RELATIVE COHOMOLOGY

D. G. HIGMAN

It is our purpose in this paper to present certain aspects of a cohomology theory of a ring R relative to a subring S, basing the theory on the notions of induced and produced pairs of our earlier paper **(2)**, but making the paper self-contained except for references to a few specific results of **(2)**. The cohomology groups introduced occur in dual pairs. Generic cocycles are defined, and the groups are related to the protractions and retractions of R-modules. Our cohomology groups are modules over the center of R, and in the final section we record some facts concerning their annihilators. Attention is given to the case in which R is a self dual S-ring in the sense of **(2)**. Applications of the theory to the study of orders in algebras will be found in **(4)**, where, in particular, results of **(3)** are generalized.

Since this paper was first submitted, Professor Hochschild has kindly given the author the opportunity of seeing the manuscript of his paper **(8)** in which the methods of Cartan-Eilenberg **(1)** are generalized to give a theory of relative homological algebra. The present paper (as well as **(2)**) has some results in common with the book of Cartan-Eilenberg and overlaps to some extent with the paper of Hochschild; we have indicated some of the relations in footnotes. Our point of view and methods differ rather widely from Hochschild's.

We are indebted to the referee for suggestions simplifying the notation and increasing the generality somewhat, and for the references to **(1)**.

1. Induced and produced pairs. Let R be a ring with identity element. We shall use the terms *right, left* and *two-sided R-module* in the customary way, but always assuming that the identity element of R acts as the identity operator. We shall abbreviate "right R-module" to "R-module."

Let S be a ring with identity element, χ a homomorphism of S into R mapping the identity element of S into that of R. Then every R-module is also an $S\chi$-module and hence an S-module.

If M is an S-module, the product $M \otimes_S R$ becomes an R-module when one defines $(u \otimes r)x = u \otimes rx$ for $u \in M$, and $r, x \in R$. The $\otimes$ notation is that of **(1)**, $M \otimes_S R$ denoting the tensor product over S of the S-module M with the left S-module R. The pair consisting of this R-module and the natural homomorphism $\kappa: M \to M \otimes_S R$, which is an S-homomorphism, we shall refer to as the *canonical* (R, S, χ)*-produced pair determined by* M. We shall omit the (R, S, χ) when no confusion can occur.

The term (R, S, χ)*-produced pair determined by* M will be used to refer to

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those pairs consisting of an R-module $P(M)$ and an S-homomorphism $\kappa_M : M \rightarrow P(M)$ which satisfy the condition

(P) *for each R-module N there exists a homomorphism* $\alpha \rightarrow \alpha^*$ *of* $\mathrm{Hom}_S(M, N)$ *into* $\mathrm{Hom}_R(P(M), N)$ *such that for* $\alpha \in \mathrm{Hom}_S(M, N)$, α^* *is the unique element of* $\mathrm{Hom}_R(P(M), N)$ *which makes the diagram*

$$\begin{tikzcd} P(M) \arrow[dr, "\alpha^*"] & \\ M \arrow[u, "\kappa_M"] \arrow[r, "\alpha"'] & N \end{tikzcd}$$

commutative.[1]

The Hom notation is that of **(1)**, $\mathrm{Hom}_S(M, N)$, for example, denoting the module of S-homomorphisms of M into N.

Taking $*$ to be the natural homomorphism of $\mathrm{Hom}_S(M, N)$ into $\mathrm{Hom}_R(M \otimes_S R, N)$, we find that the canonical-produced pair satisfies (I). It follows that any produced pair $(P(M), \kappa_M)$ determined by M is isomorphic with the canonical one, i.e., that there exists an R-isomorphism ϕ of $P(M)$ onto $M \otimes_S R$ such that the diagram

$$\begin{tikzcd} P(M) \arrow[rr, "\phi"] & & M \otimes_S R \\ & M \arrow[ul, "\kappa_M"] \arrow[ur, "\kappa"'] & \end{tikzcd}$$

is commutative **(2)**.

The *canonical* (R, S, χ)-*induced pair determined by* M consists of the module $\mathrm{Hom}_S(R,M)$, made into an R-module by setting $f^x(r) = f(rx)$ for $f \in \mathrm{Hom}_S(R,M)$ and $r, x \in R$, together with the natural homomorphism ϵ: $\mathrm{Hom}_S(R, M) \rightarrow M$, which is an S-homomorphism. An (R, S, χ)-*induced pair determined by*

[1]The considerations in **(2)** were conceived of primarily as generalizations to rings and algebras of certain parts of the theory of representations of finite groups. The term *induced module* was used in connection with $M \otimes_S R$ because of its relation to the classical construction for the induced representation in that theory. This turns out to be unfortunate because produced modules are then injective, induced modules projective, and because of the conflict with the terminology of the representation theory of topological groups. We have therefore switched notation and terminology in the present paper, interchanging "produced" with "induced" and "P" with "I" throughout.

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M consists of an R-module $I(M)$ and an S-homomorphism $\epsilon_M\colon I(M) \to M$, satisfying the dual of condition (P), namely

(I) *for each R-module N there exists a homomorphism $\beta \to \beta^+$ of* $\mathrm{Hom}_S(N, M)$ *into* $\mathrm{Hom}_R(N, I(M))$ *such that for $\beta \in$* $\mathrm{Hom}_S(N, M)$, *β^+ is the unique element of* $\mathrm{Hom}_R(N, I(M))$ *making the diagram*

$$\begin{tikzcd} I(M) \arrow[d, "\epsilon_M"'] & \\ M & N \arrow[l, "\beta"] \arrow[ul, "\beta^+"'] \end{tikzcd}$$

commutative.

The canonical-produced pair satisfies (I) with the natural homomorphism

$$\mathrm{Hom}_S(N, M) \to \mathrm{Hom}_R(N, \mathrm{Hom}_S(R, M))$$

as $+$. Every produced pair determined by M is isomorphic with the canonical one **(2)**.

Henceforth, $(P(M), \kappa_M)$ and $(I(M), \epsilon_M)$ will denote respectively an (R, S, χ)-produced pair and induced pair determined by M. A subscript (R, S) or (R, S, χ) will be used to obtain a more explicit notation when desired; thus $I_{(R,S)}(M)$ for $I(M)$, and so on.

