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JEFFERY-WILLIAMS LECTURE, 1976 NON-STATIC RADIALLY SYMMETRIC DISTRIBUTIONS OF MATTER

BY
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1. **Introduction.** When I agreed to give the 1976 Jeffery–Williams Lecture to the Canadian Mathematical Congress, I did not envisage that it would be my lot to give what might now be called the first memorial lecture in this series of lectures. During the past year, Ralph Jeffery and Lloyd Williams both died, and their deaths brought forward many deserved tributes to these two wonderful men. My lecture today will forge a link with my mathematical past, a past in which both of these men played important roles.

At the first meeting of the Canadian Mathematical Congress, the title of my talk was "Isotropic Solutions of Einstein's Field Equations". In that paper, it was shown that the line-element had the form

$$ds^2 = v^2\, dt^2 - (d\ell^2/w^2), \tag{1.1}$$

where

$$d\ell^2 = dx^2 + dy^2 + dz^2, \tag{1.2}$$

and the gravitational potentials v and w are both functions of x, y, z, and t. Further, it was shown that only four distributions of matter could lead to isotropic solutions of the Einstein field equations, one of which corresponds to a spherically symmetric distribution of matter. Although this work has been of interest to me for over thirty years, it is only in the past five years that I have been able to find a significant number of rigorous solutions of the field equations that represent non-static radially symmetric distributions of matter. Obtaining these new solutions forms the basis of the present paper.

For convenience, the line-element is taken to have the form

$$ds^2 = \exp(\nu)\, dt^2 - \exp(\mu)[dr^2 + r^2 \sin^2\theta\, d\theta^2 + r^2\, d\theta^2]. \tag{1.3}$$

2. **The field equations.** If the pressure and density of the spherically symmetric distribution of matter are respectively denoted by $p = p(r, t)$, $\rho = \rho(r, t)$, then the conditions $T_1^1 = T_2^2 = T_3^3 = -8\pi p$, $T_4^4 = 8\pi\rho$ and $T_j^i = 0$, $i \neq j$ on the

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energy-momentum tensor yield the equations

$$(2.1)\quad 8\pi p=\exp(-\mu)\left[\tfrac{1}{4}\mu'^2+\tfrac{1}{2}\mu'\nu'+\frac{\mu'+\nu'}{r}\right]-\exp(-\nu)[\ddot{\mu}+\tfrac{3}{4}\dot{\mu}^2-\tfrac{1}{2}\dot{\mu}\dot{\nu}]+\Lambda;$$

$$(2.2)\quad 8\pi p=\exp(-\mu)\left[\tfrac{1}{2}\mu''+\tfrac{1}{2}\nu''+\tfrac{1}{4}\nu'^2+\frac{\mu'+\nu'}{2r}\right]-\exp(-\nu)[\ddot{\mu}+\tfrac{3}{4}\dot{\mu}^2-\tfrac{1}{2}\dot{\mu}\dot{\nu}]+\Lambda;$$

$$(2.3)\quad 8\pi\rho=-\exp(-\mu)\left[\mu''+\tfrac{1}{4}\mu'^2+\frac{2\mu'}{r}\right]+\tfrac{3}{4}\dot{\mu}^2\exp(-\nu)-\Lambda;$$

$$(2.4)\qquad 2\dot{\mu}'-\dot{\mu}\nu'=0;$$

where the prime denotes partial differentiation with respect to r and the dot represents partial differentiation with respect to t. Λ is the so-called cosmological constant. If $\dot{\mu}=0$, the resulting equations are identical in form to the equations obtained for a static line element. For this reason, it will be assumed throughout this paper that $\dot{\mu}\neq 0$. With this restriction, (2.4) can be integrated to give

$$(2.5)\qquad \nu=2\log\dot{\mu}+\phi(t)$$

where $\phi(t)$ is an arbitrary function of t. Using (2.5) to eliminate ν, (2.1), (2.2), and (2.3) respectively become:

$$(2.6)\qquad 8\pi p=A\frac{\partial}{\partial t}\left[\exp(\tfrac{1}{2}\mu)\left\{\tfrac{1}{2}\mu'^2+\frac{2\mu'}{r}\right\}-B\right];$$

$$(2.7)\qquad 8\pi p=A\frac{\partial}{\partial t}\left[\exp(\tfrac{1}{2}\mu)\left\{\mu''+\frac{\mu'}{r}\right\}-B\right];$$

$$(2.8)\qquad 8\pi\rho=-\exp(-\mu)\left[\mu''+\tfrac{1}{4}\mu'^2+\frac{2\mu'}{r}\right]+\tfrac{3}{4}\exp(-\phi)-\Lambda,$$

where $A=\exp[-3\mu/2]/\dot{\mu}$, and $B=\exp[3\mu/2][\tfrac{1}{2}\exp(-\phi)-\tfrac{2}{3}\Lambda]$.

