SOME RESULTS ON UNIQUENESS AND SUCCESSIVE APPROXIMATIONS

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1. In the theory of ordinary differential equations, there is a strange relationship between uniqueness of solutions and convergence of the successive approximations. There are examples of differential equations with unique solutions for which the successive approximations do not converge **(8)** and of differential equations with non-unique solutions for which the successive approximations do converge **(2)**. However, in spite of the known logical independence of these two properties, almost all conditions which assure uniqueness also imply the convergence of the successive approximations. For example, the hypotheses of Kamke's general uniqueness theorem **(5)**, have been shown by Coddington and Levinson to suffice for the convergence of successive approximations, after the addition of one simple monotonicity condition **(4)**. There is one counterexample to this "principle," a generalization of Kamke's result, to which another condition in addition to a monotonicity assumption must be added before convergence of the successive approximations can be proved **(2)**.

In this paper, we shall prove a pair of theorems, generalizing the results of Kamke, and Coddington and Levinson, and conforming to the "principle" mentioned above. The generalization here is in a different direction from that given in **(2)**, but we shall alsoindicate how the results obtained here can be formulate and proved in the more general setting used there.

The statements of our results involve a pair of conditions, each relating to a first order differential equation. We assume the existence of two functions, controlling, in a sense, the behaviour of the solutions of each one of these equations, and a relation between the growths of these functions near the origin. The complexity of the results is compensated for by the large number of special cases which can be obtained by appropriate choice of these functions.

2. We consider the initial value problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \qquad \mathbf{x}(0) = 0, \tag{1}$$

where $\mathbf{x}$ and $\mathbf{f}$ are n-dimensional vectors. As usual, the norm $|\mathbf{x}|$ of a vector $\mathbf{x}$ will denote the sum of the absolute values of the components of $\mathbf{x}$ **(3)**. Let $\psi_i(t, r)$ $(i = 1, 2)$ be continuous non-negative functions defined for $0 < t < a$,

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$r \geqslant 0$, which are monotone non-decreasing in r for each fixed t. We will always assume that $\mathbf{f}$ satisfies the pair of conditions

$$|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)| \leqslant \psi_i(t, |\mathbf{x}_1 - \mathbf{x}_2|), \qquad (i = 1, 2), \tag{2}$$

for $(t, \mathbf{x}_1)$ and $(t, \mathbf{x}_2)$ in a region $0 < t < a$, $|\mathbf{x}| < b$.

THEOREM 1. *Let* $\mathbf{f}(t, \mathbf{x})$ *be continuous and satisfy* (2) *in a region* $0 < t < a$, $|\mathbf{x}| \leqslant b$. *Suppose* $A(t)$ *and* $B(t)$ *are functions on* $0 \leqslant t < a$, *with* $A(0) = B(0) = 0$, *such that*

$$\lim_{t \to 0} A(t)/B(t) = 0. \tag{3}$$

Suppose also that all solutions $u(t)$ *of*

$$u' = \psi_1(t, u) \tag{4}$$

with $u(0) = 0$ *obey* $u(t) \leqslant A(t)$ *on* $0 \leqslant t < a$, *and that the only solution* $v(t)$ *of*

$$v' = \psi_2(t, v) \tag{5}$$

on $0 \leqslant t < a$ *such that*

$$\lim_{t \to 0} v(t)/B(t) = 0 \tag{6}$$

is the trivial solution. Then there is at most one solution of (1) *on* $0 \leqslant t < a$.

Proof. Suppose there are two solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ of (1) on $0 \leqslant t < a$, and let $m(t) = |\mathbf{x}_1(t) - \mathbf{x}_2(t)|$. Then

$$|m'(t)| \leqslant |\mathbf{x}_1'(t) - \mathbf{x}_2'(t)| \leqslant |\mathbf{f}(t, \mathbf{x}_1(t)) - \mathbf{f}(t, \mathbf{x}_2(t))| \leqslant \psi_1(t, m(t)), \tag{7}$$