In case M is an S-module by virtue of being an R-module, there exist by (P) and (I) unique R-homomorphisms

$$t_M\colon P(M) \to M,\ j_M\colon M \to I(M),\quad \kappa_M t_M = j_M \epsilon_M = 1.$$

Then κ_M and j_M are $1-1$ while t_M and ϵ_M are onto. For the canonical pairs, t_M is the natural homomorphism $M \otimes_S R \to M$, j_M the natural homomorphism $M \to \mathrm{Hom}_S(R, M)$. We shall denote by $K(M)$ the kernel of t_M, $K(M) = P(M)(1 - t_M\kappa_M)$, and by $L(M)$ the cokernel of j_M, $L(M) = I(M)/I(M)\epsilon_M j_M$. Thus $K(M)$ and $L(M)$ are R-modules determined up to R-isomorphisms independently of the particular choice of induced and produced pairs. It will be convenient to introduce the notation η_M for the injection $K(M) \to I(M)$ and π_M for the projection $P(M) \to L(M)$. Note that there exists a unique S-homomorphism

$$\lambda_M\colon L(M) \to P(M),\ \lambda_M\pi_M = 1,\ \pi_M\lambda_M = 1 - \epsilon_M j_M.$$

2. The Z_R-modules $H^i(M, N)$ and $\bar{H}^i(M, N)$. Given a ring X, Z_X will denote the center of X. If M is an S-module and N an R-module, the module $\mathrm{Hom}_S(M, N)$ becomes a Z_R-module when we define $f^z(u) = f(u)z$ for $f \in \mathrm{Hom}_S(M, N)$, $u \in M$ and $z \in Z$. If M is an R-module, $\mathrm{Hom}_R(M, N)$ is a Z_R-submodule of $\mathrm{Hom}_S(M, N)$. We can verify that the homomorphisms

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$$*: \quad \mathrm{Hom}_S(M, N) \to \mathrm{Hom}_R(P(M), N), \quad +: \ \mathrm{Hom}_S(N, M) \to \mathrm{Hom}_R(N, I(M))$$

of conditions (P) and (I) are Z_R-homomorphisms.

Suppose now that M and N are R-modules. We obtain a Z_R-homomorphism $\delta_{M,N}$ of $\mathrm{Hom}_S(M, N)$ into $\mathrm{Hom}_R(K(M), N)$ by following * with the homomorphism of $\mathrm{Hom}_R(P(M), N)$ into $\mathrm{Hom}_R(K(M), N)$ induced by the injection η_M of $K(M)$ into $P(M)$. Thus, for $f \in \mathrm{Hom}_S(M, N)$,

$$f^{\delta_{M,N}} = \eta_M f^*.$$

The kernel of $\delta_{M,N}$ is $\mathrm{Hom}_R(M, N)$. In fact, if f is in this kernel,

$$0 = (1 - t_M \kappa_M)\, \eta_M f^* = f^* - t_M f.$$

Hence $t_M f = f^*$ is an R-homomorphism, and, since t_M is an R-homomorphism onto, f is an R-homomorphism. On the other hand, if $f \in \mathrm{Hom}_R(M, N)$, $f^* = t_M f$ and

$$f^{\delta_{M,N}} = \eta_M\, t_M f = 0.$$

Dually, we may define a Z_R-homomorphism $\bar{\delta}_{M,N}$ of $\mathrm{Hom}_S(N, M)$ into $\mathrm{Hom}_R(N, L(M))$, namely, the product of $+$ with the homomorphism of $\mathrm{Hom}_R(N, I(M))$ into $\mathrm{Hom}_R(N, L(M))$ induced by π_M;

$$g^{\bar{\delta}_{M,N}} = g^+ \pi_M$$

for $g \in \mathrm{Hom}_S(N, M)$. The kernel of $\bar{\delta}_{M,N}$ is $\mathrm{Hom}_R(N, M)$.

The R-module M determines R-modules $K^i(M)$ and $P^i(M)$ $(i = 0, 1, \ldots)$, defined by the recursive formulas

$$K^0(M) = P^0(M) = M, \quad K^{i+1}(M) = K(K^i(M)), \quad P^{i+1}(M) = P(K^i(M)).$$

Dually, M determines R-modules $L^i(M)$ and $I^i(M)$, $(i = 0, 1, \ldots)$, according to[2]

$$L^0(M) = I^0(M) = M, \ L^{i+1}(M) = L(L^i(M)), \ I^{i+1}(M) = I(L^i(M)).$$

We shall now define a Z_R-complex $(C(M, N), \delta)$, determined by the ordered pair M, N of R-modules, by letting

$$C^i(M, N) = \mathrm{Hom}_S(K^i(M), N), \ \delta^i = \delta_{K^i(M),N} \qquad \text{for} \quad i \geqslant 0,$$

and

$$C(M, N) = (0), \quad \delta = 0 \text{ for } i < 0.$$

We have $\delta^{i-1}\delta^i = 0$ for all i, since, for $i \geqslant 0$, the image $B^i(M, N)$ of δ^{i-1} is contained in $\mathrm{Hom}_R(K^i(M), N)$, which is the kernel of δ^i. The cohomology groups of this complex, which are Z_R-modules, and which are determined up to Z_R-isomorphism independently of the particular choice of induced pair determined by M, we shall denote by $H^i(M, N)$. More explicit notation

[2]This amounts to constructing the standard (R, S)-projective and injective resolutions of M as in **(8)**.

such as $H^i{}_{(R,S)}(M, N)$ will be used where desirable.[3] It is immediate that $H^0(M, N) = \mathrm{Hom}_R(M, N)$, and, for $i > 0$, $\alpha \geqslant 0$,

$$H^{i+\alpha}(M, N) = H^i(K^\alpha(M), N).$$

We may dualize the above construction to obtain a second Z_R-complex $(\bar{C}(M, N), \bar{\delta})$, taking

$$\bar{C}^i(M, N) = \mathrm{Hom}_S(N, L^i(M)), \ \bar{\delta}^i = \bar{\delta}_{L^i(M),N}, \qquad i \geqslant 0,$$

and $\bar{C}^i(M, N) = (0)$, $\bar{\delta}^i = 0$ for $i < 0$. We shall denote the image of δ^{i-1} by $\bar{B}^i(M, N)$, and the cohomology groups of this complex by $\bar{H}^i(M, N)$. These are again Z_R-modules, determined up to Z_R-isomorphism independently of the choice of produced pair determined by M. We have

$$\bar{H}^0(M, N) = \mathrm{Hom}_R(N, M), \ \ \bar{H}^{i+\alpha}(M, N) = \bar{H}^i(L^\alpha(M), N)$$

for all $i > 0$, $\alpha \geqslant 0$.

