $\mathscr{T}^w$ and $\mathscr{T}^s$ for the weak and strong closures of $\mathscr{T}$. We write "wk lim" and "st lim" for limits in the weak and strong topologies.

Lemma 1. *$\mathscr{T}$ is relatively weakly compact in $L(X)$ if and only if $\mathscr{T}x$ is relatively weakly compact in X (for every x in X).*

Proof. The proof is as indicated in (**3**, VI, 9.2).

Let $\{T_\alpha: \alpha \in \sigma\}$ be a net in $L(X)$. Following (**1**) we say that y is a weak x-cluster point of $\{T_\alpha\}$ if y is a weak cluster point of the net $\{T_\alpha x: \alpha \in \sigma\}$.

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An operator E in $L(X)$ is called a projection if $E^2 = E$. We write $E \leqq F$ when E and F are two projections such that $EF = FE = E$. When E and F are commuting projections both $E \vee F (= E+F-EF)$ and $E \wedge F (= EF)$ are also projections.

A net $\{E_\alpha: \alpha \in \sigma\}$ of projections is said to be naturally ordered if $E_\alpha \leqq E_\beta$ whenever $\alpha \leqq \beta$.

Lemma 2. *Let $\{E_\alpha: \alpha \in \sigma\}$ be a naturally ordered uniformly bounded net of projections on X. Then $\bigvee_\sigma E_\alpha$ exists, and is equal to* $\operatorname{st\,lim}_\sigma E_\alpha$, *if and only if $\{E_\alpha\}$ has a weak x-cluster point for each x in X.*

Proof. (1, Theorem 1.)

We shall extend this terminology and say that the net $\{E_\alpha: \alpha \in \sigma\}$ is a naturally ordered net of operators if $E_\alpha = E_\alpha E_\beta = E_\beta E_\alpha$ whenever $\alpha < \beta$; we do not require the operators E_α to be projections.

Lemma 3. *Let $\{E_\alpha: \alpha \in \sigma\}$ be a naturally ordered uniformly bounded net of operators on X. Then* $\operatorname{st\,lim}_\sigma E_\alpha$ *exists if and only if $\{E_\alpha\}$ has a weak x-cluster point for each x in X.*

Proof. The proof is a straightforward adaptation of the proof of (**1**, Theorem 1) and is therefore omitted.

We say that E: $\boldsymbol{R} \to L(X)$ is a naturally ordered function if

$$E(s) = E(s)E(t) = E(t)E(s)$$

for $s<t$. We write $E(s+)$ for $\operatorname{st\,lim}_{t \to s+} E(t)$ and $E(s-)$ for $\operatorname{st\,lim}_{t \to s-} E(t)$ when the limits exist.

Let T be an operator on X. By an ($\mathscr{F}$-)operational calculus for T we mean a bounded algebra homomorphism ψ: $\mathscr{F} \to L(X)$, where $\mathscr{F}$ is a normed algebra of functions on a compact subset of the complex plane $\boldsymbol{C}$, $\mathscr{F}$ contains the functions $\lambda \mapsto 1$, $\lambda \mapsto \lambda$, and $\psi(\lambda \mapsto 1) = I$, $\psi(\lambda \mapsto \lambda) = T$. We write $\psi(f)$ and $f(T)$ interchangeably. For each x in X and x' in X' we define ψ_x: $\mathscr{F} \to X$: $f \mapsto \psi(f)x$ and $\psi_{x,x'}$: $\mathscr{F} \to \boldsymbol{C}$: $f \mapsto \langle \psi(f)\,x, x' \rangle$.

Let $J = [a, b]$ be a compact interval in the real line $\boldsymbol{R}$. Let $\mathscr{B}_J$ be the Banach algebra of complex-valued functions of bounded variation on J with norm $||| \cdot |||_J$, $||| f |||_J = |f(b)| + \operatorname{var}(f, J)$. Let $\mathscr{A}_J$ be the Banach subalgebra of absolutely continuous functions on J. For f in $\mathscr{A}_J$,

$$||| f |||_J = |f(b)| + \int_a^b |f'(\lambda)|\, d\lambda.$$

Let $\mathscr{A}_J^0$ and $\mathscr{B}_J^0$ be the Banach subalgebras

$$\{f \in \mathscr{A}_J:\ f(b) = 0\} \text{ and } \{f \in \mathscr{B}_J:\ f(b) = 0\}$$

of $\mathscr{A}_J$ and $\mathscr{B}_J$. Let $\mathscr{N}_J$ be the Banach subalgebra of $\mathscr{B}_J$ consisting of the functions in $\mathscr{B}_J$ which are left continuous on $(a, b]$. Let $\mathscr{P}_J$ be the subalgebra of $\mathscr{A}_J$ consisting of the polynomials on J. $\mathscr{P}_J$ is dense in $\mathscr{A}_J$.

3. Integration theory

The integrals described here are based on the modified Stieltjes integrals of Krabbe (5).

Let $\mathscr{E}_J$ $(J = [a, b])$ be the family of functions $E: \mathbf{R} \to L(X)$ satisfying

(i) $E(s-)$ exists, $s \in \mathbf{R}$;
(ii) $E(s) = E(s+)$, $s \in \mathbf{R}$;
(iii) $E(s) = 0$, $s < a$;
(iv) $E(s) = E(b)$, $s \geqq b$.

It is clear that $\sup_{\mathbf{R}} \| E(s)\| = \sup_{J} \| E(s)\| < \infty$ for E in $\mathscr{E}_J$.