The only condition imposed by the field equations is the equality of (2.6) and (2.7), and this implies

$$(2.9)\qquad \exp[\tfrac{1}{2}\mu]\left[\mu''-\tfrac{1}{2}\mu'^2-\frac{\mu'}{r}\right]=\psi(r),$$

where $\psi(r)$ is an arbitrary function of its argument. For every non-static, radially symmetric, isotropic solution, the gravitational potential μ, must be a solution of (2.9), and the gravitational potential ν can be calculated from $\nu=2\log\dot{\mu}+\phi(t)$. Conversely, for every non-static solution of (2.9), $\mu=\mu_0(r,t)$, a solution of the field equations is provided by $\mu=\mu_0(r,t)$ and $\nu=2\log\dot{\mu}_0(r,t)+\phi(t)$. For this reason, the remaining part of our discussion can be confined to the determination of non-static solutions of (2.9). The non-static general solution of (2.9) will have the form $\mu=\mu(r,a,b)$, where a

and b are arbitrary functions of "t". For the moment "t" will be held constant, and (2.9) will be considered to be a total, rather than partial, differential equation. Our interest can then centre on the total differential equation

$$\exp(\tfrac{1}{2}\mu)\left[\frac{d^2\mu}{dr^2}-\frac{1}{2}\left(\frac{d\mu}{dr}\right)^2-\frac{1}{r}\frac{d\mu}{dr}\right]=\psi(r). \tag{2.10}$$

The substitution $x=\frac{1}{2}r^2$, $\mu=-2\log w$ yields

$$\frac{d^2w}{dx^2}=F(x)w^2. \tag{2.11}$$

Differential equations of this type have been investigated in considerable detail by Ince.[1] In particular, Ince was interested in determining conditions that would ensure that the general solution of (2.11) would have fixed critical points (branch points) and essential singularities. His investigation involved substitutions of the form

$$w=\lambda y+\tau, \qquad z=\phi(x), \tag{2.12}$$

where z becomes the independent variable, y the dependent variable, and $\lambda=\lambda(x)$, $\tau=\tau(x)$, $\phi=\phi(x)$ are chosen to satisfy

$$2\lambda'\phi'+\lambda\phi''=0, \qquad F\lambda=6\phi'^2, \qquad \lambda''=2F\tau\lambda, \tag{2.13}$$

where the prime denotes differentiation with respect to x. These conditions and substitutions place (2.11) into the form

$$\frac{d^2y}{dz^2}=6y^2+S(z), \tag{2.14}$$

where

$$S(z)=(F\tau^2-\tau'')/\lambda\phi'^2. \tag{2.15}$$

In order to satisfy the condition for fixed critical points, Ince proved that $S(z)$ must have the form

$$S(z)=pz+q, \tag{2.16}$$

where p and q are constants. If $p=0$, the general solution of (2.14) can be obtained in terms of the Weierstrass elliptic function. If $p\neq 0$, the corresponding solution can be expressed in terms of a Painléve transcendental function.

There is a sense in which the Ince pattern for solution of the problem posed by him is complete. If $F(x)$ of (2.11) is considered to be a known function of x, then equations (2.13) can be solved for λ, ϕ, and τ. The first part of equations

[1] E. L. Ince, "Ordinary Differential Equations", Longmans, Green and Co., Ltd., 1927, pp. 328–330.

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(2.13) can be integrated with respect to x to give

(2.17) $\lambda^2\phi'=1$, (an arbitrary constant of integration does not add to the generality of the ultimate solution).

This combined with the second of the equations gives

$$\lambda=(6/F)^{1/5}. \tag{2.18}$$

Coupling (2.17) and (2.18) yields

$$\phi'=(F/6)^{2/5}, \qquad z=\int(F/6)^{2/5}\,dx. \tag{2.19}$$

Finally the third equation of (2.13) and (2.18) will yield

$$\tau=\tfrac{1}{2}F^{-4/5}\frac{d^2}{dx^2}(F^{-1/5}) \tag{2.20}$$

and λ, ϕ, τ all become known functions of x. For this reason $(F\tau^2-\tau'')/\lambda\phi'^2$ becomes a known function of x. From (2.15), $S(z)$ will become known by inverting (2.19) and determining x as a function of z. If $S(z)$ takes the form (2.16), then the general solution of (2.11) can be determined. This pattern is illustrated in detail for the following differential equation:

$$\frac{d^2w}{dx^2}=6x^{-5n}w^2, \quad n \text{ an arbitrary real number.} \tag{2.21}$$

