

CONVOLUTION ORTHOGONALITY AND THE JACOBI POLYNOMIALS

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ABSTRACT. Let α and β be any two real numbers and let $\{P_n^{\alpha,\beta}(x)|n = 0, 1, 2, \dots\}$ be the Jacobi polynomial sequences. For any non-zero real number a , $\{P_n^{\alpha,\beta}(ax + b)|n = 0, 1, 2, \dots\}$ is an orthogonal polynomial sequence with respect to convolution if and only if either (i) $b = 1$, $\alpha = 0$ and $\beta + 1$ is not equal to a negative integer or (ii) $b = -1$, $\beta = 0$ and $\alpha + 1$ is not equal to a negative integer.

1. Introduction. Let $\mathfrak{R}[x]$ be the usual vector space of all polynomials in the indeterminate x over the field of real numbers \mathfrak{R} . We will call $\{p_n(x)|n = 0, 1, 2, \dots\}$ a *polynomial sequence* if for $n = 0, 1, 2, \dots$, the degree of $p_n(x)$ is n . A polynomial sequence $\{p_n(x)|n = 0, 1, 2, \dots\}$ is called an *orthogonal polynomial sequence* if there exists a linear functional $\mathbf{L} : \mathfrak{R}[x] \rightarrow \mathfrak{R}$, such that for all non-negative integers n and m ,

$$(1.1) \quad \mathbf{L}[p_n(x)p_m(x)] = k_n\delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta and $k_n \neq 0$, for $n = 0, 1, 2, \dots$. See Brezinski [2] or Chihara [3] for an excellent introduction to orthogonal polynomials.

According to Amerbaev and Nauzhaev [1], a polynomial sequence $\{q_n(x)|n = 0, 1, 2, \dots\}$ is an *orthogonal polynomial sequence with respect to convolution* if there exists a linear functional $\mathbf{L} : \mathfrak{R}[x] \rightarrow \mathfrak{R}$, such that for all non-negative integers n and m ,

$$(1.2) \quad \mathbf{L} \left[\frac{d}{dx} \int_0^x q_n(x-t)q_m(t)dt \right] = k_n\delta_{n,m},$$

where $k_n \neq 0$, for $n = 0, 1, 2, \dots$. If we denote the convolution product on $\mathfrak{R}[x]$, that is used in Equation (1.2), by $\langle c \rangle$ and observe that

$$(1.3) \quad x^n \langle c \rangle x^m \equiv \frac{d}{dx} \int_0^x (x-t)^n t^m dt = \frac{n!m!}{(n+m)!} x^{n+m},$$

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then the convolution product $\langle c \rangle$ is commutative, associative and distributive over ordinary polynomial addition on $\mathfrak{R}[x]$. Equation (1.2) can be written in the form

$$(1.4) \quad \mathbf{L}(q_n(x)\langle c \rangle q_m(x)) = k_n \delta_{n,m},$$

which is Equation (1.1) with the traditional polynomial product replaced by the convolution product $\langle c \rangle$. It is interesting to note that the convolution product $\langle c \rangle$ is intimately related to the convolution associated with the Laplace transform. Specifically, if the Laplace Transform \mathcal{L} is defined by

$$\mathcal{L}[f(x)] = \int_0^\infty e^{-st}f(t)dt = F(s),$$

then it is well known that

$$(1.5) \quad \frac{d}{dx} \int_0^x f(x-t)g(t)dt = \mathcal{L}^{-1}\{s(\mathcal{L}[f(x)])(\mathcal{L}[g(x)])\}.$$

By letting $f(x) = x^n$ and $g(x) = x^m$, Equation (1.5) becomes Equation (1.3).

We will have occasion to use the Gamma function $\Gamma(z)$ with the negative independent variable z , thus we follow Weierstrass [4, Page 9] in defining the Gamma function $\Gamma(z)$ by

$$\frac{1}{\Gamma(z)} = z \exp(\gamma z) \prod_{n=1}^\infty \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right],$$

where z is any finite complex number and γ is the Euler Constant. When $\Gamma(z)$ is defined in this matter it is well known that $1/\Gamma(z)$ is an entire function that has simple zeros at $z = 0, -1, -2, -3, \dots$

Also define

$$(\alpha + 1)_n = \begin{cases} \prod_{i=1}^n (\alpha + i) & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}.$$