We say that a sequence $u = (u_k\colon 0 \leqq k \leqq m)$ is a subdivision of J if $a = u_0 < u_1 < \ldots < u_m = b$. The set U_J of all subdivisions of J admits a partial order $\geqq$ defined by refinement: we write

$$u = (u_k\colon 0 \leqq k \leqq m) \geqq v = (v_j\colon 0 \leqq j \leqq n)$$

when u refines v; that is, when each $[u_{k-1}, u_k](1 \leqq k \leqq m)$ is contained in some $[v_{j-1}, v_j](1 \leqq j \leqq n)$.

Let $M(u)$ be the family of sequences $u^* = (u_k^*\colon 1 \leqq k \leqq m)$ such that

$$u_{k-1} \leqq u_k^* \leqq u_k \ (1 \leqq k \leqq m),$$

for each u in U_J.

A pair $\bar{u} = (u, u^*)$ with $u \in U_J$ and $u^* \in M(u)$ is called a marked partition of J. We write π_J for the family of marked partitions of J and define the pre-order $\geqq$ on π_J by setting $(u, u^*) \geqq (v, v^*)$ if and only if $u \geqq v$.

Let $\pi_J^i = \{\bar{u} = (u, u^*) \in \pi_J\colon u_{k-1} < u_k^* < u_k,\ 1 \leqq k \leqq m\}$ and let

$$\pi_J^r = \{\bar{u} = (u, u^*) \in \pi_J\colon u_k^* = u_k,\ 1 \leqq k \leqq m\}.$$

The sets U_J, π_J, π_J^i and π_J^r are directed by $\geqq$. Also, π_J^i and π_J^r are cofinal in π_J.

We define π_J^g for each g in $\mathscr{B}_J$ thus:

$$\pi_J^g = \begin{cases} \pi_J, & g \in \mathscr{N}_J, \\ \pi_J^i, & g \in \mathscr{B} \backslash \mathscr{N}_J. \end{cases}$$

Let $E_u = \sum_1^m E(u_{k-1})\chi_{[u_{k-1}, u_k)} + E(b)\chi_{[b, \infty)}$ when $E \in \mathscr{E}_J$ and $u \in U_J$.

Let $g_{\bar{u}} = g(a)\chi_{\{a\}} + \sum_1^m g(u_k^*)\chi_{(u_{k-1}, u_k]}$ when $g \in \mathscr{B}_J$ and $\bar{u} \in \pi_J$.

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Let Φ and Ψ be functions on J, one taking values in C, the other in $L(X)$ or in C. When $\bar{u} \in \pi_J$ we define

$$\Sigma\Phi(\Psi\Delta\bar{u}) = \sum_1^m \Phi(u_k^*)(\Psi(u_k) - \Psi(u_{k-1})).$$

The following two inequalities are evident:

$$\| \Sigma E(g\Delta\bar{u}) - \Sigma F(g\Delta\bar{u}) \| \leqq \text{var}\,(g, J) \sup_J \| E(s) - F(s) \|, \; g \in \mathscr{B}_J, \; E, F \in \mathscr{E}_J; \qquad (1)$$

$$\| \Sigma E(g\Delta\bar{u})x - \Sigma F(g\Delta\bar{u})x \| \leqq \text{var}\,(g, J) \sup_J \| E(s)x - F(s)x \|,$$
$$x \in X, \; g \in \mathscr{B}_J, \; E, \; F \in \mathscr{E}_J. \qquad (2)$$

The following integrals are defined as net limits in the strong topology (when they exist):

$$\int_J \Phi d\Psi = \text{st} \lim_{\pi_J} \Sigma\Phi(\Psi\Delta\bar{u});$$

$$\int_J^r \Phi d\Psi = \text{st} \lim_{\pi_J^r} \Sigma\Phi(\Psi\Delta\bar{u});$$

$$\int_J^l \Phi d\Psi = \text{st} \lim_{\pi_J^l} \Sigma\Phi(\Psi\Delta\bar{u});$$

$$\oint_J Edg = \text{st} \lim_{\pi_J^q} \Sigma E(g\Delta\bar{u}), \quad g \in \mathscr{B}_J, \; E \in \mathscr{E}_J.$$

Lemma 4. $\lim_{U_J} \sup_J \| E(s)x - E_u(s)x \| = 0, \; x \in X, \; E \in \mathscr{E}_J.$

Proof. Let $E \in \mathscr{E}_J$, $x \in X$. Let $\varepsilon > 0$.

For each s in $[a, b)$ there exists r_s $(s < r_s < b)$ such that $\| E(t)x - E(t')x \| \leqq \varepsilon$ when $t, t' \in [s, r_s)$, since $E(s) = E(s+)$.

For each s in $(a, b]$ there exists l_s $(a < l_s < s)$ such that $\| E(t)x - E(t')x \| \leqq \varepsilon$ when $t, t' \in [l_s, s)$, since $E(s-)$ exists.

The sets $[a, r_a)$, $(l_b, b]$, (l_s, r_s) $(a < s < b)$ form an open cover of J. Let $[a, r_a)$, $(l_b, b]$, (l_{s_j}, r_{s_j}) (j in some finite set) be a finite subcover.

Let u be the partition of J with points a, b, r_a, l_b, s_j, l_{s_j}, r_{s_j} (j in the finite set). Then $\sup_J \| E(s)x - E_u(s)x \| \leqq \varepsilon$.

Lemma 5.

