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ARITHMETIC MEAN OF VALUES AND VALUE AT MEAN OF ARGUMENTS FOR CONVEX FUNCTIONS

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Abstract

We give here some extensions of inequalities of Popoviciu and Rado. The idea is to use an inequality [C. P. Niculescu and L. E. Persson, *Convex functions. Basic theory and applications* (Universitaria Press, Craiova, 2003), Page 4] which gives an approximation of the arithmetic mean of n values of a given convex function in terms of the value at the arithmetic mean of the arguments. We also give more general forms of this inequality by replacing the arithmetic mean with others. Finally we use these inequalities to establish similar inequalities of Popoviciu and Rado type.

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1. Introduction

For $x_1, \dots, x_k > 0$, $k \geq 1$ we put

$$A_k = \frac{x_1 + \cdots + x_k}{k}, \quad G_k(x_1 \dots x_k)^{1/k}.$$

Inequalities involving the arithmetic mean and the geometric mean are of great interest. We discuss here the inequality of Popoviciu

$$\left(\frac{A_n}{G_n}\right)^n \geq \left(\frac{A_{n-1}}{G_{n-1}}\right)^{n-1} \geq \cdots \geq \left(\frac{A_1}{G_1}\right)^1 = 1 \tag{1.1}$$

and the inequality of Rado

$$n(A_n - G_n) \geq (n-1)(A_{n-1} - G_{n-1}) \geq \cdots \geq 1 \cdot (A_1 - G_1) = 0. \tag{1.2}$$

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For proofs and other consequences, see [1]. We prove that the inequality (1.1) remains true if we replace A_k with

$$A_k^p = \left(\frac{x_1^p + \cdots + x_k^p}{k}\right)^{1/p}, \quad p \geq 1.$$

Moreover, we prove that (1.2) holds also if we replace the geometric means G_k with other means.

One way to attack these inequalities is to use a nice inequality related to convex functions (for example [2, Page 4]) denoted here by (1.3).

Let $I \subseteq \mathbb{R}$ be an interval. We say that a function $f : I \to \mathbb{R}$ is convex if for all $x, y \in I$ and for each $\alpha \in [0, 1]$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y).$$

The function f is called concave if the above inequality holds with reverse sense.

If f is convex, then for

$$\alpha = \frac{n-1}{n}, \quad x = \frac{x_1 + \cdots + x_{n-1}}{n-1}, \quad y = x_n$$

we deduce that

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{1}{n} f(x_n) + \frac{n-1}{n} f\left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right).$$

Therefore

$$n\left[\frac{f(x_1) + \cdots + f(x_n)}{n} - f\left(\frac{x_1 + \cdots + x_n}{n}\right)\right] \geq (n-1)\left[\frac{f(x_1) + \cdots + f(x_{n-1})}{n-1} - f\left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right)\right]. \tag{1.3}$$

In other words, the sequence

$$a_n = n\left[\frac{f(x_1) + \cdots + f(x_n)}{n} - f\left(\frac{x_1 + \cdots + x_n}{n}\right)\right], \quad n \geq 1 \tag{1.4}$$

is monotonically increasing, where $(x_n)_{n\geq 1} \subset I$ is arbitrarily given.

The inequalities (1.1)–(1.2) follow now by applying (1.3) to $f(x) = -\ln x$ and $f(x) = e^x$ respectively.

2. The results

We give some general results concerning the sequence (1.4), then we study some particular cases to obtain results of the form (1.1) and (1.2). We use the following well-known result.

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LEMMA 2.1. *Let $I, J \subseteq \mathbb{R}$ be intervals. Assume that $\phi : I \to J$ is convex and $g : J \to \mathbb{R}$ is convex and monotonically increasing. Then $g \circ \phi : I \to \mathbb{R}$ is convex.*

Then we have the following result.

THEOREM 2.2. *Let $g : I \subseteq (0, \infty) \to \mathbb{R}$ be convex and increasing and let $(x_n)_{n\geq 1} \subseteq (0, \infty)$. Then for every $p \in (-\infty, 1] \setminus \{0\}$, the sequence*

$$a_n = n\left[\frac{g(x_1) + \cdots + g(x_n)}{n} - g\left(\left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1/p}\right)\right], \quad n \geq 1$$

is increasing.

PROOF. According to Lemma 2.1, the function $f(x) = g(x^{1/p})$, $x > 0$ is convex, because $\phi(x) = x^{1/p}$ is convex for $p \in (-\infty, 1] \setminus \{0\}$. Thus (1.4) becomes

$$a_n = n\left[\frac{g(x_1^{1/p}) + \cdots + g(x_n^{1/p})}{n} - g\left(\left(\frac{x_1 + \cdots + x_n}{n}\right)^{1/p}\right)\right]$$

and the conclusion follows by replacing x_k with x_k^p. □

An interesting case is $g(x) = x$ and $p = -1$ in Theorem 2.2. The respective sequence becomes

$$a_n = n\left[\frac{x_1 + \cdots + x_n}{n} - \left(\frac{x_1^{-1} + \cdots + x_n^{-1}}{n}\right)^{-1}\right].$$

Now, using the monotony of this sequence, we obtain the inequality

$$n(A_n - H_n) \geq (n-1)(A_{n-1} - H_{n-1}) \geq \cdots \geq 1 \cdot (A_1 - H_1) = 0,$$

where A_k and H_k are the arithmetic and harmonic means respectively:

$$A_k = \frac{x_1 + \cdots + x_k}{k}, \quad H_k = \frac{k}{1/x_1 + \cdots + 1/x_k}, \quad 1 \leq k \leq n.$$

For $p \in (-\infty, 1] \setminus \{0\}$ and $y_1, \ldots, y_k > 1$, consider the mean,

$$W_k = \exp\left[\left(\frac{\ln^p y_1 + \cdots + \ln^p y_k}{k}\right)^{1/p}\right],$$

to obtain the following result.