From (2.18) and (2.19)

$$\lambda=x^n, \qquad \phi'=x^{-2n}, \qquad z=-1/(2n-1)x^{2n-1}, \quad (n\neq\tfrac{1}{2}), \quad z=\log x \quad \text{if} \quad n=\tfrac{1}{2} \tag{2.22}$$

τ is determined by (2.20) to be

$$\tau=n(n-1)x^{5n-2}/12, \tag{2.23}$$

and therefore

$$\begin{aligned}(F\tau^2-\tau'')/\lambda\phi'^2&=-x^{8n-4}n(n-1)(7n-3)(7n-4)/24,\\ &=-n(n-1)(7n-3)(7n-4)/24(2n-1)^4z^4, \qquad n\neq\tfrac{1}{2},\\ &=-\tfrac{1}{384}, \quad \text{if} \quad n=\tfrac{1}{2}.\end{aligned} \tag{2.24}$$

Clearly $S(z)=(F\tau^2-\tau'')/\lambda\phi'^2$ will have the form required in (2.24) providing $n=0$, 1, $\frac{3}{7}$, $\frac{4}{7}$, *or* $\frac{1}{2}$. In each of these cases the general solution of (2.14)

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can be given in terms of the Weierstrass elliptic function. These solutions are

$$n=0: -w = \wp(x+\alpha, 0, \beta)$$

$$n=1: -w = x\wp\left(-\frac{1}{x}+\alpha, 0, \beta\right)$$

$$n=\tfrac{1}{2}: -w = x^{1/2}[\wp(\log x+\alpha, 0, \beta)-\tfrac{1}{48}]$$

$$n=\tfrac{3}{7}: -w = x^{3/7}\wp(7x^{-1/7}+\alpha, 0, \beta)-\tfrac{1}{49}x^{1/7}$$

$$n=\tfrac{4}{7}: -w = x^{4/7}\wp(-7x^{1/7}+\alpha, 0, \beta)-\tfrac{1}{49}x^{6/7}.$$

These are not independent solutions because pairs of them are related by spherical inversions.

For all other values of n, the equation (2.14) takes the form

$$\frac{d^2y}{dz^2}=6y^2+bz^{-4}, \quad b \text{ a constant,} \tag{2.25}$$

a differential equation that does not belong to the class of equations considered by Ince.

From the point of view of the present paper, the Ince pattern of solution is far from complete. The function $F(x)$ is not a known function. Indeed, it is an arbitrary function of its argument. Our problem is the determination of the class of functions to which $F(x)$ must belong in order that $S(z)$, given by (2.15) does in fact have the form specified in (2.16). This is the problem that will be investigated in this paper.

Certain simplifications take place if the Ince substitutions are slightly modified and taken in the form

$$w=\lambda(y+\tau), \qquad z=\phi(x). \tag{2.26}$$

Hence (2.11) takes the form

$$\lambda\phi'^2\left(\frac{d^2y}{dz^2}+\frac{d^2\tau}{dz^2}\right)+(2\lambda'\phi'+\lambda\phi'')\left(\frac{dy}{dz}+\frac{d\tau}{dz}\right)+\lambda''(y+\tau) = F\lambda^2y^2+2F\lambda^2\tau y+F\lambda^2\tau^2. \tag{2.27}$$

In order to obtain a standard form, we follow Ince and choose λ, τ, and ϕ to satisfy:

$$2\lambda'\phi'+\lambda\phi''=0 \tag{2.28}$$

$$F\lambda=6\phi'^2 \tag{2.29}$$

$$\lambda''=2F\lambda^2\tau. \tag{2.30}$$

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Hence (2.27) takes the form

$$\frac{d^2y}{dz^2}=6y^2-\left(\frac{d^2\tau}{dz^2}+\frac{\lambda''\tau-F\lambda^2\tau^2}{\lambda\phi'^2}\right). \tag{2.31}$$