3. The isomorphism $\tilde{\sigma}$. Let M and N be S-modules. There is a $Z_R \cap S_\chi$-isomorphism σ of $\mathrm{Hom}_S(P(M), N)$ onto $\mathrm{Hom}_S(M, I(N))$ mapping

$$f \in \mathrm{Hom}_S(P(M), N)$$

onto $f^\sigma = \kappa_M f^+$. The inverse τ of σ maps $g \in \mathrm{Hom}_S(M, I(N))$ onto $g^\tau = g^* \epsilon_N$. In fact, $(\kappa_M f^+)^* = f^+$, and hence $f^{\sigma\tau} = (\kappa_M f^+)^* \epsilon_N = f^+ \epsilon_N = f$, so that $\sigma\tau = 1$. Dually, $\tau\sigma = 1$. In the case of the canonical-induced and -produced pairs, σ is the natural homomorphism of $\mathrm{Hom}_S(M \otimes_S R, N)$ onto $\mathrm{Hom}_S(M, \mathrm{Hom}_S(R, M))$.

Now assume that M and N are R-modules so that $K(M)$ and $L(N)$ are defined. We then have a $Z_R \cap S_\chi$-isomorphism $\tilde{\sigma}$ of $\mathrm{Hom}_S(K(M), N)$ onto $\mathrm{Hom}_S(M, L(N))$, mapping $f \in \mathrm{Hom}_S(K(M), N)$ onto

$$f^{\tilde{\sigma}} = [(1 - t_M \kappa_M) f]^\sigma \pi_N.$$

Inverse to $\tilde{\sigma}$ is $\tilde{\tau}$ defined by

$$g^{\tilde{\tau}} = \eta_M (g \lambda_N)^\tau$$

for $g \in \mathrm{Hom}_S(M, L(N))$. Indeed,

$$\begin{aligned} f^{\tilde{\sigma}} \lambda_N &= \kappa_N [(1 - t_M \kappa_M) f]^+ \pi_N \lambda_N = \kappa_M [(1 - t_M \kappa_M) f]^+ (1 - \epsilon_N j_N) \\ &= \kappa_M [(1 - t_M \kappa_M) f]^+ - \kappa_M (1 - t_M \kappa_M) f j_N = [(1 - t_M \kappa_M) f]^\sigma. \end{aligned}$$

Hence

$$f^{\tilde{\sigma}\tilde{\tau}} = \eta_M (f^{\tilde{\sigma}} \lambda_N)^\tau = \eta_M [(1 - t_M \kappa_M) f]^{\sigma\tau} = f,$$

whence $\tilde{\sigma}\tilde{\tau} = 1$. Dually, $\tilde{\tau}\tilde{\sigma} = 1$.

[3]It can easily be proved that $H^i{}_{(R,S)}(M, N)$ is isomorphic with $Ext^i{}_{(R,S_\chi)}(M, N)$ as defined by Hochschild **(8)**. When this is taken into consideration, the connection between the results of §§2–7 of the present paper and **(8)** will be seen.

We shall show that the diagram

$$\begin{tikzcd} \mathrm{Hom}_S(M,N) \arrow[r, "\delta_{M,N}"] \arrow[dr, "\bar{\delta}_{N,M}"'] & \mathrm{Hom}_S(K(M),N) \arrow[d, "\tilde{\sigma}"] \\ & \mathrm{Hom}_S(M,L(N)) \end{tikzcd}$$

is anti-commutative. If $f \in \mathrm{Hom}_S(M, N)$,

$$(1 - t_M\kappa_M)f^{\delta} = (1 - t_M\kappa_M)\eta_M f^* = f^* - t_M f.$$

Hence

$$f^{\delta\tilde{\sigma}} = (f^* - t_M f)^{\sigma}\pi_N = f^{*\sigma}\pi_N - (t_M f)^{\sigma}\pi_N.$$

But

$$f^{*\sigma}\pi_N = \kappa_M(f^*)^+\pi_N = \kappa_M f^* j_N\pi_N = 0,$$

and

$$(t_M f)^{\sigma}\pi_N = \kappa_M(t_M f)^+\pi_N = \kappa_M t_M f^+\pi_N = f^+\pi_N = f^{\bar{\delta}}.$$

Hence

$$f^{\delta\tilde{\sigma}} = -f^{\bar{\delta}}.$$

Moreover, the diagram

$$\begin{tikzcd} \mathrm{Hom}_S(K(M),N) \arrow[d, "\tilde{\sigma}"'] \arrow[dr, "\delta_{N,K(M)}"] & \\ \mathrm{Hom}_S(M,L(N)) \arrow[r, "\delta_{M,L(N)}"'] & \mathrm{Hom}_S(K(M),L(N)) \end{tikzcd}$$

is commutative. For, if $f \in \mathrm{Hom}_S(K(M), N)$,

$$f^{\tilde{\sigma}} = \kappa_{K(M)}[(1 - t_{K(M)}\kappa_{K(M)})f]^+\pi_N,$$

hence

$$(f^{\tilde{\sigma}})^* = [(1 - t_{K(M)}\kappa_{K(M)})f]^+\pi_N.$$

Thus

$$\begin{aligned} f^{\tilde{\sigma}\delta} &= \eta_{K(M)}(f^{\tilde{\sigma}})^* = \eta_{K(M)}[(1 - t_{K(M)}\kappa_{K(M)})f]^+\pi_N \\ &= \eta_{K(M)}[(1 - t_{K(M)}\kappa_{K(M)})f]^+\pi_N = f^+\pi_N = f^{\bar{\delta}}, \end{aligned}$$

proving the desired result.