(i) $\oint_J T\chi_{[b, \infty)} dg = 0, \quad g \in \mathscr{B}_J, \; T \in L(X);$

(ii) $\oint_J T\chi_{[s, t)} dg = (g(t) - g(s))T, \quad g \in \mathscr{B}_J, \; T \in L(X), \; a \leqq s < t \leqq b;$

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(iii) $\oint_J E_u dg = \sum_1^m E(u_{k-1})(g(u_k)-g(u_{k-1}))$

$$= \text{st}\lim_{\pi_J^g} \Sigma E_u(g\Delta\bar{v}), \quad g \in \mathscr{B}_J,\ E \in \mathscr{E}_J,\ u \in U_J.$$

Proof. (i) Let $\bar{u} \in \pi_J^g$. Then

$$\Sigma\chi_{[b,\infty)}(g\Delta\bar{u}) = \sum_1^m \chi_{[b,\infty)}(u_k^*)(g(u_k)-g(u_{k-1}))$$
$$= \begin{cases} g(b)-g(u_{m-1}), & u_m^* = b, \\ 0, & u_m^* < b. \end{cases}$$

Hence $\text{st}\lim_{\pi_J^g} \Sigma T\chi_{[b,\infty)}(g\Delta\bar{u}) = 0$.

(ii) Let $a < s \leqq b$, $\bar{u} \in \pi_J^g$, $u \geqq (a, s, b)$ (no condition if $s = b$). Then $s = u_n$ for some n with $1 \leqq n \leqq m$, and

$$\Sigma\chi_{[a,s)}(g\Delta\bar{u}) = \sum_1^{n-1}(g(u_k)-g(u_{k-1})) + \chi_{[a,s)}(u_n^*)(g(s)-g(u_{n-1}))$$
$$= \begin{cases} g(u_{n-1})-g(a), & u_n^* = u_n = s, \\ g(s)-g(a), & u_n^* < u_n = s. \end{cases}$$

Hence $\text{st}\lim_{\pi_J^g} \Sigma T\chi_{[a,s)}(g\Delta\bar{u}) = (g(s)-g(a))T$. Since

$$\chi_{[s,t)} = \chi_{[a,t)} - \chi_{[a,s)} \ (a \leqq s < t \leqq b),$$

(ii) is proved.

(iii) is a direct corollary of (i) and (ii).

Theorem 1. *Let g be in $\mathscr{B}_J$ and E in $\mathscr{E}_J$. Then $\oint_J Edg$ exists, and*

$$\oint_J Edg = \text{st}\lim_{U_J} \oint_J E_u dg.$$

Also,

$$\left\| \oint_J Edg \right\| \leqq \text{var}(g, J) \sup_J \| E(s) \|,$$

$$\left\| \oint_J Edg\, x \right\| \leqq \text{var}(g, J) \sup_J \| E(s)x \|, \quad x \in X.$$

Proof. By (1) above,

$$\| \Sigma E(g\Delta\bar{v}) \| \leqq \text{var}(g, J) \sup_J \| E(s) \|, \quad \bar{v} \in \pi_J.$$

Let $u \in U_J$, let $\bar{v}$, $\bar{w} \in \pi_J^g$ and let $x \in X$. Then

$$\|\Sigma E(g\Delta\bar{v})x - \Sigma E(g\Delta\bar{w})x\| \leqq \|\Sigma E(g\Delta\bar{v})x - \Sigma E_u(g\Delta\bar{v})x\|$$
$$+\|\Sigma E(g\Delta\bar{w})x - \Sigma E_u(g\Delta\bar{w})x\| + \|\Sigma E_u(g\Delta\bar{v})x - \Sigma E_u(g\Delta\bar{w})x\|$$
$$\leqq 2 \operatorname{var}(g, J) \sup_J \|E(s)x - E_u(s)x\| + \|\Sigma E_u(g\Delta\bar{v})x - \Sigma E_u(g\Delta\bar{w})x\|.$$

Lemmas 4 and 5 now show that $\|\Sigma E(g\Delta\bar{v})x - \Sigma E(g\Delta\bar{w})x\| \to 0$ as $\bar{v}$ and $\bar{w}$ increase in π_J^g. Therefore $\{\Sigma E(g\Delta\bar{v});\ \bar{v} \in \pi_J^g\}$ is a uniformly bounded strongly Cauchy net in $L(X)$, and so converges to its unique strong limit.

The other statements of the theorem are immediate from (1) and (2).

Theorem 2. *Let $E \in \mathscr{E}_J$, $g \in \mathscr{B}_J$ and let $\{g_\alpha\colon \alpha \in \sigma\}$ be a net in $\mathscr{B}_J$ with* $\sup_\sigma \operatorname{var}(g_\alpha, J) < \infty$ *and* $g(s) = \lim_\sigma g_\alpha(s) (s \in J)$. *Then*

$$\oint_J E dg = \operatorname{st}\lim_\sigma \oint_J E dg_\alpha.$$

Proof. Let $K = \operatorname{var}(g, J) + \sup_\sigma \operatorname{var}(g_\alpha, J)$. Let $u \in U_J$. Then

$$\oint_J E dg - \oint_J E dg_\alpha = \oint_J (E - E_u) dg - \oint_J (E - E_u) dg_\alpha + \oint_J E_u d(g - g_\alpha).$$

Let $x \in X$. Then

$$\left\| \oint_J E dg\, x - \oint_J E dg_\alpha\, x \right\| \leqq K \sup_J \|E(s)x - E_u(s)x\|$$
$$+ \sup_J \|E(s)x\| \sum_1^m |(g - g_\alpha)(u_k) - (g - g_\alpha)(u_{k-1})|,$$

and this expression can be made arbitrarily small by choosing u fine enough (Lemma 4), then α large enough.