Equation (2.28) can be integrated with respect to x to give

$$\lambda^2\phi'=1. \tag{2.32}$$

Therefore,

$$F=6\lambda^{-5}. \tag{2.33}$$

Coupling these results with (2.30) gives

$$\frac{\lambda''\tau-F\lambda^2\tau^2}{\lambda\phi'^2}=6\tau^2, \tag{2.34}$$

and (2.31) becomes

$$\frac{d^2y}{dz^2}=6y^2-\left(\frac{d^2\tau}{dz^2}+6\tau^2\right). \tag{2.35}$$

This implies that the standard form used by Ince can only be obtained if τ is chosen to be a solution of

$$\frac{d^2\tau}{dz^2}=-6\tau^2-pz-q. \tag{2.36}$$

In this instance, (2.35) becomes

$$\frac{d^2y}{dz^2}=6y^2+pz+q. \tag{2.37}$$

Depending on the values of p and q, three canonical forms can be obtained. If $p=q=0$, then

$$\frac{d^2y}{dz^2}=6y^2, \tag{2.38}$$

an equation whose general solution can be expressed in terms of the Weierstrass elliptic function $\wp(z, g_2, g_3)$. Explicitly,

$$y=\wp(z+a, 0, b), \quad a, b \text{ arbitrary constants.} \tag{2.39}$$

If $p=0$, $q\neq 0$, then (2.35) can be placed into the form

$$\frac{d^2y}{dz^2}=6y^2+\tfrac{1}{2}, \tag{2.40}$$

an equation whose general solution is

$$y=\wp(z+a, -1, b). \tag{2.41}$$

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Finally if $p \neq 0$, then (2.35) can be placed in the canonical form

$$\frac{d^2y}{dz^2} = 6y^2 + z, \tag{2.42}$$

an equation whose general solution is

$$y = P(z, a, b), \tag{2.43}$$

where $P(z, a, b)$ is a Painléve transcendental function.

When (2.36) is placed in similar canonical forms, $\tau = -y$ will provide the general solution for (2.36). It is essentially at this stage that Ince leaves the problem, making no attempt to solve equations (2.28), (2.29), and (2.30), given that τ is a solution of (2.36). The solution of these three equations will now be obtained.

It has already been shown that

$$\lambda^2 \phi' = \lambda^2 \frac{dz}{dx} = 1, \tag{2.44}$$

and

$$F = 6\lambda^{-5}. \tag{2.45}$$

Therefore (2.30) becomes

$$\frac{d^2\lambda}{dx^2} = 12\lambda^{-3}\tau. \tag{2.46}$$

From $d\lambda/dx = (d\lambda/dz)(dz/dx) = \lambda^{-2}(d\lambda/dz)$, one readily obtains

$$\frac{d^2\lambda}{dx^2} = \lambda^{-4}\frac{d^2\lambda}{dz^2} - 2\lambda^{-5}\left(\frac{d\lambda}{dz}\right)^2 = 12\lambda^{-3}\tau, \tag{2.47}$$

and therefore

$$\frac{d^2}{dz^2}\left(\frac{1}{\lambda}\right) = -12\tau\left(\frac{1}{\lambda}\right). \tag{2.48}$$

The equation (2.48) is a linear differential equation in $(1/\lambda)$ where τ is a known function of z. In order to find the general solution of (2.48), we return to (2.36),

$$\frac{d^2\tau}{dz^2} = -6\tau^2 - pz - q, \tag{2.49}$$

an equation whose general solution will be denoted by

(2.50) $\tau = \tau(z, \alpha, \beta)$, α, β arbitrary, independent, parameters.

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τ is expressible in terms of the transcendental functions mentioned above. By differentiating (2.49) partially with respect to α one obtains:

$$\frac{d^2}{dz^2}\left(\frac{\partial\tau}{\partial\alpha}\right) = -12\tau\left(\frac{\partial\tau}{\partial\alpha}\right). \tag{2.51}$$

Hence $\lambda^{-1} = \partial\tau/\partial\alpha$ is a particular solution of (2.51). Similarly $\lambda^{-1} = \partial\tau/\partial\beta$ is a second such solution. Neither $\partial\tau/\partial\alpha$ nor $\partial\tau/\partial\beta$ can vanish identically. When they are considered as functions of z, they will be linearly dependent if and only if a function $\rho(\alpha, \beta)$ exists for which

$$\frac{\partial\tau}{\partial\alpha} = \rho(\alpha, \beta)\frac{\partial\tau}{\partial\beta}. \tag{2.52}$$

If some solution $\tau = \tau_0(\alpha, \beta)$ exists for (2.52), then the general solution of (2.52) is

$$\tau = f(\tau_0(\alpha, \beta), z), \tag{2.53}$$

where f is an arbitrary function of its arguments. However the general solution of (2.49) cannot be placed into the form of (2.52) because α, β were assumed to be independent parameters. Hence, $\partial\tau/\partial\alpha$ and $\partial\tau/\partial\beta$ are linearly independent solutions of (2.48), and its general solution is

$$\lambda = 1\bigg/\left(A\frac{\partial\tau}{\partial\alpha} + B\frac{\partial\tau}{\partial\beta}\right), \tag{2.54}$$

where A and B can be arbitrary functions of α, β.