Applying these results to the complexes $(C(M, N), \delta)$ and $(\bar{C}(N, M), \bar{\delta})$ we obtain an anti-commutative diagram

$$\begin{array}{ccccccccc} \ldots \to C^{-1}(M, N) & \xrightarrow{\delta^{-1}} & C^0(M, N) & \xrightarrow{\delta^0} & C'(M, N) & \xrightarrow{\delta'} & C^2(M, N) \to \ldots \\ \| & & \| & & \downarrow\bar{\sigma} & & \downarrow\bar{\sigma} \\ \ldots \to \bar{C}^{-1}(N, M) & \xrightarrow{\bar{\delta}^{-1}} & \bar{C}^0(N, M) & \xrightarrow{\bar{\delta}^0} & \bar{C}'(N, M) & \xrightarrow{\bar{\delta}'} & \bar{C}^2(N, M) \to \ldots \end{array}$$

Consequently

THEOREM 1. *$\bar{\sigma}$ induces a $Z_R \cap S_\chi$ isomorphism of $H^i(M, N)$ onto $\bar{H}^i(N, M)$ for $i \geqslant 0$.*

COROLLARY. *$H^{i+\alpha}(M, N) \simeq H^i(M, L^\alpha(N))$ and $\bar{H}^{i+\alpha}(M, N) \simeq \bar{H}^i(M, K^\alpha(N))$ for $i > 0$, $\alpha \geqslant 0$, the isomorphisms being $Z_R \cap S_\chi$-isomorphisms.*

4. Some special induced and produced pairs. To give explicit constructions for the cohomology groups introduced above one need only supply particular induced and produced pairs, the canonical ones not always being the most suitable. Thus, for example, suppose that B and A are rings, and let T be a subring of A containing the identity element thereof. Let M be a $B' \otimes T$-module. Here the $'$ indicates mirror image, and $\otimes$ the tensor product over the ring of the rational integers. A $B' \otimes T$-module may be considered as a left B-, and a right T-module, and we shall use corresponding notation where convenient. The module $M \otimes_T A$ becomes a $B' \otimes A$-module when we let $b(u \otimes a) = bu \otimes a$ and $(u \otimes a)x = ax$ for $b \in B$, $u \in M$ and a, $x \in A$, while $\mathrm{Hom}_T(A, M)$ becomes a $B' \otimes A$-module when we let ${}^bf(a) = b[f(a)]$ and $f^x(a) = f(xa)$ for $b \in B, f \in \mathrm{Hom}_T(A, M)$ and $x, a \in A$. As may be seen by verifying (I) and (P), combining these modules with the natural homomorphisms $M \to M \otimes_T A$ and $\mathrm{Hom}_T(A, M) \to M$ yields respectively a $(B' \otimes A, B' \otimes T, \chi)$-induced and produced pair determined by M, where χ is the natural homomorphism $B' \otimes T \to B' \otimes A$. If M is a $B' \otimes A$-module, $K(M)$ and $L(M)$ are the respective kernels of the natural homomorphisms $M \otimes_T A \to M$ and $M \to \mathrm{Hom}_T(A, M)$.

Now let M be a $B' \otimes T$-module, N a $B' \otimes A$-module. Further, let U be a subring of $T \cap Z_A$, containing the identity element of A. The module $\mathrm{Hom}_{B' \otimes U}(M, N)$ attains the status of a $T' \otimes_U A$-module when we define

$${}^tf(u) = f(ut) \text{ and } f^x(u) = f(u)x$$

for

$$f \in \mathrm{Hom}_{B' \otimes U}(M, N), \qquad t \in T, x \in A, u \in M;$$

let us denote it by $\Phi(M, N)$. If M is a $B' \otimes A$-module we may replace t above by $y \in A$, turning $\Phi(M, N)$ into an $A' \otimes_U A$-module. We shall outline how one may prove[4]

[4]Combining Theorems 1 and 2 gives Theorem 2 of **(8)**.

THEOREM 2. *If M and N are $B' \otimes A$-modules, then there exists a natural Z_A-isomorphism of*

$$H^i_{(B'\otimes A,\, B'\otimes T)}(M, N) \quad \text{onto} \quad \bar{H}^i_{(C,\, D)}(\Phi(M, N), A),$$

where $C = A' \otimes_U A$, $D = T' \otimes_U A$.

The first of the cohomology groups mentioned is a $Z_{B'\otimes A}$-module, the second a Z_C-module, so it makes sense to speak of a Z_A-isomorphism between them. The natural homomorphism $\eta\colon D \to C$ is understood.

The main step in the proof consists in the construction of a suitable (C, D, η)-induced pair for $\Phi(M, N)$. To this end, consider the diagram

$$\begin{tikzcd}
\Phi(P(M),N) \arrow[d, "\epsilon"'] & \\
\Phi(M,N) & H \arrow[l, "\beta"] \arrow[ul, "\beta^+"']
\end{tikzcd}$$

where (a) ϵ is the D-homomorphism induced by $\kappa_M\colon M \to P(M)$, $(P(M), \kappa_M)$ being a $(B' \otimes A,\ B' \otimes T,\ \chi)$-produced pair determined by M, and (b) β is a D-homomorphism of the C-module H. If now β^+ is a C-homomorphism such that the diagram is commutative, then for $h \in H$, $u \in M$, $(h\beta^+)(u\kappa_M) = (h\beta)(u)$, hence for $a \in A$,

$$(h\beta^+)(u\kappa_M.a) = {}^a(h\beta^+)(u\kappa_M) = [(ah)\beta](u),$$

i.e.,

4.1 $$(h\beta^+)(u\kappa_M.a) = [(ah)\beta](u).$$

On the other hand, we may verify that the formula 4.1 does indeed define a C-homomorphism making the diagram commutative. Consequently the pair $(\Phi(P(M), N), \epsilon)$ is a (C, D, η)-induced pair for $\Phi(M, N)$.

If M is a $B' \otimes A$-module, $\Phi(M, N)$ is a C-module, and the homomorphism $t_M\colon P(M) \to M$ induces the unique $A' \otimes_U A$-homomorphism $j\colon \Phi(M, N) \to \Phi(P(M), N)$ such that $j\epsilon = 1$, as is seen by taking $\beta = 1$ and $H = \Phi(M, N)$ in 4.1. Consequently the sequence

$$(0) \quad \to \Phi(M, N) \xrightarrow{j} \Phi(P(M), N) \xrightarrow{\gamma} \Phi(K(M), N) \to (0)$$

is exact, where γ is induced by the injection $K(M) \to P(M)$. In the complex for constructing the groups $\bar{H}^i(\Phi(M, N), A)$ from the produced pair $(\Phi(I(M), N), \epsilon)$ we may therefore identify $L(\Phi(M, N))$ with $\Phi(K(M), N)$ and the projection of $\Phi(I(M), N)$ onto $L(\Phi(M, N))$ with γ.

