A NOTE ON COUNTABLY COMPACT SEMIGROUPS

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It is well known that every compact topological semigroup has an idempotent and every compact bicancellative semigroup is a topological group. Also every locally compact semigroup which is algebraically a group, is a topological group. In this note we extend these results to the case of countably compact semigroups satisfying the I^{st} axiom of countability. Some of our results are valid under the weaker condition of sequential compactness. Finally some structure theorems concerning the closure of all integral powers of an element are extended to the countably (sequentially) compact case.

1. Introduction

In what follows S is a Hausdorff topological semigroup. (The associative 'multiplication' $(x, y) \to xy$ is 'jointly' continuous in the product topology of $S \times S$). S is called countably compact (resp. sequentially compact) if every infinite subset has a limit point (resp. every sequence has a convergent subsequence). Note that sequential implies countable compactness and that both are equivalent in the presence of I^{st} countability. An example of a I^{st}-countable, countably compact, locally compact, normal, non-compact semigroup is the semigroup of all ordinals less than the first uncountable ordinal with composition $xy \equiv \max(x, y)$ and the order topology. A problem of Wallace [9, p. 101] raises the question whether a countably compact semigroup which is also algebraically a group, is a topological group. (See also [4, p. 813]). It is well known that every compact semigroup has an idempotent [8, p. 103]. The main purpose of this note is to prove corresponding results in the countably (sequentially) compact case. In some of the theorems the I^{st} countability axiom is assumed in addition – although we have conjectured the existence of an idempotent in an arbitrary (Hausdorff) countably (sequentially) compact semigroup. It is hoped that the present results might give some insight to the general problem and cause some research in the direction of semi-topological (multiplication separately continuous) semigroups. Also some structure theorems concerning the semigroups $\Gamma(a)$ and $K(a)$ are given. For each $a \in S$, these semigroups are defined as follows:

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$P(a) = \{a, a^2, \cdots\}$

$\Gamma_n(a) = \overline{\{a^n, a^{n+1}, \cdots\}}$, the bar indicating closure in S.

$\Gamma(a) = \Gamma_1(a) = \overline{P(a)}$

$K(a) = \cap \{\Gamma_n(a); n \geqq 1\}$

$K_s(a) = \{\text{all subsequential limits of } P(a)\}$

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By a left (right) fiber in S we mean any set of the form $x^{-1}x$ (resp. $xx^{-1} = \{s \in S; sx = x\}$), $x \in S$.

THEOREM 1. *If S is sequentially compact, then for every $a \in S$,*

(a) $K_s(a) = K_s(a)a^k$ *for all* $k \geqq 1$.

(b) *For every* $y \in K_s(a)$, $yy^{-1} \neq \emptyset$ and $y^{-1}y \neq \emptyset$.

(c) $\overline{K_s(a)} = \overline{\overline{K_s(a)}y}$ *for every* $y \in K_s(a)$.

PROOF. Clearly $K_s(a)$ is a non-empty (commutative) semigroup. (a). We prove that if $y \in K_s(a)$, then $y \in K_s(a)a^k$. There is a subsequence $\{a^m\} \to y$. Also the sequence $\{a^{m-k}; m > k\}$ has a subsequence $\{a^{m_i-k}\} \to z$ for some $z \in K_s(a)$. Then $a^{m_i-k}a^k \to za^k$, so that by continuity and Hausdorff property, $y = za^k$. (b) and (c). Suppose $u, y \in K_s(a)$. Let $\{a^k\} \to y$. Then for each k, $u = z_k a^k$, $z_k \in K_s(a)$, by Part (a). Now some subsequence $z_{k_i} \to w \in \overline{K_s(a)}$. Then $z_{k_i}a^{k_i} \to wy = u$, so that $K_s(a) \subset \overline{K_s(a)}y$ from which ($\overline{K_s(a)}$ being a semigroup) Part (c) follows. By taking $u = y$ in the argument above, one shows that $yy^{-1} \neq \emptyset$.