using (2). Suppose there exists σ, $0 < \sigma < a$, such that

$$m(\sigma) > A(\sigma). \tag{8}$$

Then there is a solution $u(t)$ of (4) passing through the point $(\sigma, m(\sigma))$ and existing on some interval to the left of σ. As far to the left of σ as $u(t)$ exists, it satisfies

$$u(t) \leqslant m(t). \tag{9}$$

To prove (9), we observe that

$$u' = \psi_1(t, u) + \epsilon, \; u(\sigma) = m(\sigma), \tag{10}$$

has solutions $u(t, \epsilon)$ for all sufficiently small $\epsilon > 0$, existing as far to the left of σ as $u(t)$ exists, and $\lim_{\epsilon \to 0+} u(t, \epsilon) = u(t)$ (**5**, p. 83). Thus it suffices to prove

$$u(t, \epsilon) \leqslant m(t), \tag{11}$$

for all $\epsilon > 0$ and all solutions $u(t, \epsilon)$ of (10). If this inequality does not hold, there is a least upper bound ζ of numbers $t \leqslant \sigma$ for which (11) is false. Since $m(\sigma) = u(\sigma) = u(\sigma, \epsilon)$ and the functions $m(t)$ and $u(t, \epsilon)$ are continuous,

$$m(\zeta) = u(\zeta, \epsilon), \qquad m'(\zeta) \geqslant u'(\zeta, \epsilon). \tag{12}$$

Then

$$\psi_1(\zeta, m(\zeta)) + \epsilon = \psi_1(\zeta, u(\zeta, \epsilon)) + \epsilon = u'(\zeta, \epsilon) \leqslant m'(\zeta) \leqslant \psi_1(\zeta, m(\zeta)),$$

using (7), (10), and (12). This contradiction proves (11), which, as we have remarked, implies (9).

The solution $u(t)$ can be continued to $t = 0$. If $u(c) = 0$ for some $c, 0 < c < \sigma$, we can effect the continuation by defining $u(t) = 0$ for $0 < t < c$; otherwise (9) ensures the possibility of the continuation. Since $m(0) = 0$, $\lim_{t\to 0} u(t) = 0$, and we define $u(0) = 0$. Now we have a solution $u(t)$ of (4) with $u(0) = 0$, and by hypothesis $u(t) \leqslant A(t)$. However, in view of (8) and the definition of $u(t)$, we have $u(\sigma) > A(\sigma)$, a contradiction which proves

$$m(t) \leqslant A(t), \qquad 0 \leqslant t < a. \tag{13}$$

To prove the uniqueness of solutions, we must show that $m(t)$ vanishes identically on $0 \leqslant t < a$. To complete the proof, we must show that this is implied by (13). Proceeding as before but using ψ_2 in place of ψ_1, we obtain $|m'(t)| \leqslant \psi_2(t, m(t))$. The assumption that $m(\tau) > 0$ for some $\tau, 0 < \tau < a$, yields, by the same argument as before, a solution $v(t)$ of (5) on $0 \leqslant t \leqslant \tau$ such that $v(\tau) = m(\tau)$, $0 \leqslant v(t) \leqslant m(t)$, $v(0) = 0$. Then

$$0 \leqslant \lim_{t\to 0} v(t)/B(t) \leqslant \lim_{t\to 0} m(t)/B(t) \leqslant \lim_{t\to 0} A(t)/B(t) = 0,$$

using (3) and (13). But by hypothesis this implies that $v(t)$ is identically zero, which contradicts $v(\tau) = m(\tau) > 0$, and therefore $m(t)$ vanishes identically on $0 \leqslant t < a$, which completes the proof of the theorem.

3. The successive approximations to the solution of (1) are defined by

$$\mathbf{x}_0(t) = 0, \mathbf{x}_{j+1}(t) = \int_0^t \mathbf{f}(s, \mathbf{x}_j(s))ds, \qquad (j = 0, 1, \ldots). \tag{14}$$

THEOREM 2. *Let* $\mathbf{f}(t, \mathbf{x})$ *be continuous in a region* $0 < t < a$, $|\mathbf{x}| \leqslant b$ *and bounded in norm by* M *in this region. If the hypotheses of Theorem* 1 *are satisfied then the successive approximations* (14) *converge uniformly on the interval* $0 \leqslant t < \alpha$, *where* $\alpha = \min\,(a, b/M)$, *to the unique solution of* (1).