Let $S(g, E) = g(b)E(b) - \oint_J E dg$ when $g \in \mathscr{B}_J$, $E \in \mathscr{E}_J$.

Lemma 6.

(i) $S(g, \chi_{[s,\infty)}T) = g(s)T$, $g \in \mathscr{B}_J$, $T \in L(X)$, $a \leqq s \leqq b$;

(ii) $\|S(g, E)\| \leqq |\!|\!| g |\!|\!|_J \sup_J \|E(s)\|$, $g \in \mathscr{B}_J$, $E \in \mathscr{E}_J$;

(iii) $\|S(g, E)x\| \leqq |\!|\!| g |\!|\!|_J \sup_J \|E(s)x\|$, $g \in \mathscr{B}_J$, $E \in \mathscr{E}_J$, $x \in X$;

(iv) $S(\chi_{[a,s]}, E) = E(s)$, $E \in \mathscr{E}_J$, $s \in J$.

Proof. (i), (ii) and (iii) follow directly from Lemma 5 and Theorem 1. As to (iv):

(a) $s = b$. $S(\chi_J, E) = 1 . E(b) - \oint_J E d\chi_J = E(b)$.

(b) $s<b$. Then

$$
\begin{aligned}
S(\chi_{[a,s]}, E) &= -\oint_J E d\chi_{[a,s]} \\
&= -\operatorname{st}\lim_{U_J} \oint_J E_u d\chi_{[a,s]} \qquad \text{(Theorem 1)} \\
&= -\operatorname{st}\lim_{U_J} \sum_1^m E(u_{k-1})(\chi_{[a,s]}(u_k)-\chi_{[a,s]}(u_{k-1})) \\
&= -\operatorname{st}\lim_{U_J} (-E(u_{n-1})) \text{ where } s \in [u_{n-1}, u_n) \\
&= \operatorname{st}\lim_{U_J} E_u(s) \\
&= E(s) \qquad \text{(Lemma 4).}
\end{aligned}
$$

Lemma 7. *Let $g \in \mathscr{B}_J$ and $\bar{u} \in \pi_J^g$. Then $g_{\bar{u}} \in \mathscr{B}_J$, and*

$$g(s) = \lim_{\pi_J^g} g_{\bar{u}}(s), \quad s \in J.$$

Also, var $(g_{\bar{u}}, J) \leqq 2 \sup_J |g(s)|$ *if g is real monotonic increasing.*

Proof. $g_{\bar{u}} \in \mathscr{B}_J$, trivially; and $g_{\bar{u}}(a) = g(a)$. If $a<s \leqq b$ and $u \geqq (a, s, b)$ (no condition if $s = b$) then $s = u_n$ for some $n\,(1 \leqq n \leqq m)$ and

$$g_{\bar{u}}(s) = g(u_n^*) = \begin{cases} g(u_n^*), & g \in \mathscr{N}_J, \\ g(u_n), & g \in \mathscr{B}_J \backslash \mathscr{N}_J. \end{cases}$$

Therefore $\lim_{\pi_J^g} g_{\bar{u}}(s) = g(s)\,(s \in J)$.

If g is real monotonic increasing, then var $(g, J) \leqq 2 \sup_J |g(s)|$ and $g_{\bar{u}}$ is also monotonic increasing. Hence var $(g_{\bar{u}}, J) \leqq 2 \sup_J |g_{\bar{u}}(s)| \leqq 2 \sup_J |g(s)|$.

Theorem 3. *Let $E \in \mathscr{E}_J$. Then*

$$S(g, E) = \begin{cases} g(a)E(a) + \int_J g dE, & g \in \mathscr{N}_J, \\ g(a)E(a) + \int_J^r g dE, & g \in \mathscr{B}_J. \end{cases}$$

Proof. Let $\bar{u} \in \pi_J^g$. Then

$$
\begin{aligned}
S(g_{\bar{u}}, E) &= g(a)S(\chi_{\{a\}}, E) + \sum_1^m g(u_k^*)(S(\chi_{[a,u_k]}, E) - S(\chi_{[a,u_{k-1}]}, E)) \\
&= g(a)E(a) + \Sigma g(E\Delta\bar{u}).
\end{aligned}
$$

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We can assume that g is real monotonic increasing, without loss of generality. Theorem 2 and Lemma 7 now give the result.

We shall write $\int_J^{\oplus} g\,dE$ instead of $S(g, E)$ when $g \in \mathscr{B}_J$ and $E \in \mathscr{E}_J$.

The proofs above give us analogous scalar integrals defined for functions of bounded variation on J and functions satisfying the scalar version of the definition of $\mathscr{E}_J$.

Let $\mathscr{D}_J$ be the algebra of complex-valued functions on $\boldsymbol{R}$ generated by the functions $\chi_{[s,\,t)}$ $(a \leqq s < t \leqq b)$. Let $\mathscr{Q}_J$ be the closure of $\mathscr{D}_J$ in the supremum norm. Then $\mathscr{Q}_J$ is the algebra of functions which vanish on $(-\infty, a)$ and on $[b, \infty)$ and are right continuous and left limitable on $\boldsymbol{R}$ (Lemma 4 or (4, Theorem 4.5)).