If S is left simple ($Sx = S$ for all $x \in S$) and contains an idempotent, then we call S a pre-topological left group. In such a space the right translation $x \to xa$, $a \in S$, is a homeomorphism as it is seen in Lemma 2. If in addition, left inversion with respect to a fixed right identity of S is continuous, then S is called a topological left group. (Note that in an algebraic left group every idempotent e is a right identity and for every $a \in S$, there is $b \in S$ such that $ba = e$ and ab is also idempotent. [2, I, p. 37] and [1, p. 45]).

LEMMA 2. *If S is a pre-topological left group, then the right translation $s \to sx$, $x \in S$, is a homeomorphism.*

The proof follows since each right translation has a suitable right translation as inverse, and all right translations are continuous.

LEMMA 3. (a) *S is a topological left group iff $S \cong E(S) \times G$, where $E(S)$ is a topological left zero semigroup and $G \equiv eS$ is a topological group with e being any (fixed) idempotent in S.*

(b) *A pretopological left group is topological iff any maximal subgroup is topological.*

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PROOF. The functions $\phi : E(S)\times G \to S$ and $\psi : S \to E(S)\times G$ defined by $\phi(f, g) = fg$ and $\psi(s) = (s(es)^{-1}, es)$ are continuous semigroup morphisms which are inverses of each other. Thus $S = E(S)\times G$. The converse is clear, ($E(S)$ is usually chosen as the set of all idempotents in S). (b). Since eS is a maximal subgroup, the claim follows from (a).

THEOREM 4. *If S is sequentially compact with right cancellation, then S is a pretopological left group. If in addition S is either I^{st}-countable or locally compact, then S is a topological left group.*

PROOF. Let $a, b \in S$, such that $ab = b$. (Theorem 1). Then by right cancellation, a is a right identity for S. Let $y \in S$; since Sy is also sequentially compact, it has also a right identity $y_0 = sy$ for some $s \in S$. Then for any $z \in S$, $zysy = zy$ or $zys = z$, so that ys is a right identity for S also. Since $Sys \supset Ss$, we have $Sy \supset S$, and S is a left group.

By Lemma 3 a left group is topological iff any maximal subgroup (which is necessarily of the form eS, with e an idempotent) is topological. Thus local compactness will make the (closed) subgroup eS topological and Lemma 3 applies.

In the first countable case it suffices to show that eS is topological, in view of Lemma 3. We show that inversion in eS is continuous. Let $x_n \to x_0 \in eS$; then some subsequence $x_{n_i}^{-1} \to u \cdot (x_n^{-1}x_n = e$ for all n). Hence $e = x_{n_i}^{-1}x_{n_i} \to ux_0$, which means $u = x_0^{-1}$. Now if $\{x_n^{-1}\}$ does not converge to $u = x_0^{-1}$, then there is a neighbourhood V_u of u such that some subsequence $x_k^{-1} \in V_u^c$ (= complement of V_u). By sequential compactness, there is a subsequence $x_{k_i}^{-1} \to w$, and therefore $e = x_{k_i}^{-1}x_{k_i}$ converges to wx_0 which implies $w = x_0^{-1}$. This is a contradiction since $x_{k_i}^{-1} \notin V_u$.

REMARK. Since topological left and right group implies topological group, the above Lemma gives conditions under which a theorem in [4], [4, Theorem 14] becomes valid (Corollary 5 below). That theorem in [4] remains a conjecture for its proof does not seem to be justified by nets [4, p. 813, proof of Lemma 2].

COROLLARY 5. *Every countably compact I^{st}-countable bicancellative semigroup is a topological group. Also, every sequentially compact locally compact bicancellative semigroup is a topological group.*

REMARK. An interesting class of left simple semigroups is the Baer-Levi semigroup of all $1-1$ functions on a set A of infinite cardinality p such that $A-f(A)$ has also cardinality p, under composition. [2, II, p. 83]. It follows from Theorem 4 that such a semigroup cannot be topologized with a Hausdorff sequentially compact topology into a topological semigroup. For if this was so, then S would have an idempotent (since it is left cancellative), which is a contradiction to the fact that these semigroups possess no idempotents.