Proof. It follows easily from the definition (14) of the successive approximations that they satisfy $|\mathbf{x}_j(t_1) - \mathbf{x}_j(t_2)| \leqslant M|t_1 - t_2|$ in the interval $0 \leqslant t < \alpha$. This implies that the sequence $\{\mathbf{x}_j(t)\}$ is equicontinuous on this interval. Taking $t_2 = 0$, we have $|\mathbf{x}_j(t_1)| \leqslant Mt_1 \leqslant b$, and thus the sequence is also uniformly bounded. Therefore there exists a subsequence $\{\mathbf{x}_{jK}(t)\}$ which converges uniformly to a function $\mathbf{x}(t)$ on $0 \leqslant t < \alpha$. Because of (14) and the continuity of $\mathbf{f}$, the sequence $\{\mathbf{X}_{jK+1}(t)\}$ converges uniformly to a function

$$\mathbf{x}^*(t) = \int_0^t \mathbf{f}(s, \mathbf{x}(s))ds.$$

We will prove that $\mathbf{x}_{j+1}(t) - \mathbf{x}_j(t) \to 0$ as $j \to \infty$. This will imply, because of (14), that $\mathbf{x}(t) = \mathbf{x}^*(t)$, so that $\mathbf{x}(t)$ is a solution of (1). Since the solution of (1) is unique, every convergent subsequence of $\{\mathbf{x}_j(t)\}$ converges to a solution, hence to the same solution, which shows that the original sequence converges to $\mathbf{x}(t)$ on $0 \leqslant t < \alpha$. Because of the equicontinuity of the sequence, this convergence is uniform.

We define

$$\mathbf{w}_j(t) = \mathbf{x}_{j+1}(t) - \mathbf{x}_j(t), \qquad m(t) = \limsup_{j\to\infty} |\mathbf{w}_j(t)|.$$

Then $m(0) = 0$, and $m(t)$ is continuous on $0 \leqslant t < \alpha$, since it is the upper limit of a uniformly bounded equicontinuous sequence of functions. We must show that $m(t)$ vanishes identically on $0 \leqslant t < \alpha$. Using (14) and (2), we obtain

$$(15) \qquad |\mathbf{w}_{j+1}(t+h) - \mathbf{w}_{j+1}(t)| \leqslant \int_t^{t+h} |\mathbf{f}(s, \mathbf{x}_{j+1}(s)) - \mathbf{f}(s, \mathbf{x}_j(s))| ds \leqslant \int_t^{t+h} \psi_1(s, |\mathbf{w}_j(s)|)ds.$$

Because of the continuity of m and the equicontinuity of $\{\mathbf{w}_j\}$, given any $\epsilon > 0$, there exists an integer $N(\epsilon)$, independent of s and j, such that

$$(16) \qquad |\mathbf{w}_j(s)| < m(s) + \epsilon, \qquad j > N(\epsilon).$$

It follows from (15) and (16) that

$$(17) \qquad |\mathbf{w}_{j+1}(t+h) - \mathbf{w}_{j+1}(t)| \leqslant \int_t^{t+h} \psi_1(s, m(s)+\epsilon)ds, \quad j > N(\epsilon).$$

It follows easily from the definition of m that

$$|m(t+h) - m(t)| \leqslant \limsup_{j\to\infty} |\mathbf{w}_{j+1}(t+h) - \mathbf{w}_{j+1}(t)|.$$