The Jacobi polynomial sequences $\{P_n^{\alpha,\beta}(x) | n = 0, 1, 2, 3, \dots\}$ are defined in [3, Page 143] by

$$(1.6) \quad P_n^{\alpha,\beta}(x) = (-2)^{-n}(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{n+\alpha}(1+x)^{n+\beta}],$$

or by

$$(1.7) \quad P_n^{\alpha,\beta}(x) = \frac{(1+\alpha)_n}{n!} \sum_{k=0}^n \frac{(-n)_k (n+\alpha+\beta+1)_k}{(1+\alpha)_k} \left(\frac{1-x}{2}\right)^k$$

Throughout this paper we will always require that α and β are real numbers such that

$$(1.8) \quad 1 + \alpha + \beta \neq -1, -2, -3, \dots$$

This is a reasonable restriction on α and β because if $1 + \alpha + \beta$ equals a negative integer $-m$, then the degree of $P_m^{\alpha,\beta}(x)$ is not equal m and thus in this case $\{P_m^{\alpha,\beta}(x)|n = 0, 1, 2, 3, \dots\}$ is *not* a polynomial sequence. The Jacobi polynomials satisfy the orthogonality relation

$$(1.9) \quad \int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x) dx = \frac{2^{1+\alpha+\beta} \Gamma(1+\alpha+n) \Gamma(1+\beta+n)}{n!(1+\alpha+\beta+2n) \Gamma(1+\alpha+\beta+n)} \delta_{n,m},$$

if $\alpha > -1$ and $\beta > -1$. See [3, Page 148].

The main result of this paper is the following theorem.

THEOREM 1.1. *Define the polynomial sequence $\{Q_n(x; \alpha, \beta)|n = 0, 1, 2, 3, \dots\}$ by*

$$Q_n(x; \alpha, \beta) = P_n^{\alpha,\beta}(1 - 2x).$$

(a) *If $\beta > -1$, then*

$$(1.10) \quad \int_0^1 (1-x)^\beta (Q_n(x; 0, \beta) {}_{(c)}Q_m(x; 0, \beta)) dx = \frac{(-1)^n n!}{(1+\beta)_n (1+\beta+2n)} \delta_{n,m},$$

where $n, m = 0, 1, 2, 3, \dots$

(b) *If $\beta = -1$, then*

$$(1.11) \quad \mathbf{E}_1(Q_n(x; 0, \beta) {}_{(c)}Q_m(x; 0, \beta)) = (-1)^n \delta_{m,n}/2,$$

where $\mathbf{E}_1 : \mathfrak{R}[x] \rightarrow \mathfrak{R}$ is the evaluational functional at 1 defined by

$$\mathbf{E}_1(x^n) = 1,$$

where $n, m = 0, 1, 2, 3, \dots$

(c) *If $\beta \neq -1, -2, -3, \dots$, then*

$$(1.12) \quad \mathbf{F}_\beta(Q_n(x; 0, \beta) {}_{(c)}Q_m(x; 0, \beta)) = \frac{(-1)^n n!}{(1+\beta)_n (1+\beta+2n)} \delta_{n,m},$$

where the linear functional $\mathbf{F}_\beta : \mathfrak{R}(x) \rightarrow \mathfrak{R}$ is defined by

$$(1.13) \quad \mathbf{F}_\beta(x^n) = \frac{n!}{(\beta+1)_{n+1}}$$

and $n, m = 0, 1, 2, 3, \dots$

(d) $\{Q_n(x; \alpha, \beta) | n = 0, 1, 2, \dots\}$ is an orthogonal polynomial sequence with respect to convolution if and only if $\alpha = 0$ and $\beta + 1 \neq -1, -2, -3, \dots$

2. **Proof of Theorem 1.1.** (a) Because the convolution product $\langle c \rangle$ is commutative and distributive over polynomial addition, in order to prove the convolution orthogonality relation (1.10), we need only show that

$$(2.1) \quad \int_0^1 (1-x)^\beta (x^k \langle c \rangle Q_n(x; 0, \beta)) dx = \frac{(n!)^2 \Gamma(1 + \beta + n)}{(1 + \beta)_n \Gamma(2 + \beta + 2n)} \delta_{k,n},$$

for $k = 0, 1, 2, \dots, n$.