COROLLARY 2.3. *For any sequence $(y_n)_{n\geq 1} \subset (0, \infty)$,*

$$n(A_n - W_n) \geq (n-1)(A_{n-1} - W_{n-1}) \geq \cdots \geq 1 \cdot (A_1 - W_1) = 0. \tag{2.1}$$

PROOF. We apply Theorem 2.2 to $g(x) = e^x$. Thus the sequence

$$a_n = n\left[\frac{e^{x_1} + \cdots + e^{x_n}}{n} - \exp\left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1/p}\right], \quad n \geq 1$$

is increasing. The result follows by writing $e^{x_k} = y_k$, $1 \leq k \leq n$. □

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For $y_1, \ldots, y_k > 1$ write

$$V_k = \exp[(\ln y_1 \cdot \cdots \cdot \ln y_k)^{1/k}]$$

to give the following result.

COROLLARY 2.4. *For any sequence* $(y_n)_{n\geq 1} \subset (1, \infty)$,

$$n(A_n - V_n) \geq (n-1)(A_{n-1} - V_{n-1}) \geq \cdots \geq 1 \cdot (A_1 - V_1) = 0.$$

PROOF. The result follows from (2.1) by taking the limit as $p \to 0$. Indeed, it is known that

$$\lim_{p\to 0} \left(\frac{\alpha_1^p + \cdots + \alpha_n^p}{n} \right)^{1/p} = (\alpha_1 \cdots \alpha_n)^{1/n}$$

so $\lim_{p\to 0} W_k = V_k$, for all $1 \leq k \leq n$. □

Further we use the following well-known result.

LEMMA 2.5. *Let* $I, J \subseteq \mathbb{R}$ *be intervals. Assume that* $\phi : I \to J$ *is concave and* $g : J \to \mathbb{R}$ *is convex and monotonically decreasing. Then* $g \circ \phi : I \to \mathbb{R}$ *is convex.*

Now by applying (1.4) for the function $g \circ \phi$, we obtain the following result.

THEOREM 2.6. *Let* $I, J \subseteq \mathbb{R}$ *be intervals and assume that* $\phi : I \to J$ *is concave and* $g : J \to \mathbb{R}$ *is convex and monotonically decreasing. Then the sequence*

$$a_n = n\left[\frac{g(\phi(x_1)) + \cdots + g(\phi(x_n))}{n} - g\left(\phi\left(\frac{x_1 + \cdots + x_n}{n}\right)\right)\right], \quad n \geq 1 \tag{2.2}$$

is increasing.

First let us put $g(x) = -\ln x$ in (2.2), assuming that $\phi > 0$. Hence

$$\begin{aligned} a_n &= n\left[\ln \phi\left(\frac{x_1 + \cdots + x_n}{n}\right) - \frac{\ln \phi(x_1) + \cdots + \ln \phi(x_n)}{n} \right] \\ &= n \ln \frac{\phi(x_1 + \cdots + x_n)/n}{\sqrt[n]{\phi(x_1) \cdot \cdots \cdot \phi(x_n)}}. \end{aligned}$$

Thus we proved the following result.

COROLLARY 2.7. *Let* $I \subseteq \mathbb{R}$ *be an interval and assume that* $\phi : I \to (0, \infty)$ *is concave. Then the sequence*

$$a_n = \left[\frac{\phi(x_1 + \cdots + x_n)/n}{\sqrt[n]{\phi(x_1) \cdot \cdots \cdot \phi(x_n)}} \right]^n, \quad n \geq 1 \tag{2.3}$$

is increasing.

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For $p \in \mathbb{R} \setminus \{0\}$ and $x_1, \ldots, x_k > 0$ we put

$$A_k^p = \left(\frac{x_1^p + \cdots + x_k^p}{k}\right)^{1/p}.$$

COROLLARY 2.8. *Let $p \geq 1$ and $x_1, \ldots, x_n > 0$. Then*

$$\left(\frac{A_n^p}{G_n}\right)^n \geq \left(\frac{A_{n-1}^p}{G_{n-1}}\right)^{n-1} \geq \cdots \geq \left(\frac{A_1^p}{G_1}\right)^1 = 1. \tag{2.4}$$

PROOF. Let us take in (2.3) $\phi(x) = x^{1/p}$, with $p \geq 1$. Obviously, ϕ is concave and (2.3) becomes

$$a_n = \left[\frac{((x_1 + \cdots + x_n)/n)^{1/p}}{\sqrt[n]{x_1^{1/p} \cdots x_n^{1/p}}}\right]^n.$$

By replacing x_k with x_k^p, we obtain that the sequence

$$a_n = \left[\frac{((x_1^p + \cdots + x_n^p)/n)^{1/p}}{\sqrt[n]{x_1 \cdot \cdots \cdot x_n}}\right]^n = \left(\frac{A_n^p}{G_n}\right)^n, \quad n \geq 1$$

is increasing. Hence (2.4) is true. □

Finally, note that T. Popoviciu's inequality is the particular case with $p = 1$ of (2.4).

References

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