Hewitt showed in (4) that the integral $\int_{[a,\,b)} \omega\,dg$ can be defined for g in $\mathscr{B}_J$ and ω in $\mathscr{Q}_J$ as the limit of any sequence $\left\{\int_{[a,\,b)} \omega_n\,dg\right\}$ where $\{\omega_n\} \subset \mathscr{D}_J$, $\omega = \lim \omega_n$ in the supremum norm, and $\int_{[a,\,b)} \chi_{[s,\,t)}\,dg = g(t) - g(s)\,(a \leqq s < t \leqq b)$ by definition. The scalar version of Theorem 1 shows that

$$\int_{[a,\,b)} \omega\,dg = \int_J^{i} \omega\,dg \quad (g \in \mathscr{B}_J,\ \omega \in \mathscr{Q}_J).$$

Let Σ_J be the algebra of subsets of $[a, b)$ generated by sets of the form $[s, t)\,(a \leqq s < t \leqq b)$. Theorem 4.10 of (4) shows that $(\mathscr{Q}_J)'$ may be identified with the space of bounded finitely additive measures on Σ_J. Each such measure μ defines a function g_μ in $\mathscr{B}_J^0$ by $g_\mu(s) = -\mu([s, b))\,(s \in J)$: conversely, each function g in $\mathscr{B}_J^0$ defines such a measure μ_g by $\mu_g([s, t)) = g(t) - g(s)(a \leqq s < t \leqq b)$. This correspondence is one-to-one.

We can therefore identify $(\mathscr{Q}_J)'$ and $\mathscr{B}_J^0$. The pairing between $\mathscr{Q}_J$ and $\mathscr{B}_J^0$ is given by

$$\langle \omega, g \rangle = \int_{[a,\,b)} \omega\,dg, \qquad = \int_J^{i} \omega\,dg, \quad \omega \in \mathscr{Q}_J,\ g \in \mathscr{B}_J^0.$$

Lemma 8. *Let $\{g_\alpha : \alpha \in \sigma\}$ be a bounded net in $\mathscr{B}_J^0$ and let $g \in \mathscr{B}_J^0$. Then $g = \lim_\sigma g_\alpha$ in the $\mathscr{Q}_J$-topology of $\mathscr{B}_J^0$ if and only if $g(s) = \lim_\sigma g_\alpha(s)\,(a \leqq s < b)$.*

Proof. (Cf. (3, IV, 13.35).) Note that $\langle \chi_{[s,\,b)}, g \rangle = -g(s)\,(a \leqq s < b)$ and apply Theorem 2.

The following result seems not to be known generally.

Theorem 4. *Let $g \in \mathscr{B}_J$. Then there is a net $\{g_\alpha: \alpha \in \sigma\}$ in $\mathscr{A}_J$ such that $g = \lim_\sigma g_\alpha$ pointwise on J and $\sup ||| g_\alpha |||_J \leqq ||| g |||_J$.*

Proof. Since g can be written as $(g - g(b)\chi_J) + g(b)\chi_J$ we see that it is enough to show that if $g \in \mathscr{B}_J^0$ then there is a net $\{g_\alpha: \alpha \in \sigma\}$ in $\mathscr{A}_J^0$ such that $g = \lim_\sigma g_\alpha$ pointwise on $[a, b)$ and $\sup_\sigma \mathrm{var}\,(g_\alpha, J) \leqq \mathrm{var}\,(g, J)$.

Now each ω in $\mathscr{Q}_J$ defines a bounded functional on $\mathscr{A}_J^0$ by $f \mapsto \int_J \omega df$. Therefore $\mathscr{Q}_J$ can be identified with a subspace of $(\mathscr{A}_J^0)'$, that is, with $L^\infty(J)$ under the ess-sup norm.

Each function g in $\mathscr{B}_J^0$ defines a bounded functional L_g on $\mathscr{Q}_J$ by

$$L_g(\omega) = \int_J^i \omega dg \quad (\omega \in \mathscr{Q}_J).$$

Since the sup and ess-sup norms agree on $\mathscr{Q}_J$, we see from Theorem 1 that

$$\| L_g \| \leqq ||| g |||_J = \mathrm{var}\,(g, J).$$

The Hahn-Banach theorem allows us to extend L_g to a functional (also denoted by) L_g on $(\mathscr{A}_J^0)'$ without increasing its norm. So $L_g \in (\mathscr{A}_J^0)''$. By Goldstine's theorem (3, V, 4.5) there is a net $\{g_\alpha: \alpha \in \sigma\}$ in $\mathscr{A}_J^0$ converging to L_g in the $(\mathscr{A}_J^0)'$-topology of $(\mathscr{A}_J^0)''$ and satisfying $\sup ||| g_\alpha |||_J \leqq \| L_g \| \leqq ||| g |||_J$. Then $g = \lim_\sigma g_\alpha$ in the $\mathscr{Q}_J$-topology of $\mathscr{B}_J^0$, so $g(s) = \lim_\sigma g_\alpha(s)\,(a \leqq s < b)$ (Lemma 8).

4. Well-bounded operators of type (B)

Let T be a bounded operator on the Banach space X. We define $p(T)$ in the natural way for each polynomial p by setting $p(T) = \Sigma a_n T^n$ when $p(s) = \Sigma a_n s^n$. The map $p \mapsto p(T)$ is an algebra homomorphism.

We say that T is well-bounded (on J) if there is a compact interval J such that $\psi: \mathscr{P}_J \to L(X): p \mapsto p(T)$ is an operational calculus; that is, T is well-bounded if there exist a compact interval J and a constant K such that

$$\| p(T) \| \leqq K ||| p |||_J, \quad p \in \mathscr{P}_J.$$

If T is well-bounded, so is T' (with the same J and K).

Smart introduced this definition and proved the following result.