By noting that for all polynomials $\Pi(x)$,

$$\frac{d}{dt} [t^k \langle c \rangle \Pi(t)] = kt^{k-1} \langle c \rangle \Pi(t) \quad \text{if } k > 0,$$

we have by using integration by parts k times on the left hand side of Equation (2.1)

$$\begin{aligned} \int_0^1 (1-t)^\beta [t^k \langle c \rangle Q_n(t; 0, \beta)] dt &= \frac{k!}{(1 + \beta)_k} \int_0^1 (1-t)^\beta (1-t)^k Q_n(t; 0, \beta) dt \\ &= \frac{k!}{(\beta + 1)_k} \int_{-1}^1 \frac{(1+x)^\beta (1+x)^k}{2^{\beta+k+1}} P_n^{0,\beta}(x) dx. \\ &= \frac{k!}{(\beta + 1)_k} \int_{-1}^1 \frac{(1+x)^\beta x^k}{2^{\beta+k+1}} P_n^{0,\beta}(x) dx, \\ &= \frac{n!}{(\beta + 1)_n} \frac{n! \Gamma(1 + \beta + n)}{\Gamma(2 + \beta + 2n)} \delta_{k,n}. \end{aligned}$$

Equation (1.10) follows because the leading coefficient of $Q_n(x; 0, \beta)$ is

$$\frac{(1 + \beta)_{2n} (-1)^n}{n! (1 + \beta)_n}.$$

(b) The c -orthogonality relation, Equation (1.11), follows from Equation (1.10) and the fact that

$$\lim_{\beta \rightarrow -1} (\beta + 1) \int_0^1 (1-x)^\beta x^m dx = 1 = \mathbf{E}_1(x^m),$$

for $m = 0, 1, 2, \dots$

(c) By hypothesis $\beta \neq -1, -2, \dots$. Thus the linear functional $\mathbf{F}_\beta : \mathfrak{R}[x] \rightarrow \mathfrak{R}$ is well-defined by Equation (1.13). For $n = 0, 1, 2, \dots$,

$$(2.2) \quad \mathbf{F}_\beta(Q_n(x; 0, \beta)) = \frac{1}{\beta + 1} \sum_{k=0}^n \frac{(-n)_k (1 + \beta + n)_k}{(\beta + 2)_k} \frac{1}{k!} = \frac{(-1)^n (0)_n}{(\beta + 1)_{n+1}} = \frac{\delta_{n,0}}{\beta + 1}$$

Because of Equation (2.1), we have for $\beta > -1$

$$(2.3) \quad x^{k(c)}Q_n(x; 0, \beta) = \sum_{i=n-k}^{n+k} a_i(\beta, n, k)Q_i(x; 0, \beta),$$

where $0 \leq k \leq n$. Equation (2.3) is not only a polynomial identity in x but is also a polynomial identity in β and therefore it is valid for all real numbers β . By using Equations (2.2) and (2.3), it follows that for $\beta \neq -1, -2, \dots$

$$F_\beta(x^{k(c)}Q_n(x; 0, \beta)) = \frac{(n!)^2\Gamma(1 + \beta + n)}{[(1 + \beta)_n]\Gamma(2 + \beta + 2n)}\delta_{k,n},$$

where $0 \leq k \leq n$, which implies Equation (1.12).

(d) if $\alpha = 0$ and $\beta \neq -1, -2, \dots$, then by parts (b) and (c) we have that $\{Q_n(x; \alpha, \beta)|n = 0, 1, 2, \dots\}$ is an orthogonal polynomial sequence with respect to convolution.

To prove the converse, let α and β be any two real numbers such that $\{Q_n(x; \alpha, \beta)|n = 0, 1, 2, \dots\}$ is an orthogonal polynomial sequence with respect to convolution. Thus, there exist a linear functional $S_{\alpha,\beta} : \mathfrak{R}[x] \rightarrow \mathfrak{R}$ defined by

$$(2.4) \quad S_{\alpha,\beta}(Q_n(x; \alpha, \beta)) = \delta_{0,n},$$

where $n = 0, 1, 2, \dots$. Because $\{Q_n(x; \alpha, \beta)|n = 0, 1, 2, \dots\}$ is orthogonal with respect to convolution it follows from Inequality (1.8) that $\alpha + \beta + 2 \neq 0$, and thus a special case of Equation (1.4) is

$$\begin{aligned} 0 &= S_{\alpha,\beta} [t^{(c)}Q_2(t; \alpha, \beta)] \\ &= S_{\alpha,\beta} \left[\int_0^t Q_2(s; \alpha, \beta)ds \right] \\ &= S_{\alpha,\beta} \left[\frac{Q_3(t; \alpha - 1, \beta - 1) - Q_3(0; \alpha - 1, \beta - 1)}{(\alpha + \beta + 2)} \right] \\ &= \frac{(\alpha)_3}{3!(\alpha + \beta + 2)}. \end{aligned}$$

Therefore, $\alpha = 0$, or $\alpha = -1$, or $\alpha = -2$.