THEOREM 6. *Let $\Gamma(a)$ be I^{st}-countable countably compact. Then $K(a) = K_s(a)$;*

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$K(a)$ is the minimal ideal of $\Gamma(a)$ as well as the unique maximal subgroup of $\Gamma(a)$. Moreover, if $\Gamma(a)$ has an unit, then $K(a) = \Gamma(a)$.

PROOF. Since S is I^{st}-countable, $K(a) = K_s(a)$. Since the right translations $s \to sx$, $x \in S$, are closed mappings, the result follows from Theorem 1, Part (c), and the fact that $\Gamma(a)$ is commutative. Next, let e be the unit of $\Gamma(a)$. Suppose $e \notin K(a)$. Then $e \notin \Gamma_{n_0}(a)$ for some n_0. Then $e = a^m$, for some m such that $1 \leqq m \leqq n_0$. Then $\Gamma_n(a) = \Gamma(a)$ for every n and hence $K(a) = \Gamma(a)$ which is a contradiction. Therefore we have $e \in K(a)$ and so $\Gamma(a) = \Gamma(a)e = K(a)$, since $K(a)$ is the minimal ideal of $\Gamma(a)$.

COROLLARY 7. *Every countably compact I^{st}-countable topological semigroup contains an idempotent.*

COROLLARY 8. *A I^{st}-countable countably compact semigroup with a unit and no other idempotents is a topological group. Also, the closure of a subgroup of a I^{st}-countable countably compact semigroup, is a topological group.*

PROOF. Let $a \in S$. Then from Theorem 6, $\Gamma(a) = K(a)$, so that a has an inverse with respect to the unit e of S. Then as in Lemma 3, we can show that the mapping $a \to a^{-1}$ is continuous on S. Hence the first part follows. It is easy to check the second part.

COROLLARY 9. *Let S be normal I^{st}-countable with the right translations $s \to sx$, $x \in S$, as closed mappings. Further let S have a non-empty right fiber with empty interior. Then S has an idempotent. (Here we do not assume countably compact).*

PROOF. Assume $A = xx^{-1} \neq \emptyset$, Interior $(A) = \emptyset$. Then by [7, p. 10], Boundary (A) is countably compact. By Corollary 7, A has an idempotent.

REMARK. In view of the work of [7, p. 10], a stronger version of the above Corollary is true. Namely: Let S be I^{st}-countable normal. Let $x \in S$, such that $xx^{-1} \neq \emptyset$, Interior $(xx^{-1}) = \emptyset$, and the right translates by x of every countable closed subset of S be closed. Then the fiber xx^{-1} is a countably compact semigroup with idempotent.

REMARK. Let $E =$ the set of all idempotents, in a I^{st}-countable countably compact semigroup S. If E is finite, then S has a completely simple closed kernel (= unique minimal ideal). This can be proved as in [8, p. 104–105], since Lemma 1 of [8, p. 100] is valid in our case. We have conjectured that S has such a kernel if E is compact.

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Added in proof. In this note throughout, joint continuity of multiplication has been used in the proofs. However, some results can be obtained in the 'separately continuous multiplication' case by following Ellis' beautiful work on transformation groups (see R. Ellis, Locally compact transformation groups, Duke Math. Journal, 1957). For example, by noting that a countably compact regular space is a Baire space and by following Ellis' paper above, one can prove that a countably compact Ist countable completely regular group with separately continuous multiplication is topological and hence compact since a Ist countable topological group is metrizable.

Finally we remark that some of the results in this note have been known in the compact case for some time (see, for instance, Koch's Thesis, Tulane University, 1953). But the arguments usually followed in the compact case don't carry over in the non-compact situation.

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