The modules $\mathrm{Hom}_{B'\otimes T}(M, N)$ and $\mathrm{Hom}_D(A, \Phi(M, N))$ are in particular Z_A-modules, the first being a $Z_{B'\otimes A}$- and the second a Z_C-module. The natural isomorphism between them is a Z_A-isomorphism. One may now verify the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{B'\otimes T}(M, N) & \rightarrow & \mathrm{Hom}_{B'\otimes T}(K(M), N) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_D(A, \Phi(M, N)) & \rightarrow & \mathrm{Hom}_D(A, \Phi(K(M), N)) \end{array}$$

where (1) the arrows pointing down represent the natural isomorphisms, (2) the top arrow represents $\delta_{M,N}$, and (3) the bottom arrow represents the product of

$$+: \ \mathrm{Hom}_D(A, \Phi(M, N)) \rightarrow \mathrm{Hom}_D(A, \Phi(P(M), N))$$

with the homomorphism of the second of these modules into

$$\mathrm{Hom}_D(A, \ \Phi(K(M), N))$$

induced by γ. Application of this fact to the appropriate complexes gives Theorem 2.

5. Generic cocycles. Let M, N and X be R-modules, and let

$$f \in \mathrm{Hom}_S(K(M), N).$$

Then $\mu_f: g \rightarrow gf$ for $g \in \mathrm{Hom}_S(X, K(M))$ defines a $Z_R \cap S_X$-homomorphism μ_f of $\mathrm{Hom}_S(X, K(M))$ into $\mathrm{Hom}_S(X, N)$. If

$$f \in \mathrm{Hom}_R(K(M), N)$$

we see that μ_f is a Z_R-homomorphism, and moreover, that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_S(X, K(M)) & \overset{\delta_{X,K(M)}}{\rightarrow} & \mathrm{Hom}_S(K(X), K(M)) \\ \downarrow \mu_f & & \downarrow \mu_f \\ \mathrm{Hom}_S(X, N) & \overset{\delta_{X,N}}{\rightarrow} & \mathrm{Hom}_S(K(X), N) \end{array}$$

is commutative. For then

$$g^{\mu_f \delta} = \eta_X(gf)^* = \eta_X g^* f = g^{\delta\mu_f}.$$

Application of this to the complexes $(C(X, K(M)), \delta)$ and $(C(X, N), \delta)$ proves that μ_f induces a Z_R-homomorphism of $H^\alpha(X, K(M))$ into $H^\alpha(X, N)$ for all $\alpha \geqslant 0$. In particular:

If $f \in \mathrm{Hom}_S(K^i(M), N)$, μ_f induces a Z_R-homomorphism of $H^\alpha(M, K^i(M))$ into $H^\alpha(M, N)$ for all $\alpha \geqslant 0$, and μ_f: $I^i \rightarrow f$, where I^i is the identity automorphism of $K^i(M)$.

We shall refer to the element $I^i \in B^i(M, K^i(M))$ as the *first generic i-cocycle determined by M.*

Dually, if

$$f \in \mathrm{Hom}_S(N, L(M)), \quad \lambda_f: g \rightarrow fg$$

defines a $Z_R \cap S_X$-homomorphism λ_f of $\mathrm{Hom}_S(L(M), X)$ into $\mathrm{Hom}_S(N, X)$, which is a Z_R-homomorphism if $f \in \mathrm{Hom}_R(N, L(M))$. In the latter case, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_S(L(M), X) & \xrightarrow{\bar{\delta}_{X,L(M)}} & \mathrm{Hom}_S(L(M), L(X)) \\ \lambda_f \downarrow & & \lambda_f \downarrow \\ \mathrm{Hom}_S(N, X) & \xrightarrow{\bar{\delta}_{X,N}} & \mathrm{Hom}_S(N, L(X)) \end{array}$$

is commutative. Hence λ_f induces a Z_R-homomorphism of $H^\alpha(X, L(M))$ into $H^\alpha(X, N)$, $\alpha \geqslant 0$. In particular:

If $f \in \mathrm{Hom}(N, L^i(M))$, λ_f induces a Z_R-homomorphism of $H^\alpha(N, L^i(M))$ into $H^\alpha(M, N)$ for all $\alpha \geqslant 0$, and λ_f: $J^i \to f$, where J^i is the identity automorphism of $L^i(M)$.

We shall refer to the element $J^i \in \bar{B}^i(M, K^i(M))$ as the *second generic i-cocycle determined by M.*

As a consequence of the above considerations we obtain

THEOREM 3. *If i is a positive integer, and M is an R-module, then the following conditions imply each other.*

(a) $H^i(M, N) = (0)$ *for all R-modules N.*

(b) $H^i(M, K^i(M)) = (0)$.

(c) $I^i \in B^i(M, N)$.

(d) $H^{i+\alpha}(M, N) = (0)$ *for all R-modules N and all* $\alpha \geqslant 0$.

The dual Theorem 3′ is obtained by replacing H with $\bar{H}$, K with L, B with $\bar{B}$, and I with J.

6. Protractions and retractions.[5] Let M be an R-module. If H is an R-module and $\phi: M \to H$ is an R-homomorphism, we shall call the pair (H, ϕ) and *(R, S)-protraction of M* provided that there exists an S-homomorphism $\lambda: M \to H$ such that $\lambda\phi = 1$. The kernel $N = H(1 - \phi\lambda)$ of ϕ will be called the *kernel of* (H, ϕ). Two (R, S)-protractions (H_1, ϕ_1) and (H_2, ϕ_2) of M with kernel N will be called *R-isomorphic* if there exists an R-isomorphism μ of H_1 onto H_2 such that $\phi_1 = \mu\phi_2$.

Corresponding to an (R, S)-protraction of M with kernel N there is an element $f_\lambda \in \mathrm{Hom}_R(K(M), N)$ defined by

$$f_\lambda = \eta_M \lambda^*(1 - \phi\lambda).$$

It can be seen that the correspondence $(H, \phi) \to f_\lambda$ induces a 1–1 mapping of the set of classes of R-isomorphic (R, S)-protractions of M with kernel N onto $H'(M, N)$, which becomes a group isomorphism when the Baer composition is introduced into the set of classes of protractions.