Lemma 9. *Let T in $L(X)$ be a well-bounded operator with natural operational calculus $\psi: \mathscr{P}_J \to L(X)$ of norm K. Then ψ has a unique extension to an operational calculus (also denoted by) $\psi: \mathscr{A}_J \to L(X)$, of norm K, such that*

(i) *if S in $L(X)$ satisfies $TS = ST$, then $Sf(T) = f(T)S, f \in \mathscr{A}_J$;*
(ii) *$f(T') = f(T)', f \in \mathscr{A}_J$.*

Proof. (9, Lemma 2.1.)

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The notion of a decomposition of the identity was introduced by Ringrose in (7). A decomposition of the identity for X (on J) is a family $\{F(s): s \in \boldsymbol{R}\}$ of projections on X' such that

(i) $F(s) = 0, \quad s < a,$

$F(s) = I, \quad s \geqq b;$

(ii) $F(s)F(t) = F(t)F(s) = F(s), \quad s \leqq t;$

(iii) there is a positive constant $K(\geqq 1)$ such that

$$\| F(s)\| \leqq K, \quad s \in \boldsymbol{R};$$

(iv) the function $s \mapsto \langle x, F(s)x' \rangle$ is Lebesgue measurable for each $x \in X$ and $x' \in X'$;

(v) if $x \in X$, $x' \in X'$, $s \in [a, b)$, and if the function $t \mapsto \int_a^t \langle x, F(u)x' \rangle du$ is right differentiable at s, then the right derivative at s is $\langle x, F(s)x' \rangle$;

(vi) for each x in X, the map $X' \to L^\infty(a, b)$: $x' \mapsto \langle x, F(\ )x \rangle$ is continuous when X' and $L^\infty(a, b)$ are given their weak* topologies (as duals of X and $L^1(a, b)$).

An operator T in $L(X)$ is said to be decomposable (on J) if there is a decomposition of the identity for X on J such that

$$\langle Tx, x' \rangle = b\langle x, x' \rangle - \int_a^b \langle x, F(s)x' \rangle ds, \quad x \in X,\ x' \in X'.$$

Theorems 2 and 6 of (7) show that T is decomposable on J if and only if T is well-bounded on J; and the two constants K coincide. Also, we can choose $F(\cdot)$ so that $S'F(s) = F(s)S'$ $(s \in \boldsymbol{R})$ for all S in $L(X)$ satisfying $ST = TS$. Furthermore, the operational calculus of Lemma 9 is given by

$$\langle f(T)x, x' \rangle = f(b)\langle x, x' \rangle - \int_a^b \langle x, F(s)x' \rangle f'(s) ds,$$

where $x \in X$, $x' \in X'$, $f \in \mathscr{A}_J$.

Let T be a well-bounded operator on X. T is said to be decomposable in X if there is a family $\{E(s): s \in \boldsymbol{R}\}$ of projections on X such that $\{E'(s): s \in \boldsymbol{R}\}$ is a decomposition of the identity for T. If an operator is decomposable in X, then it has a unique decomposition of the identity (7, Theorem 8).

Let T be decomposable in X with unique decomposition of the identity $\{E'(s): s \in \boldsymbol{R}\}$. Following Berkson and Dowson (2), we say that T is well-bounded of type (B) if $E \in \mathscr{E}_J$; that is, if $E(s+) = E(s)$ $(s \in \boldsymbol{R})$ and $E(s-)$ is defined at each s in $\boldsymbol{R}$.

The following theorem (cf. (2, Theorem 4.2)) characterises the well-bounded operators of type (B) in a manner similar to the characterisation of scalar-type spectral operators in (10).

Theorem 5. *Let T be a bounded operator on the Banach space X. The following five conditions are equivalent*:

(i) *T is well-bounded of type (B) with an $\mathscr{A}_J$-operational calculus of norm K;*

(ii) *there exist a compact interval J and a naturally ordered family $\{E(s): s \in \boldsymbol{R}\}$ of projections on X such that $E \in \mathscr{E}_J$, $E(b) = I$, $K = \sup_{\boldsymbol{R}} \| E(s)\|$ and*

$$T = \int_J^{\oplus} s dE(s);$$

(iii) *T is well-bounded with an operational calculus ψ: $\mathscr{A}_J \to L(X)$ of norm K, such that $\psi((\mathscr{A}_J)_1)$ is weakly relatively compact in $L(X)$;*

(iv) *T is well-bounded with an operational calculus ψ: $\mathscr{A}_J \to L(X)$ of norm K, and ψ_x is weakly compact for each x in X;*

(v) *T is well-bounded with an operational calculus ψ: $\mathscr{A}_J \to L(X)$ of norm K, and ψ_x is compact for each x in X.*

Proof. We show that (i)⇒(v)⇒(iv)⇒(iii)⇒(ii)⇒(i).

(i)⇒(v). Let $\{E'(s): s \in \boldsymbol{R}\}$ be the unique decomposition of the identity for T. By the definition of a type (B) operator, $E \in \mathscr{E}_J$. We define ψ: $\mathscr{A}_J \to L(X)$ by

$$\psi(f) = \int_J^{\oplus} f dE, \quad f \in \mathscr{A}_J.$$

The map ψ is linear and bounded; also,

$$\psi(s \mapsto 1) = \int_J^{\oplus} dE = I;$$

moreover,

$$\begin{aligned} \langle \psi(s \mapsto s)x, x' \rangle &= \int_J^{\oplus} s d\langle E(s)x, x' \rangle \\ &= b\langle x, x' \rangle - \int_J \langle E(s)x, x' \rangle ds \\ &= \langle Tx, x' \rangle, \quad x \in X, x' \in X'. \end{aligned}$$

Hence $\psi(s \mapsto s) = T$.