Combining this with (17) and then letting $\epsilon \to 0$, we obtain

$$(18) \qquad |m(t+h) - m(t)| \leqslant \int_t^{t+h} \psi_1(s, m(s))ds,$$

using the continuity and monotonicity of $\psi_1(t, r)$ in r. The inequality (18) implies that $m'(t)$ exists on any interval $(t, t+h)$, for $t \geqslant 0$, and that $|m'(t)| \leqslant \psi_1(t, m(t))$. The argument used in the proof of Theorem 1, beginning with (7), proves

$$(19) \qquad m(t) \leqslant A(t).$$

To complete the proof, we must show that (19) implies $m(t) \equiv 0$. The argument is much the same as the last stage of the proof of Theorem 1. Suppose $m(\tau) > 0$ for some $\tau, 0 < \tau < \alpha$. A repetition of the first part of the proof, using ψ_2 in place of ψ_1, gives a solution $v(t)$ of (5) on $0 \leqslant t \leqslant \tau$ such that $v(\tau) = m(\tau)$, $0 \leqslant v(t) \leqslant m(t)$, $v(0) = 0$. Then $0 \leqslant \lim_{t\to 0} v(t)/B(t) \leqslant \lim_{t\to 0} m(t)/B(t) \leqslant \lim_{t\to 0} A(t)/B(t) = 0$, using (3) and (19). By hypothesis, this implies that $v(t)$ vanishes identically, contradicting $v(\tau) = m(\tau) > 0$, and it follows that $m(t) = 0, 0 \leqslant t < \alpha$.

4. The results of this paper were originally suggested by consideration of a function $\mathbf{f}(t, \mathbf{x})$ satisfying the pair of conditions

$$|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)| \leqslant C|\mathbf{x}_1 - \mathbf{x}_2|^\alpha, \tag{20}$$

$$|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)| \leqslant k|\mathbf{x}_1 - \mathbf{x}_2|/t, \tag{21}$$

where C, α, and k are constants with $0 < k, 0 < \alpha < 1, k(1 - \alpha) < 1$. It was shown by Krein and Krasnosel'skii **(6)** that this pair of conditions implies the uniqueness of solutions of (1), and by Luxemburg **(7)** that the successive approximations (14) converge. In **(1)** these results were proved by methods analogous to those used here. They are special cases of Theorems 1 and 2 obtained by taking $\psi_1(t, r) = Cr^\alpha$, $\psi_2(t, r) = kr/t$. It is easily verified that the hypotheses of Theorems 1 and 2 are satisfied, with $A(t) = C(1 - \alpha)t^{1/(1-\alpha)}$, $B(t) = t^k$, and the condition $k(1 - \alpha) < 1$ gives (3). Thus the results of Krein–Krasnosel'skiĭ and Luxemburg are contained in those of this paper.

The general uniqueness theorem of Kamke **(5)** and the successive approximation theorem of Coddington and Levinson **(4)** are also contained in this paper. In fact, the proofs used here are quite similar to the proofs of these results **(3**, chapter 2**)**. The central hypothesis there is the existence of a function $\psi(t, r)$, continuous and non-negative in $0 < t < a$, $r \geqslant 0$, and monotone non-decreasing in r for each fixed t, such that there is no non-trivial solution of

$$r' = \psi(t, r), \qquad r(0) = r'(0) = 0, \tag{22}$$

on $0 \leqslant t < a$, and such that

$$|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)| \leqslant \psi(t, |\mathbf{x}_1 - \mathbf{x}_2|),$$

for $(t, \mathbf{x}_1)$, $(t, \mathbf{x}_2)$ in a region $0 < t < a$, $|\mathbf{x}| \leqslant b$. That this result is contained in the present results is seen by taking $\psi_1(t, r) = \psi(t, r)$ and $\psi_2(t, r) = 2M$, where M is a bound for $|\mathbf{f}(t, \mathbf{x})|$ in $0 < t < a$, $|\mathbf{x}| \leqslant b$. Then we can take $B(t) = t$. If $A(t)$ is a solution of (4) with $A(0) = A'(0) = 0$, the condition (3) follows from

$$0 = A'(0) = \lim_{t\to 0} A(t)/t = \lim_{t\to 0} A(t)/B(t),$$

and the hypotheses of Theorems 1 and 2 are satisfied.