The last equation requiring solution is (2.32), which can be written

$$x = \int \lambda^2\,dz = \int \frac{dz}{\left(A\frac{\partial\tau}{\partial\alpha} + B\frac{\partial\tau}{\partial\beta}\right)^2}, \tag{2.55}$$

an integral that can be evaluated in the following way.

From (2.51) and its companion equation for $\partial\tau/\partial\beta$, one has

$$\frac{d^2}{dz^2}\left(\frac{\partial\tau}{\partial\alpha}\right) = -12\tau\frac{\partial\tau}{\partial\alpha} \tag{2.56}$$

$$\frac{d^2}{dz^2}\left(\frac{\partial\tau}{\partial\beta}\right) = -12\tau\frac{\partial\tau}{\partial\beta}. \tag{2.57}$$

Multiplying (2.56) by $\partial\tau/\partial\beta$ and (2.57) by $\partial\tau/\partial\alpha$, and subtracting these results gives

$$\frac{\partial\tau}{\partial\beta}\frac{d^2}{dz^2}\left(\frac{\partial\tau}{\partial\alpha}\right) - \frac{\partial\tau}{\partial\alpha}\frac{d^2}{dz^2}\left(\frac{\partial\tau}{\partial\beta}\right) = 0 \tag{2.58}$$

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or

$$\frac{\partial\tau}{\partial\beta}\frac{d}{dz}\left(\frac{\partial\tau}{\partial\alpha}\right)-\frac{\partial\tau}{\partial\alpha}\frac{d}{dz}\left(\frac{\partial\tau}{\partial\beta}\right)=\psi(\alpha,\beta), \tag{2.59}$$

where $\psi(\alpha, \beta)$ is an arbitrary function of its arguments. If $\psi(\alpha, \beta)=0$, then (2.59) can be written

$$\frac{d}{dz}\left(\frac{\partial\tau}{\partial\alpha}\Big/\frac{\partial\tau}{\partial\beta}\right)=0, \tag{2.60}$$

or

$$\frac{\partial\tau}{\partial\alpha}=c(\alpha,\beta)\frac{\partial\tau}{\partial\beta}. \tag{2.61}$$

Therefore $\partial\tau/\partial\alpha$ and $\partial\tau/\partial\beta$ would not be linearly independent functions of z. Hence $\psi(\alpha, \beta)\not\equiv 0$. When $\psi(\alpha, \beta)$ does not vanish identically, there is no loss in generality in taking $\psi(\alpha, \beta)\equiv 1$. The choice of two new arbitrary parameters

$$\alpha'=\int \psi(\alpha,\beta)\,d\alpha, \qquad \beta'=\beta, \tag{2.62}$$

will imply $\psi(\alpha', \beta')\equiv 1$. Hence (2.59) becomes,

$$\frac{\partial\tau}{\partial\beta}\frac{d}{dz}\left(\frac{\partial\tau}{\partial\alpha}\right)-\frac{\partial\tau}{\partial\alpha}\frac{d}{dz}\left(\frac{\partial\tau}{\partial\beta}\right)=1, \tag{2.63}$$

a result that can be written as

$$\frac{d}{dz}\left(\frac{\partial\tau}{\partial\alpha}\Big/\frac{\partial\tau}{\partial\beta}\right)=1\Big/\left(\frac{\partial\tau}{\partial\beta}\right)^2 \tag{2.64}$$

or

$$\frac{d}{dz}\left(\frac{\partial\tau}{\partial\beta}\Big/\frac{\partial\tau}{\partial\alpha}\right)=-1\Big/\left(\frac{\partial\tau}{\partial\alpha}\right)^2. \tag{2.65}$$

Returning to (2.55)

$$x=\int\frac{dz}{\left(A\dfrac{\partial\tau}{\partial\alpha}+B\dfrac{\partial\tau}{\partial\beta}\right)^2}=\int\frac{dz}{\left(\dfrac{\partial\tau}{\partial\alpha}\right)^2\left\{A+B\left(\dfrac{\partial\tau}{\partial\beta}\Big/\dfrac{\partial\tau}{\partial\alpha}\right)\right\}^2}. \tag{2.66}$$

By using (2.65), (2.66) becomes

$$x=-\int\frac{d\left(\dfrac{\partial\tau}{\partial\beta}\Big/\dfrac{\partial\tau}{\partial\alpha}\right)}{\left\{A+B\left(\dfrac{\partial\tau}{\partial\beta}\Big/\dfrac{\partial\tau}{\partial\alpha}\right)\right\}^2}=\frac{\dfrac{\partial\tau}{\partial\alpha}}{B\left(A\dfrac{\partial\tau}{\partial\alpha}+B\dfrac{\partial\tau}{\partial\beta}\right)}+C \tag{2.67}$$

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where C is also an arbitrary function of α, β.