To show that $\alpha \neq -1$, we first write $Q_n(x; \alpha, \beta)$ as a hypergeometric function and use a well known summation formula for the hypergeometric function of unit argument [4, Page 69], to show that $S_{\alpha,\beta}$ has the following equivalent definition

$$S_{\alpha,\beta}(x^n) = \frac{(\alpha + 1)_n}{(\alpha + \beta + 2)_n},$$

for $n = 0, 1, 2, \dots$. Thus,

$$\begin{aligned} 0 \neq S_{\alpha,\beta}(x^{(c)}Q_1(x; \alpha, \beta)) &= S_{\alpha,\beta}((1 + \alpha)x + (2 + \alpha + \beta)x^2/2) \\ &= \frac{(1 + \alpha)_2}{2} \frac{2 + \alpha(\alpha + \beta + 4)}{(\alpha + \beta + 2)_2}, \end{aligned}$$

which along with restriction (1.8) implies that $\alpha \neq -1$.

The only remaining case to consider is $\alpha = -2$. If this is the case,

$$Q_2(x; -2, \beta) = x^2(\beta + 1)_2$$

and thus because of the Restriction (1.8),

$$0 \neq S_{\alpha, \beta}(Q_2(x; -2, \beta) \langle c \rangle Q_2(x; -2, \beta)) = [(1 + \beta)_2]^2(\alpha + 1)_4 / (\alpha + \beta + 2)_4 = 0,$$

which is a contradiction. Thus $\alpha \neq -2$ and therefore the only remaining possible value for α is 0. Q.E.D.

Part (b) of Theorem (1.1) is especially interesting because it gives an example of a polynomial sequence that is not an orthogonal polynomial sequence but it is a polynomial sequence that is orthogonal with respect to convolution.

3. Some Consequences of Theorem 1.1.

(3.1) *Convolution Three Term Recursion Formula*; It is well known [3; Pages 18, 21, 22] that a polynomial sequence $\{P_n(x) | n = 0, 1, 2, 3, \dots\}$ is an orthogonal polynomial sequence if and only if $\{P_n(x) | n = 0, 1, 2, 3, \dots\}$ satisfies a three term recursion relation of the form

$$P_{-1}(x) = 0, P_0(x) = \gamma_0, \\ xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n = 0, 1, 2, 3, \dots,$$

where $\gamma_n \neq 0$ and $\alpha_n \neq 0$. By using the same technique as was used in the proof of Favard's Theorem [3; Page 21], it follows that $\{Q_n(x) | n = 0, 1, 2, 3, \dots\}$ is orthogonal with respect to convolution if and only if

$$Q_{-1}(x) = 0, Q_0(x) = c_0, \\ x \langle c \rangle Q_n(x) = a_n Q_{n+1}(x) + b_n Q_n(x) + c_n Q_{n-1}(x), \quad n = 0, 1, 2, 3, \dots,$$

where $a_n \neq 0$ and $c_n \neq 0$. Thus $Q_n(x; 0, \beta) \equiv P_n^{0, \beta}(1 - 2x)$, for $n = 0, 1, 2, 3, \dots$, satisfies a "convolution" three term recursion relation

$$Q_{-1}(x; 0, \beta) = 0, Q_0(x; 0, \beta) = 1,$$

$$(3.1) \quad \int_0^x Q_n(t; 0, \beta) dt = a_n Q_{n+1}(x; 0, \beta) + b_n Q_n(x; 0, \beta) + c_n Q_{n-1}(x; 0, \beta),$$

for $n = 0, 1, 2, \dots$. By using well known formulae for the Jacobi Polynomials (see [4; Page 263 Formula 2, and Page 265 Formulae 14 & 15]), it is easy to show that for all real numbers α and β satisfying the Inequality (1.8)