Since E is naturally ordered,

$$\{f(a)E(a) + \Sigma f(E\Delta\bar{u})\}\{g(a)E(a) + \Sigma g(E\Delta\bar{u})\} = fg(a)E(a) + \Sigma fg(E\Delta\bar{u}), \quad f, g \in \mathscr{A}_J, \quad \bar{u} \in \pi_J.$$

Thus ψ is an algebra homomorphism. By Lemma 9, ψ is the unique $\mathscr{A}_J$-operational calculus for T.

For each x in X, the function $s \mapsto E(s)x$ is right continuous. Hence its range $\mathscr{E}_x = \{E(s)x: s \in \boldsymbol{R}\}$ is separable, and its set of discontinuities is countable (**9**, 330).

Let $\{E(s_n)x: n \in N\}$ be a sequence of points in $\mathscr{E}_x$. Since $E(s) = 0\,(s<a)$ and $E(s) = I\,(s \geqq b)$, we can assume that $s_n \in [a-\varepsilon, b]$ (any $\varepsilon>0$). We can

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therefore extract a monotone (convergent) subsequence $\{s_n'\}$ from $\{s_n\}$. The sequence $\{E(s_n')x\colon n \in N\}$ converges because $E \in \mathscr{E}_J$: thus $\mathscr{E}_x$ is relatively compact in X. Let $\mathscr{K}_x$ be the closed absolutely convex hull of $\mathscr{E}_x$. Then $\mathscr{K}_x$ is compact (by the argument of (3, V, 2.6)).

Let $f \in \mathscr{A}_J$, $|||f|||_J \leqq 1$; let $\bar{u} \in \pi_J$. Then

$$f(b)E(b)x - \Sigma E(f\Delta\bar{u})x = f(b)x - \sum_1^m (f(u_k) - f(u_{k-1}))E(u_k^*)x \in \mathscr{K}_x.$$

Therefore $\psi_x(f) \in \mathscr{K}_x$. Thus ψ_x is a compact map.

(v)⇒(iv). Trivial.

(iv)⇒(iii). This is immediate from Lemma 1.

(iii)⇒(ii). We define the function $k_{s,h}$ on R for each s in R and $h>0$ by

$$k_{s,h}(t) = \begin{cases} 1, & t \leqq s, \\ 1+(s-t)/h, & s \leqq t \leqq s+h, \\ 0, & s+h \leqq t; \end{cases}$$

then $k_{s,h} \in \mathscr{A}_J$ and $|||k_{s,h}|||_J \leqq 1$. Also, $\chi_{(-\infty,s]} = \lim_{h\to 0} k_{s,h}$, pointwise.

Let $\mathscr{U}$ be an ultrafilter on $(0, \infty)$ converging to 0 in the usual topology of $\boldsymbol{R}$. When f is a continuous function on $(0, \infty)$ we write $\lim_{\substack{h\to 0\\ \mathscr{U}}} f(h)$ for the value at $\mathscr{U}$ of the extension of f to the Stone-Čech compactification of $(0, \infty)$.

Lemmas 2 and 4 of (7) show that every bounded functional on $\mathscr{A}_J$ has the form

$$f \mapsto L(f) = m_L f(b) - \int_a^b \omega_L(s)f'(s)ds, \quad f \in \mathscr{A}_J,$$

where $m_L \in \boldsymbol{C}$, $\omega_L \in L^\infty(J)$ and

$$\omega_L(s) = \lim_{\substack{h\to 0\\ \mathscr{U}}} \int_0^1 \omega_L(s+ht)dt, \quad a \leqq s < b.$$

We define $L_{x,x'}$ on $\mathscr{A}_J$ for x in X and x' in X' by

$$L_{x,x'}(f) = \langle \psi(f)x, x' \rangle, \quad x \in X,\ x' \in X',\ f \in \mathscr{A}_J.$$

Let $m_{x,x'}$ and $\omega_{x,x'}$ be the associated constant and $L^\infty(J)$ function:

$$L_{x,x'}(f) = m_{x,x'}f(b) - \int_a^b \omega_{x,x'}(s)f'(s)ds, \quad x \in X,\ x' \in X',\ f \in \mathscr{A}_J.$$

Then

$$\langle \psi(s \mapsto 1)x, x' \rangle = \langle x, x' \rangle = m_{x,x'}.$$

Also

$$L_{x,x'}(k_{s,h}) = \int_0^1 \omega_{x,x'}(s+ht)dt, \quad a \leqq s < s+h < b.$$

The set

$$\mathscr{K} = \{\psi(k_{s,h})\colon a \leqq s < s+h < b\}^w$$

is compact and Hausdorff in the weak topology of $L(X)$ because $\psi((\mathscr{A}_J)_1)$ is weakly relatively compact. $\mathscr{K}$ may therefore be considered as a complete Hausdorff uniform space with the uniformity defined by the functions

$$\{S \mapsto \langle Sx, x' \rangle : x \in X, x' \in X'\}.$$

Since

$$\langle \psi(k_{s,h})x, x' \rangle = \int_0^1 \omega_{x,x'}(s+ht)dt \underset{\mathscr{U}}{\overset{h\to 0}{\to}} \omega_{x,x'}(s), \quad a \leqq s < b,$$

the filter $\{\psi(k_{s,h}): h \in U, U \in \mathscr{U}\}$ is Cauchy, and therefore has a unique weak limit point, say $E(s)$, in $\mathscr{K}$. Let $E(s) = 0\,(s<a)$ and $E(s) = I\,(s \geqq b)$.