The (R, S)-protraction (H, ϕ) of M with kernel N is said to *split* if there exists an R-homomorphism $\alpha: M \to H$ such that $\alpha\phi = 1$. Two R-isomorphic protractions split, or do not, together. Let $f_\lambda \in \mathrm{Hom}_R(K(M), N)$ correspond

[5]The material of this section overlaps considerably with Cartan and Eilenberg (**1**, §6, Ch. II), as well as with Hochschild (**8**).

to (H, ϕ) as above. Then it is easy to see that each of the following conditions is necessary and sufficient for (H, ϕ) to split:

(a) there exists an R-submodule N^* of H such that $H = N \oplus N^*$,

(b) $H \simeq N \oplus M$ as an R-module,

(c) $f_\lambda \in B^1(M, N)$.

The pair $(I(M), t_M)$ is an (R, S)-protraction of M with kernel $K(M)$, and corresponds to the class of the first generic 1-cocycle

$$I^1 = f_{\kappa_M}$$

An R-module M will be called (R, S)-*projective* if, whenever (H', ϕ) is an (R, S)-protraction of an R-module H and $\alpha\colon M \to H$ is an R-homomorphism, there exists an R-homomorphism $\bar{\alpha}\colon M \to H$ such that $\alpha = \bar{\alpha}\phi$. From **(2**, Theorem 6**)** Theorem 3, and the above remarks we conclude

THEOREM 4. *Each of the following conditions is necessary and sufficient for an R-module M to be* (R, S)*-projective.*

(a) *The* (R, S)*-protraction* $(P(M), t_M)$ *splits.*

(b) *Every* (R, S)*-protraction of M splits.*

(c) $H^1(M, N) = (0)$ *for all R-modules N.*

If U is an S-module, the R-module $P(U)$ is (R, S)-projective according to **(2**, Theorem 3**)**. Hence we have

COROLLARY. *If U is an S-module,* $H^i(P(U), N) = (0)$ *for all R-modules N and all* $i > 0$.

A pair (H, ψ) consisting of an R-module H and an R-homomorphism $\psi\colon M \to H$ will be called an (R, S)*-retraction of M with kernel N* if there exists an S-homomorphism $\mu\colon H \to M$ such that $\psi\mu = 1$, and if N is the cokernel of ψ, $N = H/M\psi$. This is the dual of the concept of (R, S)-protraction. We define R-isomorphism between retractions by dualizing the corresponding concept for protractions, and obtain a 1–1 correspondence between the set of classes of isomorphic (R, S)-retractions of M with cokernel N and the elements of $\bar{H}^1(M, N)$ (and hence of $H^1(N, M)$ by Theorem 1). The definition of splitting for (R, S)-retractions is dual to that for protractions. Of course there is a 1–1 correspondence between the set of (R, S)-protractions of M with kernel N and the (R, S)-retractions of N with cokernel N, such that a protraction splits if and only if the corresponding retraction splits.

$(P(M), j_M)$ is an (R, S)-retraction of M with cokernel $L(M)$, and corresponds to the class of the second generic 1-cocycle J^1.

Dual to (R, S)-projective modules we have (R, S)-injective modules, and dual to Theorem 4 we have

THEOREM 4′. *The following conditions*

(a) *The* (R, S)*-retraction* $(I(M), j_M)$ *of M splits.*

(b) *Every* (R, S)*-retraction of M splits.*

(c) $\bar{H}^1(M, N) = (0)$ *for all R-modules N.*

are each necessary and sufficient for an R-module M to be (R, S)*-injective.*

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If U is an S-module, $I(U)$ is (R, S)-injective according to **(2**, Theorem 3′**)**. Consequently

COROLLARY. *For every S-module U and every R-module N, $\bar{H}^i(I(U), N) = (0)$ for all $i > 0$.*

7. Cohomology dimension. If an R-module M is (R, S)-projective [injective], then according to **(2**, Theorem 6**)** so is $K(M)$ $[L(M)]$. Under certain circumstances the converse is true. We shall consider the hypothesis

$(R, S; M)$ *There exists an R-isomorphism μ_M of $I(M)$ onto $P(M)$.*

If $(R, S; M)$ holds, then M is (R, S)-projective if and only if it is (R, S)-injective, as follows from **(2**, Theorems 6, 6′**)**.

THEOREM 5. *Suppose that the hypothesis $(R, S; K(M))$ holds. Then $K(M)$ (R, S)-projective implies that M is (R, S)-projective.*

Proof. If $(R, S; K(M))$ holds and $K(M)$ is (R, S)-projective then $K(M)$ is (R, S)-injective. Hence the (R, S)-retraction $(P(M), \eta_M)$ of $K(M)$ splits by **(2**, Theorem 6′**)**. Hence $P(M) \simeq M \otimes K(M)$ as an R-module, so that M is (R, S)-projective by **(2**, Theorem 6**)**.

The dual Theorem 5′ is obtained by replacing K with L and projective with injective.

It will be convenient to denote by $d_{(R,S)}\, M$ the smallest integer $i > 0$ such that $H^i(M, N) = (0)$ for all R-modules N, if such an i exists, setting $d_{(R,S)}\, M = \infty$ otherwise. The dual $\bar{d}_{(R,S)}\, M$ is defined by replacing H by $\bar{H}$. By Theorems 4, 4′, $d_{(R,S)}\, M \leqslant i$ if and only if $K^{i-1}(M)$ is (R, S)-projective, while $\bar{d}_{(R,S)}\, M \leqslant i$ if and only if $L^{i-1}\,(M)$ is (R, S)-injective. By Theorem 5, if $(R, S; K^i(M))$ holds, then $d_{(R,S)}\, M \leqslant i + 1$ implies $d_{(R,S)}\, M \leqslant i$, while if $(R, S; L^i(M))$ holds, then $\bar{d}_{(R,S)}\, M \leqslant i + 1$ implies $\bar{d}_{(R,S)}\, M \leqslant i$.

The two conditions

(c.i) $\quad d_{(R,S)}\, M \leqslant i$ for all R-modules M.

