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A DETERMINANT FOR RECTANGULAR MATRICES

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The familiar notion of the determinant is generalised to include rectangular matrices. An expression for a normalised generalised inverse of a matrix is given in terms of its determinant and a possible generalisation of the Schur complement is discussed as a simple application.

1. Introduction

The well known concept of the determinant has been defined to cover square matrices alone. This concept is very intimately related to the concept of inversion of square nonsingular matrices in the sense that square matrices which have nonvanishing determinant can be inverted. In extending the idea of inversion to cover rectangular matrices [6], the need to generalise the concept of the determinant was not felt as a number of methods devised to compute the inverse did not use any determinants whatsoever. It is felt by the author that many of the identities from Linear Algebra can be generalised and be used in, say, the theory of permanants and its evaluation [4]. Certain results in respect of this will be communicated for publication elsewhere.

It is shown [6] that for an $m \times n$ real matrix A, there exists a unique $n \times m$ matrix $X = A^+$ which satisfies the following conditions:

(1.1) $$AXA = A ,$$

(1.2) $$XAX = X ,$$

(1.3) $$(AX)^T = AX ,$$

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(1.4) $$(XA)^T = XA .$$

The matrix A^+ is known as the Moore-Penrose generalised inverse of A. For some purposes, however, matrices which satisfy fewer than all of the above conditions are of interest. A matrix which satisfies the condition (1.1) is called a generalised inverse $A^{(g)}$ of A. A matrix satisfying conditions (1.1)-(1.2) is called a reflexive generalised inverse $A^{(r)}$ of A. A matrix which satisfies the conditions (1.1)-(1.2) and (1.3)/(1.4) is known as a right/left normalised generalised inverse $A^{(n)}$ of A. Each class of inverses mentioned above is nonempty; in fact

$$A^+ \in \left\{A^{(n)}\right\} \subset \left\{A^{(r)}\right\} \subset \left\{A^{(g)}\right\} .$$

The Moore-Penrose generalised inverse A^+ of A is primarily important because of its uniqueness [6]. The other types of inverses are, however, required in many applications [1, 7, 8, 9].

2. A determinant $|A|$

Let J_n be the set of integers $\{1, 2, \ldots, n\}$. Let the integers $m, K_{p1}, \ldots, K_{pm}$ be such that

(i) $m \le n$,

(ii) $K_{pi} \in J_n$ for all $i \in J_m$ and $p = 1, 2, \ldots$,

(iii) $K_{p1} < K_{p2} < \ldots < K_{pm}$.

For an integer d, $1 \le d \le (n-m+1)$, define a set S_d such that

$$S_d = \left\{e_p^d = \left(d, K_{p2}, \ldots, K_{pm}\right)\right\} .$$

If $N_d = {}^{n-d}C_{m-1}$, then the cardinal number of S_d is N_d. The sets S_d , $1 \le d \le n-m+1$, will be ordered as follows. A set $S_u < S_v$ whenever $u < v$. Moreover, the elements e_p^d and e_q^d will be placed in the order $e_p^d < e_q^d$ whenever $K_{ps} < K_{qs}$ for all $s = 2, 3, \ldots, m$. All the m-tuples, therefore, admit of the following order; namely,

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$$e_1^1 < \ldots < e_{N_1}^1 < e_1^2 < \ldots < e_{N_2}^2 < \ldots < e_1^{n-m+1} < \ldots < e_{N_{n-m+1}}^{n-m+1} .$$

Consider a real matrix $A = (a_{ij})$ of order $m \times n$, $m \le n$. Let A_p^d be a submatrix of order $m \times m$ of A whose columns conform to the ordering of integers in e_p^d, $1 \le d \le (n-m+1)$, $1 \le p \le N_d$.

DEFINITION 2.1. For an $m \times n$, $m \le n$, matrix A with real elements, let A_p^d be defined as above, then the number

$$\sum_{d=1}^{n-m+1} \sum_{p=1}^{N_d} \det A_p^d \tag{2.1}$$

will be defined as the determinant of A, and will be denoted by $|A|$.

DEFINITION 2.2. For an $m \times n$, $m \ge n$, matrix A with real elements, $|A|$ will be defined as $|A^T|$, where A^T denotes the transposed matrix of A.

LEMMA. (1) *If* $m = 1$, *then* $|A| = a_{11} + a_{12} + \ldots + a_{1n}$.

(2) *If* $m = n$, *then* $|A| = \det A = \det A_1^1$.

(3) *If any row of* A *is multiplied by* c, *then* $|A|$ *is multiplied by* c.

(4) *If any two rows of* A *are identical, then* $|A| = 0$.

(5) *If any row of* A *is a linear combination of the remaining rows of* A, *then* $|A| = 0$.

(6) *If any two rows of* A *are interchanged, then* $|A|$ *is multiplied by* -1.