Since $k_{s,h}k_{t,k} = k_{t,k}k_{s,h} = k_{s,h}$ for $0<h<t-s$, $0<k$, we have

$$E(s)E(t) = E(t)E(s) = E(s)$$

when $s<t$. Thus E: $\boldsymbol{R} \to L(X)$ is a naturally ordered function. By Lemma 3 and the weak compactness of $\mathscr{K}$, the strong limits $E(s+)$ and $E(s-)$ exist for all s in $\boldsymbol{R}$. Also, since

$$\langle E(s)x, x' \rangle = \omega_{x,x'}(s), \quad a \leqq s < b, x \in X, x' \in X',$$

we have

$$\begin{aligned}\langle E(s)x, x' \rangle &= \lim_{\substack{h\to 0\\ \mathscr{U}}} \int_0^1 \omega_{x,x'}(s+ht)dt \\ &= \omega_{x,x'}(s+) \\ &= \langle E(s+)x, x' \rangle, \quad a \leqq s < b.\end{aligned}$$

Therefore $E(s) = E(s+)\,(a \leqq s < b)$; hence each $E(s)$ is a projection. Thus $E \in \mathscr{E}_J$.

We define ψ': $\mathscr{A}_J \to L(X)$: $f \mapsto \int_J^{\oplus} f dE$. Then

$$\begin{aligned}\langle \psi'(f)x, x' \rangle &= f(b)\langle x, x' \rangle - \int_J \langle E(s)x, x' \rangle df(s) \\ &= f(b)\langle x, x' \rangle - \int_a^b \omega_{x,x'}(s) f'(s)ds \\ &= \langle \psi(f)x, x' \rangle, \quad x \in X, x' \in X', f \in \mathscr{A}_J.\end{aligned}$$

So $\psi = \psi'$ and $T = \int_J^{\oplus} s dE(s)$ (take f: $s \mapsto s$).

By Lemma 6, $\| \psi' \| \leqq \sup_{\boldsymbol{R}} \| E(s) \|$. Since $E(s) = \operatorname{st}\lim_{\substack{h\to 0\\ \mathscr{U}}} \psi(k_{s,h})$, we see that $\| \psi' \| = \sup_{\boldsymbol{R}} \| E(s) \|$.

(ii)⇒(i). The operational calculus ψ' constructed above shows that (ii)⇒(i). The theorem is proved.

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We note that each projection $E(s)\,(a \leqq s < b)$ is the strong limit of the sequence $\{\psi(k_{s,\,n^{-1}}):\ n \in N\}$.

Theorem 6. *Let T be a bounded operator on the Banach space X satisfying the equivalent conditions of Theorem 5. Then the operational calculus* ψ *extends to an operational calculus* ψ': $\mathscr{B}_J \to L(X)$ *having the same norm. Let* $\{g_\alpha:\ \alpha \in \sigma\}$ *be a uniformly bounded net in* $\mathscr{B}_J$ *converging pointwise to a function g in* $\mathscr{B}_J$: *then* $g(T) = \operatorname{st\,lim}_{\sigma} g_\alpha(T)$. *Also* $\{g(T):\ g \in \mathscr{B}_J\} \subset \{f(T):\ f \in \mathscr{A}_J\}^s$.

Proof. We define ψ': $\mathscr{B}_J \to L(X)$: $f \mapsto \int_J^{\oplus} f dE$. It is clear that ψ' is a linear map of norm $\sup_R \| E(s)\|$, and that ψ' extends ψ. The argument in the proof of Theorem 5 ((i)$\Rightarrow$(v)) shows that ψ' is an algebra homomorphism. $\{g(T):\ g \in \mathscr{B}_J\} \subset \{f(T):\ f \in \mathscr{A}_J\}^s$ since $\{E(s):\ s \in R\} \subset \{f(T):\ f \in \mathscr{A}_J\}^s$ and the integrals defining ψ' exist in the strong topology.

The rest of the theorem follows directly from Theorem 2.

REFERENCES

(1) J. Y. Barry, On the convergence of ordered sets of projections, *Proc. Amer. Math. Soc.* **5** (1954), 313-314.

(2) E. Berkson and H. R. Dowson, On uniquely decomposable well-bounded operators, *Proc. London Math. Soc.* (3) **22** (1971), 339-358.

(3) N. Dunford and J. T. Schwartz, *Linear operators* (Interscience Publishers Inc., New York, 1958).

(4) E. Hewitt, A problem concerning finitely additive measures, *Mat. Tidsskr.* **B** 1951 (1951), 81-94.

(5) G. L. Krabbe, A Helly convergence theorem for Stieltjes integrals, *Indagationes Math.* **27** (1965), 52-69.

(6) J. R. Ringrose, On well-bounded operators, *J. Australian Math. Soc.* **1** (1960), 334-343.

(7) J. R. Ringrose, On well-bounded operators II, *Proc. London Math. Soc.* (3) **13** (1963), 613-638.

(8) W. H. Sills, On absolutely continuous functions and the well-bounded operator, *Pacific J. Math.* **17** (1966), 349-366.

(9) D. R. Smart, Conditionally convergent spectral expansions, *J. Australian Math. Soc.* **1** (1960), 319-333.

(10) P. G. Spain, On scalar-type spectral operators, *Proc. Cambridge Phil. Soc.* **69** (1971), 409-410.

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