$$(3.2) \quad P_n^{\alpha, \beta}(x) = \frac{2(\alpha + \beta + 1 + n)}{(\alpha + \beta + 1 + 2n)(\alpha + \beta + 2 + 2n)} \frac{dP_{n+1}^{\alpha, \beta}(x)}{dx} \\ + \frac{2(\alpha - \beta)}{(\alpha + \beta + 2n)(\alpha + \beta + 2 + 2n)} \frac{dP_n^{\alpha, \beta}(x)}{dx} \\ - \frac{2(\alpha + n)(\beta + n)}{(\alpha + \beta + n)(\alpha + \beta + 2n)(\alpha + \beta + 1 + 2n)} \frac{dP_{n-1}^{\alpha, \beta}(x)}{dx},$$

for $n = 0, 1, 2, 3, \dots$, which becomes when $\alpha = 0$ and x is replaced by $1 - 2x$,

$$(3.3) \quad Q_n(x; 0, \beta) = \frac{-(\beta + n + 1)}{(\beta + 2n + 1)(\beta + 2n + 2)} \frac{dQ_{n+1}(x; 0, \beta)}{dx} + \frac{\beta}{(\beta + 2n)(\beta + 2n + 2)} \frac{dQ_n(x; 0, \beta)}{dx} + \frac{n}{(\beta + 2n)(\beta + 2n + 1)} \frac{dQ_{n-1}(x; 0, \beta)}{dx},$$

for $n = 0, 1, 2, 3, \dots$

From the equation formed by integrating both sides of Equation (3.3) and by noting that

$$\frac{-(\beta + n + 1)Q_{n+1}(0; 0, \beta)}{(\beta + 2n + 1)(\beta + 2n + 2)} + \frac{\beta Q_n(0; 0, \beta)}{(\beta + 2n)(\beta + 2n + 2)} + \frac{nQ_{n-1}(0; 0, \beta)}{(\beta + 2n)(\beta + 2n + 1)} \equiv 0,$$

we have that the coefficients in Equation (3.1) are given by

$$a_n = \frac{-(\beta + n + 1)}{(\beta + 2n + 1)(\beta + 2n + 2)},$$

$$b_n = \frac{\beta}{(\beta + 2n)(\beta + 2n + 2)},$$

$$c_n = \frac{n}{(\beta + 2n)(\beta + 2n + 1)},$$

for $n = 0, 1, 2, \dots$

(3.2) *The Evaluation of a Determinant:* Let us define $\{q_n(x; 0, \beta) | n = 0, 1, 2, 3, \dots\}$ by

$$q_n(x; 0, \beta) = \frac{(-1)^n (\beta + 1)_n}{(\beta + 1)_{2n}} Q_n(x; 0, \beta).$$

From Equation (3.1), it follows that $\{q_n(x; 0, \beta) | n = 0, 1, 2, 3, \dots\}$ is given by,

$$(3.4) \quad q_{-1}(x; 0, \beta) = 0, q_0(x; 0, \beta) = 1$$

$$\int_0^x q_n(t; 0, \beta) dt = q_{n+1}(x; 0, \beta) + \frac{\beta}{(\beta + 2n)(\beta + 2n + 2)} q_n(x; 0, \beta) - \frac{n(\beta + n)}{(\beta + 2n - 1)(\beta + 2n)^2(\beta + 2n + 1)} q_{n-1}(x; 0, \beta).$$

Because the coefficient of $x^n/n!$ in $q_n(x; 0, \beta)$ is 1 and because of Equation (1.3), the polynomial sequence $\{q_n(x; 0, \beta) | n = 0, 1, 2, 3, \dots\}$ can be thought of as the ‘‘convolution’’ monic orthogonal polynomial sequence relative to $\{Q_n(x; 0, \beta) | n = 0, 1, 2, 3, \dots\}$. By using a technique similar to that used in the classical orthogonal polynomial theory (See [3; Pages 18 and 19]), it is easy to show that for $n = 1, 2, 3, \dots$,

$$(3.5) \quad \frac{D_{n-2}D_n}{(D_{n-1})^2} = \frac{-n(\beta + n)}{(\beta + 2n - 1)(\beta + 2n)^2(\beta + 2n + 1)},$$

where D_n is the determinant defined by

$$D_n = \begin{vmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \dots & \gamma_n \\ \gamma_1 & \gamma_2 & \gamma_3 & \dots & \gamma_{n+1} \\ \gamma_2 & \gamma_3 & \gamma_4 & \dots & \gamma_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_n & \gamma_{n+1} & \gamma_{n+2} & \dots & \gamma_{2n} \end{vmatrix}, \text{ and } D_{-1} = 1,$$