Proof. For $m = 1$, $N_d = 1$ and $1 \le d \le n$. In view of Definition 2.1, (1) now follows. If $m = n$, then $d = 1 = N_d$ and $A_1^1 = A$; from this it follows that $|A| = \det A$. If the ith row of A is multiplied by c, then the ith row of the submatrix A_p^d for every value of d and p will be multiplied by c, in which case $\det A_p^d$ will be multiplied by

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e . From this (3) follows. The proof of (4), (5), (6) follows on the same lines by noting the corresponding properties of A_p^d for every value of p and d .

The det A_p^d contains $m!$ terms each of which corresponds to a permutation of e_p^d . More precisely, each term of det A_p^d is a product of m elements of A_p^d taken from its each row and column once only with positive or negative sign attached to it according as the permutation of e_p^d is even or odd. Since $|A|$ contains $m! \times {}^nC_m = m! \times \sum_{d=1}^{n-m+1} N_d = {}^nP_m$ terms in all with each one containing the entries from different rows and different columns of A , we can define $|A|$ equivalently as follows:

DEFINITION 2.3. Let A be a real matrix of order $m \times n$, $m \leq n$; then

$$|A| = \sum (-1)^n a_{1t_1} \cdots a_{mt_m}$$

where $\sum$ ranges over all the permutations of integers $t_1, \ldots, t_m$ taken m at a time from J_n and h is defined as the number of inversions required to bring $t_1, \ldots, t_m$ to its natural order.

EXAMPLE 2.1. Find the determinant of A , where

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 1 \end{bmatrix} .$$

Solution. Since $m = 3$ and $n = 4$, $d = 1$ or 2 and $N_1 = 3$ and $N_2 = 1$. The sets S_1 and S_2 contain the following elements, namely, $S_1 = \{(1, 2, 3), (1, 2, 4), (1, 3, 4)\}$, $S_2 = \{(2, 3, 4)\}$. In view of Definition 2.1, therefore,

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$
$$= -12 + 2 + 6 + 2 = -2 .$$

3. Cofactors and expansion of $|A|$

In view of Definition 2.3 of $|A|$, it is clear that $|A|$ is a linear and homogeneous function of the entries in the ith row of A. If C_{ij} denotes the coefficient of a_{ij}, $j = 1, \ldots, n$, then we get the expression

(3.1) $$|A| = a_{i1}C_{i1} + \ldots + a_{in}C_{in} .$$

The coefficient C_{ij} of a_{ij} in (3.1) is called the cofactor of a_{ij}. Let E, F, G and H be the submatrices of A of the order $(i-1) \times (j-1)$, $(i-1) \times (n-j)$, $(m-i) \times (n-j)$ and $(m-i) \times (j-1)$ respectively such that

$$\begin{array}{cc} A = \begin{pmatrix} E & \vdots & F \\ \ldots & a_{ij} & \ldots \\ H & \vdots & G \end{pmatrix} & i\text{th row;} \\ \quad\quad j\text{th column} & \end{array}$$

then the determinant of the submatrix

(3.2) $$M_{ij} = \begin{pmatrix} E & -F \\ -H & G \end{pmatrix}$$

of the order $(m-1) \times (n-1)$ corresponds to the cofactor of a_{ij}. Alternately we have

(3.3) $$C_{ij} = |M_{ij}| .$$

Using (3.1)-(3.3) we can evaluate $|A|$ in terms of the determinants of lower order. This is illustrated in the following example.

EXAMPLE 3.1. For the matrix A of Example 2.1, find $|A|$ using the cofactors of the second row.

Solution. We have $|A| = 2C_{21} + 1C_{22} + 1C_{23} + 0C_{24}$, where C_{2j} , $j = 1, \ldots, 4$ are the following determinants:

$$C_{21} = \begin{vmatrix} -2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = -2|(2\ 1)| + 1|(-3\ 1)| - 1|(-3\ -2)|$$
$$= -2(2+1) + 1(-3+1) - 1(-3-2) = -3 \ ,$$

$$C_{22} = \begin{vmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix} = 1|(2\ 1)| + 1|(1\ 1)| - 1|(1\ -2)|$$
$$= 1(2+1) + 1(1+1) - 1(1-2) = 6 \ .$$

Similarly $C_{23} = -2$ and $C_{24} = -11$, which yields $|A| = -2$. Note that we have used the cofactors of the first row of M_{ij} to evaluate C_{ij} in terms of the determinant of row vectors according to (1) of the lemma.