(c′.i) $\quad \bar{d}_{(R,S)}\, M \leqslant i$ for all R-modules M.

imply each other as we deduce at once from Theorem 1. We define *class* (R, S) to be the minimum integer $i > 0$ such that (c.i) and (c′.i) hold if such an i exists, letting *class* $(R, S) = \infty$ otherwise. From the above we have

THEOREM 6. *If the hypothesis $(R, S; M)$ holds for every R-module M, then class $(R, S) < \infty$ implies class $(R, S) = 1$.*

The hypotheses of this theorem are satisfied if R is a self dual S-ring in the sense of **(2)**.

Now let us suppose that B, A, T and U are rings as in §4, and let M and N be $B' \otimes A$-modules. If

$$d_{(B' \otimes A,\, B' \otimes T)} M = i < \infty,$$

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then $I^i(M) \simeq K^i(M) \otimes K^{i+1}(M)$ as a $B' \otimes A$-module, according to Theorem 6 of **(2)**. Consequently

$$\Phi(I^i(M), N) \simeq \Phi(K^i(M), N) \oplus \Phi(K^{i+1}(M), N)$$

as an $A' \otimes_U A$-module, whence one concludes by the construction of §4 that

$$d_{(A' \otimes_U A,\ T' \otimes_U A)}(M, N) \leqslant i.$$

In particular, if M is $(B' \otimes A, B' \otimes \text{T})$-projective then $\Phi(M, N)$ is $(A' \otimes_U A, T' \otimes_U A)$-injective **(8**, Lemma 2**)**. Moreover, we conclude that

$$\text{class}(A' \otimes_U A, T' \otimes_U A) = d_{(A' \otimes_U A,\ T' \otimes_U A)} A = \bar{d}_{(A' \otimes_U A,\ T' \otimes_U A)} A$$
$$\geqslant \text{class}(B' \otimes A, B' \otimes T),$$

considering A in the natural way as an $A' \otimes_U A$-module.

8. The ideals $\mathfrak{J}^i(M)$ and $\bar{\mathfrak{J}}^i(M)$. The results of §5 can be refined as follows. We shall denote by $\mathfrak{J}^i(M, N)$ [or $\mathfrak{J}^i_{(R,S)}(M, N)$] the annihilator of the Z_R-module $H^i(M, N)$. Letting $\mathfrak{J}^i(M)$ denote the intersection over all R-modules N of the ideals $\mathfrak{J}^i(M, N)$ we have by §5 that

$$\begin{aligned} \mathfrak{J}^i(M) &= \mathfrak{J}^i(M, K^i(M)) \\ &= \{\omega \in Z_R | \zeta\omega \in \bar{B}^i(M, K^i(M))\}. \end{aligned}$$

Here $\zeta\omega$ denotes right operation by ω; $\zeta\omega \colon u \to u\omega$ for $u \in K^i(M)$, $\omega \in Z_R$, so that $\zeta\omega = I^{i\omega}$. Condition (a) of Theorem 3 is equivalent to the condition that $\mathfrak{J}^i(M) = Z_R$.

Dually, we define $\bar{\mathfrak{J}}^i(M, N)$ to be the annihilator of the Z_R-module $\bar{H}^i(M, N)$, and $\bar{\mathfrak{J}}^i(M)$ to be the intersection over all R-modules N of these ideals. Then

$$\begin{aligned} \bar{\mathfrak{J}}^i(M) &= \bar{\mathfrak{J}}(M, L^i(M)) \\ &= \{\omega \in Z_R | \zeta\omega \in \bar{B}^i(M, L^i(M))\}. \end{aligned}$$

The condition dual to condition (a) of Theorem 3 is equivalent to the condition that $\bar{\mathfrak{J}}^i(M) = Z_R$.

We have at once that for $i > 0$, $\alpha \geqslant 0$,

$$\mathfrak{J}^{i+\alpha}(M, N) = \mathfrak{J}^i(K^\alpha(M), N), \quad \bar{\mathfrak{J}}^{i+\alpha}(M, N) = \bar{\mathfrak{J}}^i(L^\alpha(M), N).$$

Moreover, by Theorem 1,

$$\mathfrak{J}^i(M, N) \cap S_\chi = \bar{\mathfrak{J}}^i(N, M) \cap S_\chi$$

while the Corollary to Theorem 1 implies that

$$\mathfrak{J}^i(M) \cap S_\chi \subseteq \mathfrak{J}^{i+\alpha}(M), \quad \bar{\mathfrak{J}}^i(M) \cap S_\chi \subseteq \bar{\mathfrak{J}}^{i+\alpha}(M)$$

for $i > 0$, $\alpha \geqslant 0$.

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Let (H, ϕ) be an (R, S)-protraction of the R-module M with kernel N (§6). Then there exists an S-homomorphism $\lambda: M \to H$ such that $\lambda\phi = 1$, and $N = H(1 - \phi\lambda)$. The corresponding element $f_\lambda \in \mathrm{Hom}_R(M, N)$ is defined by $f_\lambda = \eta_M\lambda^*(1 - \phi\lambda)$.

PROPOSITION. *If $\omega \in Z_R$, then $f_\lambda^{\omega} \in B^1(M, N)$ if and only if there exists an R-homomorphism $\beta: M \to H$ such that $\beta\phi = \zeta\omega$, where $\zeta\omega$ denotes right operation on M by ω.*

Proof. Suppose that

$$f\lambda^{\omega} = g^{\delta_{M,N}}, \quad g \in \mathrm{Hom}_S(M, N).$$

This means that $\eta_M\lambda^*(1 - \phi\lambda)\zeta\omega = \eta_M g^*$. Then, if η is the injection $N \to H$,

$$0 = \eta_M[\lambda^*(1 - \phi\lambda)\zeta\omega - \eta_M g^*]\eta = \eta_M[\lambda^*\zeta\omega - g^*\eta]$$

$$= \eta_M[\lambda\zeta\omega - g\eta]^* = [\lambda\zeta\omega - g\eta]^{\delta_{M,N}}.$$

Consequently $\beta = \lambda\zeta\omega - g\eta$ is an element of $\mathrm{Hom}_R(M, H)$. Further, $\beta\phi = \lambda\zeta\omega\phi - g\eta\phi = \zeta\omega$.