All the required equations have now been solved. The process will yield a solution of

$$\frac{d^2w}{dz^2}=F(x)w^2 \tag{2.68}$$

if and only if

$$F=6\lambda^{-5}=6\left(A\frac{\partial\tau}{\partial\alpha}+B\frac{\partial\tau}{\partial\beta}\right)^5. \tag{2.69}$$

When this is so, one can write the general solution of (2.11), in parametric form, in terms of two transcendental functions, and their partial derivatives. To illustrate, the first case involving the Weierstrass elliptic function will be used. For $\wp=\wp(z, g_2, g_3)$ it is known that

$$\Delta\frac{\partial\wp}{\partial g_3}=\wp'(3g_2\zeta-\tfrac{9}{2}g_3z)+6g_2\wp^2-9g_3\wp-g_2^2 \tag{2.70}$$

and

$$\Delta\frac{\partial\wp}{\partial g_2}=\wp'\left(-\tfrac{9}{2}g_3\zeta+\frac{g_2^2z}{4}\right)-9g_3\wp^2+\tfrac{1}{2}g_2^2\wp+\tfrac{3}{2}g_2g_3, \tag{2.71}$$

where $\Delta=g_2^3-27g_3^2$, and ζ is the Weierstrass zeta function, given by $\zeta'(z)=-\wp(z)$.

If the simplest case,

$$\frac{d^2y}{dz^2}=6y^2, \tag{2.72}$$

is considered, then the general solution of (2.72) is

(2.73) $\quad y=\wp(z+a, 0, b)$, where a and b are arbitrary constants.

By (2.36), the corresponding equation for τ is

$$\frac{d^2\tau}{dz^2}=-6\tau^2. \tag{2.74}$$

Although it is natural to take the corresponding general solution of (2.74) to be

$$\tau=-\wp(z+\alpha, 0, \beta), \tag{2.75}$$

it must be remembered that the solution for τ was normalized to satisfy (2.63). For (2.75), (2.70) gives

$$\begin{aligned}\frac{\partial\tau}{\partial\beta}=-\frac{\partial\wp}{\partial\beta}&=-\{\wp'(-\tfrac{9}{2}\beta(z+\alpha))-9\beta\wp\}/-27\beta^2\\&=-\{(z+\alpha)\wp'+2\wp\}/6\beta.\end{aligned} \tag{2.76}$$

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Obviously

$$\frac{\partial \tau}{\partial \alpha} = -\wp'. \tag{2.77}$$

This gives

$$\frac{\partial \tau}{\partial \beta}\frac{d}{dz}\left(\frac{\partial \tau}{\partial \alpha}\right) - \frac{\partial \tau}{\partial \alpha}\frac{d}{dz}\left(\frac{\partial \tau}{\partial \beta}\right) = -\tfrac{1}{2}. \tag{2.78}$$

Hence the normalization can be obtained by

$$\alpha' = -\int \tfrac{1}{2}\, d\alpha = -\tfrac{1}{2}\alpha, \qquad \beta' = \beta. \tag{2.79}$$

Thus τ must be taken to have the form

$$\tau = -\wp(z-2\alpha, 0, \beta). \tag{2.80}$$

Hence

$$\frac{\partial \tau}{\partial \alpha} = 2\wp', \qquad \frac{\partial \tau}{\partial \beta} = -\{(z-2\alpha)\wp' + 2\wp\}/6\beta. \tag{2.81}$$

From (2.67)

$$x = \frac{2\wp'}{B\left\{2A\wp' - \dfrac{B}{6\beta}((z-2\alpha)\wp' + 2\wp)\right\}} + C. \tag{2.82}$$

Further

$$\lambda = 1\bigg/\left\{2A\wp' - \frac{B}{6\beta}((z-2\alpha)\wp' + 2\wp)\right\} \tag{2.83}$$

and

$$F = 6\left\{2A\wp' - \frac{B}{6\beta}((z-2\alpha)\wp' + 2\wp)\right\}^5. \tag{2.84}$$