4. Normalised generalised inverse of A

We recall that for an $m \times n$ matrix A , the class of $m \times n$ matrices B satisfying the conditions

$$ABA = A \ , \quad BAB = B \quad \text{and} \quad AB \quad \text{or} \quad BA \quad \text{symmetric,}$$

is not empty. In spite of the fact that such a matrix B is not unique, it is required in statistics for many estimation purposes [1, 7, 8, 9]. In what follows, we shall present a realisation of such a B in terms of the cofactors of elements of A whenever $|A| \neq 0$. In view of (3.1) and (4) of the lemma, it is easy to see that

$$\sum_{j=1}^{n} a_{ij}C_{kj} = \delta_{ik}|A| \ .$$

Let $C = (C_{ij})$, be the $m \times n$ matrix whose elements, C_{ij} , are the cofactors of a_{ij} . If $C^T = R = (r_{ij})$, then $r_{ij} = C_{ij}$. Now,

$$AR = (a_{ij})(r_{ij}) = \left(\sum_{k=1}^{n} a_{ik}r_{kj}\right) = \left(\sum_{k=1}^{n} a_{ik}C_{jk}\right) \tag{4.1}$$
$$= (|A|\delta_{ij}) = |A|I_{m\times m} \ .$$

From this it follows that for $|A| \neq 0$, the matrix

$$B = R/|A| \tag{4.2}$$

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is a normalised generalised inverse of A . For the matrix A of Example 2.1, we find that the B , as defined above, is given by

$$B = -\tfrac{1}{2}\begin{pmatrix} 1 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -2 & 0 \\ 7 & -11 & 3 \end{pmatrix} .$$

The expression (4.2) breaks down when $|A| = 0$. In the next theorem we give a necessary and sufficient condition for matrix A to be rank deficient in which case $|A| = 0$.

THEOREM 4.1. *A necessary and sufficient condition for an* $m \times n$, $m \le n$, *matrix* A *to be rank deficient is that for all* $d = 1, \ldots, (n-m+1)$ *and* $p = 1, \ldots, N_d$, $\det A_p^d = 0$.

Proof. Let $r < m \le n$ be the rank of A . This implies that r rows of A are linearly independent and form a basis for the row-space of A . The remaining $(m-r)$ rows can be expressed as linear combinations of r rows. For every value of $d = 1, \ldots, (n-m+1)$ and $p = 1, \ldots, N_d$, therefore, A_p^d contains at least $(m-r)$ rows as linear combinations of r rows. Since $r < m$, it follows that $\det A_p^d = 0$. Conversely, if $\det A_p^d = 0$ for every value of d and p , the column rank of A_p^d will be smaller than m . This implies that the column space of A is spanned by a basis containing less than m columns of A . In other words, A is column rank deficient matrix. Since the column rank of a matrix is the same as its row rank, the result now follows.

COROLLARY 4.1.1. *A necessary and sufficient condition for an* $m \times n$, $m \le n$, *matrix* A *to be of full row rank is that* $\det A_p^d$ *is not zero for at least one value of* p *and* d .

COROLLARY 4.1.2. *A necessary and sufficient condition for a square matrix* A *to be of full rank is that* $\det A \ne 0$.

5. Schur complement

Let A be an $m \times n$, $m \le n$, matrix partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is an $r \times r$ nonsingular principal submatrix of A. The Schur complement of A_{11} in A, denoted by (A/A_{11}), is defined as the matrix

$$A_{22} - A_{21}A_{11}^{-1}A_{12} .$$

Define $t_r = {}^{n-r}C_{m-r}$ and for $1 \le p \le t_r$, construct $1 \times t_r$ matrix $B = (b_{1p})$, where $b_{1p} = \det A_p^1$. We shall show that

(5.1) $$|B| = \det A_{11} \times |(A/A_{11})| .$$

The matrix A can be written as

$$\begin{aligned} A &= \begin{pmatrix} I_r & 0 \\ A_{21}A_{11}^{-1} & I_{m-r} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22}-A_{21}A_{11}^{-1}A_{12} \end{pmatrix} \\ &= PQ , \end{aligned}$$

say, where the matrices P and Q are of order $m \times m$ and $m \times n$ respectively. Let $S \subseteq S_1$ which contains the elements e_p^1 for $1 \le p \le t_r$. In view of the order relation introduced amongst the elements e_p^d in Section 2, we now note that the first r entries in e_p^1 belonging to S are the integers $1, 2, \ldots, r$ in that order. If the last $(m-r)$ entries of such e_p^1 are denoted by e_{pc}^1, then we can write $e_p^1 = \{1, \ldots, r, e_{pc}^1\}$, $1 \le p \le t_r$. Now we have

$$\begin{aligned} b_{1p} &= \det A_p^1 = \det\left(PQ_p^1\right) = \det P \det Q_p^1 \\ &= \det Q_p^1 = \det A_{11} \det(A/A_{11})_{pc}^1 . \end{aligned}$$

Hence

$$|B| = \left|\det A_{11} \det(A/A_{11})^1_{pc}\right| = \det A_{11} \sum_{p=1}^{t_r} \det(A/A_{11})^1_{pc}$$
$$= \det A_{11} |(A/A_{11})| \, .$$

The last step has followed from Definition 2.3.

If $m = n$, then $t_r = 1$ and, in view of (2) of the lemma, equation (5.1) reduces to $\det A = \det A_{11} \det(A/A_{11})$. This result has been proved in [2, 3].

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