5. Now we obtain a generalization of the Krein–Krasnosel'skiĭ–Luxemburg results by application of our theorems. This will illustrate the flexibility of the method in generating new results. We retain the condition (21), corresponding to $\psi_2(t, r) = kr/t$, $B(t) = t^k$, but replace (20) by

$$|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)| \leqslant C|\mathbf{x}_1 - \mathbf{x}_2|^\alpha t^\beta, \qquad \alpha < 1, \beta > -1, \tag{23}$$

corresponding to the choice $\psi_1(t, r) = Cr^\alpha t^\beta$. It is easy to solve (4) with this choice of ψ_1, and we find that we can take $A(t) = pt^{(\beta+1)/(1-\alpha)}$, where p is a constant depending only on α, β, C. For (3) to be satisfied, we must have $(\beta + 1)/(1 - \alpha) - k > 0$, or $\beta + 1 > k(1 - \alpha)$. The results of **(1; 6; 7)** are the special case $\beta = 0$. If $(\beta + 1)/(1 - \alpha) \leqslant 1$, the condition (23) alone would suffice, as can be seen from the Kamke-Coddington-Levinson results. Thus we assume $\beta + 1 > 1 - \alpha$, or $\beta > - \alpha$.

Slightly more generally, we can replace the condition (23) by

$$|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)| \leqslant C|\mathbf{x}_1 - \mathbf{x}_2|^\alpha \lambda(t), \tag{24}$$

where $\alpha < 1$ and $\lambda(t) = o[t^{k(1-\alpha)-1}]$ as $t \to 0$. This corresponds to the choice $\psi_1(t, r) = Cr^\alpha \lambda(t)$, which yields

$$[A(t)]^{1-\alpha} = p \int_0^t \lambda(t)dt.$$

If $\lambda(t)$ satisfies the above growth condition, we have

$$\int_0^t \lambda(t)dt = o[t^{k(1-\alpha)}],$$

and $A(t) = o(t^k) = o(B(t))$, so that the hypotheses of our theorems are satisfied.

These examples illustrate suitable choices of $\psi_1(t, r)$ and $A(t)$. There is less flexibility in the choice of $\psi_2(t, r)$, as the definition of $B(t)$ implies that there can be no non-trivial solution of (5) which vanishes on an interval near the origin.

6. The results obtained in this paper can also be given in the more general setting of **(2)**. As the proofs are quite similar to those given here, with the same alterations used in **(2)**, we shall only outline the results, without proofs. Instead of using a norm $|\mathbf{x}|$ for vectors $\mathbf{x}$, we use a function $V(t, \mathbf{x})$ defined for real t and vectors $\mathbf{x}$ with non-negative real values, which is continuous in $(t, \mathbf{x})$, has one-sided partial derivatives with respect to t and the components of $\mathbf{x}$, and whose vanishing implies $\mathbf{x} = 0$. We use V_t to denote a partial derivative of V with respect to t, V_x to denote some gradient vector of V, and . to denote the usual scalar product of vectors. Any condition which involves V_t or V_x is understood to be required for all one-sided derivatives.

The only change in the hypotheses of the theorems is that the pair of conditions (2) are replaced by

$$V_t(t, \mathbf{x}_1 - \mathbf{x}_2) + V_x . (\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)) \leqslant \psi_i(t, V(t, \mathbf{x}_1 - \mathbf{x}_2)), \ i = 1, 2,$$

in the generalization of Theorem 1. For the generalization of Theorem 2, we must add the condition

$$V(t, \int_{t-h}^{t} [\mathbf{f}(s, \mathbf{x}_1(s)) - \mathbf{f}(s, \mathbf{x}_2(s))]ds)$$

$$\leqslant \int_{t-h}^{t} \{ V_t(s, \mathbf{x}_1(s) - \mathbf{x}_2(s)) + V_x . [\mathbf{f}(s, \mathbf{x}_1(s)) - \mathbf{f}(s, \mathbf{x}_2(s))]\} \, ds,$$

for any continuous $\mathbf{f}$, $\mathbf{x}_1$, $\mathbf{x}_2$, and the condition $V(\max\,(t_1, t_2), \mathbf{x}_1 + \mathbf{x}_2) \leqslant V(\mathbf{x}_1, t_1) + V(\mathbf{x}_2, t_2)$. This condition is essentially just a weakened form of the triangle inequality for the function $V(t, \mathbf{x})$. The other hypotheses of Theorems 1 and 2 and the conclusions remain unchanged.

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