Suppose on the other hand that there exists an R-homomorphism $\beta: M \to H$ such that $\beta\phi = \zeta\omega$. Since

$$\eta_M\beta^* = \eta_M\beta\, t_M = 0,$$

if we let $\gamma = \lambda\zeta\omega - \beta$, we have

$$(f\lambda^{\omega})\eta = \eta_M(\lambda^*\zeta\omega - \beta^*) = \eta_M\gamma^* = g^{\delta_{M,N}}\eta,$$

where $g = \gamma(\zeta\omega - \phi\gamma)$ is an element of $\mathrm{Hom}_S(M, N)$. Hence

$$f\lambda^{\omega} = g^{\delta_{M,N}}.$$

There is the dual result for (R, S)-retractions of M.

Application of this proposition to the (R, S)-protraction $(P(M), t_M)$ of M gives (a) of the following theorem, (a′) being its dual.

THEOREM 7. *If $\omega \in Z_R$, then*

(a) *$\omega \in \mathfrak{J}^1(M)$ if and only if there exists $\beta \in \mathrm{Hom}_R(M, P(M))$ such that $\beta\, t_M = \zeta\omega$.*

(a′) *$\omega \in \overline{\mathfrak{J}}^1(M)$ if and only if there exists $\beta \in \mathrm{Hom}_R(I(M), M)$ such that $j_M\beta = \zeta\omega$.*

$$\begin{tikzcd}
I(M) \arrow[r, "\mu_M"] & P(M) \arrow[d, "t_M"] \arrow[dl, "\alpha^*"'] \\
M \arrow[u, "j_M"] & M \arrow[l, "\alpha"] \arrow[ul, "\alpha^+"']
\end{tikzcd}$$

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Assuming hypothesis $(R, S; M)$ of §7, namely, the existence of an R-isomorphism μ_M of $I(M)$ onto $P(M)$, we may construct the Casimir operators as in **(2)**. Thus, if $\alpha \in \mathrm{Hom}_S(M, M)$, $c(\alpha) = \alpha^+\mu_M t_M$ and $\bar{c}(\alpha) = j_M\mu_M\alpha^*$ are elements of $\mathrm{Hom}_R(M, M)$. We may call c and $\bar{c}$ the *first* and *second Casimir operators* associated with M. They are of course dependent on μ_M.

THEOREM 8. *If $(R, S; M)$ holds, then an element $\omega \in Z_R$ is contained in $\mathfrak{J}'(M)$ if and only if there exists an S-endomorphism α of M such that $c(\alpha) = \zeta\omega$.*

Proof. Suppose α is an S-endomorphism of M such that $c(\alpha) = \zeta\omega$. Then $\beta = \alpha^+\mu_M$ is an R-homomorphism of M into $I(M)$, and

$$\beta t_M = \alpha^+\mu_M t_M = c(\alpha) = \zeta\omega.$$

Hence $\omega \in \mathfrak{J}^1(M)$ by Theorem 7.

On the other hand, $\omega \in \mathfrak{J}'(M)$ implies by Theorem 7 the existence of an R-homomorphism $\beta\colon M \to I(M)$ such that $\beta t_M = \zeta\omega$. Now $\alpha = \beta\mu_M{}^{-1}\epsilon_M$ is an S-endomorphism of M such that

$$c(\alpha) = \alpha^+\mu_M t_M = (\beta\mu_M^{-1}\epsilon_M)^+\mu_M t_M = \beta t = \zeta\omega.$$

The dual of Theorem 8 is obtained by replacing $\mathfrak{J}$ by $\bar{\mathfrak{J}}$ and c by $\bar{c}$.

The condition $(R, S; M)$ may be strengthened by demanding that $c = \bar{c}$. Let us denote the resulting condition by $(R, S; M)^+$. If R is a self-dual S-ring in the sense of **(2)**, $(R, S; M)^+$ holds for all R-modules M **(2)**. Comparing Theorem 8 and its dual we have

COROLLARY 1. *If $(R, S; M)^+$ holds then $\mathfrak{J}^1(M) = \bar{\mathfrak{J}}^1(M)$.*

We can also prove

COROLLARY 2. *If $(R, S; K(M))^+$ holds then $\mathfrak{J}^2(M) \cap S_\chi = \mathfrak{J}^1(M) \cap S_\chi$, while dually, if $(R, S; L(M))^+$ holds then $\bar{\mathfrak{J}}^2(M) \cap S_\chi = \bar{\mathfrak{J}}^1(M) \cap S_\chi$.*

Proof. By Corollary 1, $\mathfrak{J}^2(M) = \mathfrak{J}^1(K(M)) = \bar{\mathfrak{J}}^1(K(M))$. By Theorem 1 there exists a $Z_R \cap S_\chi$-isomorphism of $\bar{H}^1(K(M), M)$ onto $H^1(M, K(M))$. It follows that $\bar{\mathfrak{J}}^1(K(M)) \cap S_\chi \subseteq \mathfrak{J}^1(M, K(M)) = \mathfrak{J}^1(M)$. Since $\mathfrak{J}^1(M) \cap S_\chi \subseteq \mathfrak{J}^2(M)$, the corollary is proved.

The above results may be extended by using the recursion relations $\mathfrak{J}^{i+\alpha}(M) = \mathfrak{J}^i(K^\alpha(M))$ and $\bar{\mathfrak{J}}^{i+\alpha}(M) = \mathfrak{J}^i(L^\alpha(M))$. Thus, for example, we have by Corollary 2 that if $(R, S; M)^+$ holds for all R-modules M, then for $i > 0$,

$$\mathfrak{J}^i(M) \cap S_\chi = \mathfrak{J}^1(M) \cap S_\chi$$

and

$$\bar{\mathfrak{J}}^i(M) \cap S_\chi = \bar{\mathfrak{J}}^1(M) \cap S_\chi.$$

The methods of §4 may be used to give further information concerning these ideals in the case considered there. For example, if M is an $A' \otimes_U A$-module we find that

$$\mathfrak{I}^i_{(C,D)}(A) \cap D\eta \subseteq \mathfrak{I}^i_{(C,D)}(M)$$

where the notation is that of §4.

To see how ideals of the kind considered here occur in the study of orders in algebras, see **(3)** and **(4)**.

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Montana State University and
University of Michigan