and γ_n are the ‘‘convolution’’ moments of $\{Q_n(x; 0, \beta) | n = 0, 1, 2, 3, \dots\}$ defined by

$$\gamma_n = \int_0^1 (1-t)^\beta t^n dt / n! = \frac{1}{(\beta + 1)_{n+1}}.$$

By using the recursion relations (3.5) for D_n and by noting that $D_0 = (1 + \beta)^{-1}$ and $D_1 = -(1 + \beta)^{-2}(2 + \beta)^{-2}(3 + \beta)^{-1}$, we find that

$$D_n = \frac{(-1)^{\lceil n/2 \rceil} \prod_{i=1}^n i!}{[(1 + \beta)_{n+1}]^{n+1} \prod_{i=1}^n (n + 2 + \beta)_{n+1-i}},$$

where $\lceil a \rceil$ is the smallest integer $\geq a$.

(3.3) *Convolution Associate Polynomials:* By using Equation (3.4) and direct substitution, it is easy to show that for $\beta > -1$,

$$q_{n-1}^{(1)}(x; 0, \beta) \equiv \frac{d}{dx} \int_0^1 (1-t)^\beta q_n(x+t; 0, \beta) dt.$$

is also a polynomial solution of the convolution three term recursion formula (3.4). For the case when $\beta = -1$, there is only one second solution of the convolution three term recursion formula (3.4), namely when $\alpha = 0$. It is given by

$$q_{n-1}^{(1)}(x; 0, -1) = \frac{(-1)^{n-1}}{(n + 1)_{n-1}} P_{n-1}^{1,0}(-1 - 2x).$$

In general, if $\beta \neq -1, -2, \dots$, then for $n = 1, 2, 3, \dots$,

$$q_{n-1}^{(1)}(x; 0, \beta) = \frac{d}{dx} \mathbf{F}_\beta (q_n(x+t; 0, \beta)),$$

where \mathbf{F}_β is the linear functional (acting on polynomials having t as their independent variable) given by Equation (1.13). In all these cases the convolution associate polynomial sequence $\{q_n^{(1)}(x) | n = 0, 1, 2, 3, \dots\}$ satisfies the convolution three term recursion formula

$$\int_0^x q_{n-1}^{(1)}(t; 0, \beta) dt = q_n^{(1)}(x; 0, \beta) + \frac{\beta}{(\beta + 2n)(\beta + 2n + 2)} q_{n-1}^{(1)}(x; 0, \beta) - \frac{n(\beta + n)}{(\beta + 2n - 1)(\beta + 2n)^2(\beta + 2n + 1)} q_{n-2}^{(1)}(x; 0, \beta),$$

for $n = 1, 2, 3, \dots$

(3.4) *Another Convolution Orthogonal Polynomial Sequence:* For the Jacobi polynomial sequences the following symmetry relation is well known [4, Page 256]

$$P_n^{\alpha,\beta}(-x) = (-1)^n P_n^{\beta,\alpha}(x).$$

Thus,

$$P_n^{\alpha,\beta}(1 - 2x) = (-1)^n P_n^{\beta,\alpha}(2x - 1).$$

If we define the polynomial sequence $\{R_n(x; \alpha, \beta) | n = 0, 1, 2, \dots\}$ by

$$R_n(x; \alpha, \beta) = P_n^{\beta,\alpha}(2x - 1),$$

then Theorem (1.1) is also true when $R_n(x; 0, \beta)$ replaces $Q_n(x; 0, \beta)$.

4. A Characterization. Because $\{P_n^{0,\beta}(1 - 2x) | n = 0, 1, 2, \dots\}$, $\{P_n^{\beta,0}(2x - 1) | n = 0, 1, 2, \dots\}$, and $\{P_n^{1,0}(-2x - 1) | n = 0, 1, 2, \dots\}$, are all orthogonal polynomial sequences with respect to convolution it is of interest to characterize those polynomial sequences $\{P_n^{\alpha,\beta}(ax + b) | n = 0, 1, 2, \dots\}$, which are orthogonal with respect to convolution.