Under these conditions, the general solution of $d^2w/dx^2 = F(x)w^2$ becomes, in parametric form,

$$w = \frac{\wp(z+a, 0, b) - \wp(z-\alpha, 0, \beta)}{2A\wp'(z-2\alpha, 0, \beta) - \dfrac{B}{6\beta}((z-2\alpha)\wp'(z-2\alpha, 0, \beta) + 2\wp(z-2\alpha, 0, \beta))} \tag{2.85}$$

$$x = \frac{2\wp'(z-2\alpha, 0, \beta)}{B\left\{2A\wp'(z-2\alpha, 0, \beta) - \dfrac{B}{6\beta}((z-2\alpha)\wp'(z-2\alpha, 0, \beta) + 2\wp(z-2\alpha, 0, \beta))\right\}} + C.$$

(2.86)

If we return for the moment to the original field variables r, t, then $x = r^2/2$ is independent of t. Hence α, β, A, B, C may also be taken as independent of t. The dependence on t will only emerge in the expression for w, through the parameters a and b. This solution does not actually have the generality it seems to have. It is possible, without loss of generality, to require $z=2\alpha$ to correspond to $x=0$. This would imply $A=0$. The original line element is invariant to a change in scale $r=c\bar{r}$, ($c\neq 0$ an arbitrary constant), $\mu=\bar{\mu}-2\log c$. Hence our solution loses no generality if x is replaced by $c^2\bar{x}$ and w is replaced by $\bar{w}/c$. For this reason it is possible to choose $B=1$. The variable z is of course a parameter. Hence a linear transformation of the form, $z=2\alpha+\beta^{1/6}\bar{z}$, $\beta\neq 0$, will not restrict the generality of the solution that was obtained. Using such transformations, and the known homogeneity properties of the functions involved places (2.85) and (2.86) into the form

$$w=\frac{6\beta\{\wp(z+a,0,b)-\wp(z,0,1)\}}{z\wp'(z,0,1)+2\wp(z,0,1)} \tag{2.87}$$

$$x=\frac{-12\beta^{5/6}\wp'(z,0,1)}{z\wp'(z,0,1)+2\wp(z,0,1)}+C \tag{2.88}$$

Since $\mu=-2\log w$, $x=r^2/2$, $\nu=2\log\dot{\mu}+\phi(t)$, the corresponding solution of the field equations is, in parametric form, given by

$$e^{\mu}=\frac{\{z\wp'(z,0,1)+2\wp(z,0,1)\}^2}{36\beta^2\{\wp(z+a,0,b)-\wp(z,0,1)\}^2} \tag{2.89}$$

$$e^{\nu}=\frac{4\exp(\phi(t))\left\{\dot{a}\wp'(z+a,0,b)+\dot{b}\dfrac{\partial\wp}{\partial b}(z+a,0,b)\right\}^2}{\{\wp(z+a,0,b)-\wp(z,0,1)\}^2} \tag{2.90}$$

$$r^2=\frac{-24\beta^{5/6}\wp'(z,0,1)}{z\wp'(z,0,1)+2\wp(z,0,1)}+C. \tag{2.91}$$

It is of course possible to use the Burman–Lagrange inversion process to express z as a function of r. Such an expression is so complicated that it seems hardly worth the while to obtain it. Although it is possible to use a limiting process in our solution to see what happens when $\beta=0$, it is probably easier to retrace all the steps to find the elementary solutions corresponding to this case. The equation for τ was

$$\frac{d^2\tau}{dz^2}=-6\tau^2 \tag{2.92}$$

which can be integrated to give

$$\left(\frac{d\tau}{dz}\right)^2=-4\tau^3-\beta. \tag{2.93}$$

When $\beta=0$, (2.93) has two solutions

$$\tau=0 \quad \text{and} \quad \tau=-\frac{1}{(z+\alpha)^2}. \tag{2.94}$$

As before, z is a parameter so α can be chosen to be zero without affecting the generality of our solution. The equation determining λ is

$$\frac{d^2}{dz^2}\left(\frac{1}{\lambda}\right)=-\frac{12\tau}{\lambda}. \tag{2.95}$$

Corresponding to $\tau=0$, λ has the solution

$$\lambda=1/(Az+B) \tag{2.96}$$

and corresponding to $\tau=-1/z^2$, λ has the solution

$$\lambda=z^3/(Az^7+B). \tag{2.97}$$

Similarly

$$x=\int\lambda^2\,dz \tag{2.98}$$

yields

$$x=-\frac{1}{A(Az+B)}+C \quad (\text{when } \tau=0) \tag{2.99}$$

and

$$x=\frac{1}{7A(Az^7+B)}+C \quad (\text{when } \tau=-1/z^2). \tag{2.100}$$

As before, it is possible to choose $B=0$ and $A=1$. This gives

$$\lambda=z^{-1} \quad (\text{when } \tau=0), \qquad \lambda=z^{-4} \quad (\text{when } \tau=-1/z^2) \tag{2.101}$$

$$x=-z^{-1}+C \quad (\tau=0), \qquad x=-\frac{z^{-7}}{7}+C, \quad (\tau=-1/z^2) \tag{2.102}$$

$$\begin{aligned} F(x)&=6z^5 \quad (\tau=0), \qquad F(x)=6z^{20} \quad (\tau=-1/z^2)\\ &=6(c-x)^{-5} \quad (\tau=0), \qquad F(x)=6\{7(c-x)\}^{-20/7}, \quad (\tau=-1/z^2) \end{aligned} \tag{2.103}$$

and these yield the solutions

$$w=z^{-1}\wp(z+a,0,b), \qquad z=(c-x)^{-1} \tag{2.104}$$

and

$$w=z^{-4}[\wp(z+a,0,b)-z^{-2}], \qquad z=\{7(c-x)\}^{-1/7}. \tag{2.105}$$

For $\tau=0$ the corresponding solution of the field equations are,

$$\begin{aligned} e^{\mu}&=z^2\{\wp(z+a,0,b)\}^{-2}, \qquad z=1/(c-\tfrac{1}{2}r^2),\\ e^{\nu}&=4\exp(\phi(t))\{\wp_t(z+a,0,b)/\wp(z+a,0,b)\}^2, \end{aligned} \tag{2.106}$$

where

$$\wp_t(z+a,0,b)=\frac{\partial\wp}{\partial t}=\dot{a}\wp'+\dot{b}\frac{\partial\wp}{\partial b}, \tag{2.107}$$

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$\partial\wp/\partial b$ can be expressed in terms of z, $\wp$, and $\wp'$ is desired. For $\tau=-z^{-2}$:

$$(2.108)\qquad \begin{aligned} e^{\mu} &= z^8\{\wp(z+a,0,b)-z^{-2}\}^{-2}, \qquad z=\{7(c-\tfrac{1}{2}r^2)\}^{-1/7} \\ e^{\nu} &= 4\exp(\phi(t))\{\wp_t/\wp(z+a,0,b)-z^{-2}\}^2, \end{aligned}$$

where

$$(2.109)\qquad \wp_t=\frac{\partial}{\partial t}\wp(z+a,0,b)=\dot{a}\wp'+\dot{b}\frac{\partial\wp}{\partial b}.$$

These are of course instances where explicit solutions of the field equations have been found. Solutions expressible in terms of elementary functions are easily obtainable from the known conditions under which the Weierstrass elliptic functions degenerate into an elementary function.

The case corresponding to the second canonical form involves the equation

$$(2.110)\qquad \frac{d^2\tau}{dz^2}=-6\tau^2-\tfrac{1}{2}.$$

This case can be treated in a similar manner, and the general solutions of the field equations can be obtained. For this case, the Weierstrass zeta function will enter the solutions for e^{μ} and e^{ν}.

The third case, is one whose solution involves a Painléve transcendental function. I know of no corresponding expressions for $\partial p/\partial\alpha$ and $\partial p/\partial\beta$ in terms of a combination of known functions and derivatives or integrals with respect to z of the Painléve function. Although the general solution of the field equations can be expressed in terms of a Painléve transcendental function $p(z, a, b)$, and its partial derivatives $\partial p/\partial z$, $\partial p/\partial a$, and $\partial p/\partial b$, I have been unable to carry the analysis beyond this point.

3. **Conclusion.** It has been possible to find, in parametric form, all functions $F(z)$ for which

$$(3.1)\qquad \frac{d^2w}{dx^2}=F(x)w^2$$

has solutions with fixed critical points. Although the corresponding solutions to (3.1) can be reduced to three canonical forms, there can be generated many particular solutions by allowing the transcendental functions to degenerate into elementary functions. One can, therefore, obtain a significant number of rigorous solutions to the field equations, solutions that have been found for the first time. Nevertheless, more can be accomplished by following the pattern used to find these solutions.

Although equation (2.25)

$$(3.2)\qquad \frac{d^2y}{dz^2}=6y^2+bz^{-4},$$

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was obtained by following the Ince pattern, the equation was discarded because it violated the canonical forms required for fixed critical points. This equation, however, can be transformed into an Abelian differential equation of the second kind. Solutions to this differential equation will also represent a spherically symmetric distribution of matter. Unfortunately few of the properties of the solutions of Abelian differential equations are known.

It would be of interest to study the distributions of pressure and density that the new solutions will yield. Some may be of physical interest.

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