THEOREM (4.1) *If $\{p_n(x) | n = 0, 1, 2, \dots\}$, is any polynomial sequence that is orthogonal with respect to convolution, then for all non-zero real numbers a $\{p_n(ax) | n = 0, 1, 2, \dots\}$ is an orthogonal polynomial sequence with respect to convolution.*

PROOF. Define the linear operator $\eta^b : \mathfrak{R}[x] \rightarrow \mathfrak{R}[x]$ by $\eta^b x^n = (bx)^n$ where $n = 0, 1, 2, \dots$, and note that for all polynomials $p(x)$ and $q(x)$

$$\eta^b(p(x) \text{c} q(x)) = p(ax) \text{c} q(ax).$$

Let $\{p_n(x) | n = 0, 1, 2, \dots\}$ satisfy the convolution orthogonality relation (1.4). Thus for all non-zero real numbers a

$$(\mathbf{L} \circ \eta^{1/a})(p_n(ax) \text{c} p_m(ax)) = k_n \delta_{n,m}$$

and therefore $\{p_n(ax) | n = 0, 1, 2, \dots\}$ is orthogonal with respect to convolution.

Q.E.D.

THEOREM 4.2. *If $\{P_n^{\alpha,\beta}(ax+b) | n = 0, 1, 2, \dots\}$ is an orthogonal polynomial sequence with respect to convolution, then either (i) $b = 1, \alpha = 0$ and $\beta + 1 \neq$ a negative integer or (ii) $b = -1, \beta = 0$ and $\alpha + 1 \neq$ a negative integer.*

PROOF. By integrating both sides of Equation (3.2), we obtain

$$\begin{aligned}
 (4.1) a \int_0^x P_n^{\alpha,\beta}(at+b)dt &= \frac{2(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)} \\
 &\times [P_{n+1}^{\alpha,\beta}(ax+b) - P_{n+1}^{\alpha,\beta}(b)] \\
 &+ \frac{2(\alpha-\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)} [P_n^{\alpha,\beta}(ax+b) - P_n^{\alpha,\beta}(b)] \\
 &- \frac{2(\alpha+n)(\beta+n)}{(\alpha+\beta+n)(\alpha+\beta+2n)(\alpha+\beta+2n+1)} \\
 &\times [P_{n-1}^{\alpha,\beta}(ax+b) - P_{n-1}^{\alpha,\beta}(b)] \\
 &= \frac{2(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)} P_{n+1}^{\alpha,\beta}(ax+b) \\
 &+ \frac{2(\alpha-\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)} P_n^{\alpha,\beta}(ax+b) \\
 &- \frac{2(\alpha+n)(\beta+n)}{(\alpha+\beta+n)(\alpha+\beta+2n)(\alpha+\beta+2n+1)} P_{n-1}^{\alpha,\beta}(ax+b) \\
 &- \frac{2}{\alpha+\beta+n} P_{n+1}^{\alpha-1,\beta-1}(b).
 \end{aligned}$$

Because $\{P_n^{\alpha,\beta}(ax+b)|n = 0, 1, 2, \dots\}$ is an orthogonal polynomial sequence with respect to convolution, therefore it satisfies a convolution three term recursion formula and therefore Equation (4.1) implies that $P_n^{\alpha-1,\beta-1}(b) = 0$ for $n \geq 3$. But it is well known [4, Page 263] that $\{P_n^{\alpha-1,\beta-1}(x)|n = 0, 1, 2, \dots\}$ satisfies the following ordinary three term recursion formula.

$$\begin{aligned}
 &2n(\alpha+\beta+n-2) \\
 &\times (\alpha+\beta+2n-4)P_n^{\alpha-1,\beta-1}(x) = (\alpha+\beta+2n-3) \\
 &\times [(\alpha-1)^2 + (\beta-1)^2 + x(\alpha+\beta+2n-2)(\alpha+\beta+2n-4)] \\
 &\times P_{n-1}^{\alpha-1,\beta-1}(x) - 2(\alpha+n-2)(\beta+n-2)(\alpha+\beta+2n-2) \\
 &\times P_{n-2}^{\alpha-1,\beta-1}(x),
 \end{aligned}$$

for $n = 2, 3, 4, \dots$. By letting $n = 4$ and $x = b$ in this equation we get the following four cases. (i) $\alpha + 2 = 0$, (ii) $\beta + 2 = 0$. (iii) $\alpha + \beta + 6 = 0$ or (iv) $P_2^{\alpha-1,\beta-1}(b) = 0$.

Case (iii) can be eliminated immediately because $\alpha + \beta + 6 = 0 \Rightarrow \alpha + \beta + 1 = -5$ which contradicts Inequality (1.8).

For the case $\alpha = -2$, we know from Equation (4.1) that

$$\frac{(n-2)(\beta+n)}{(\beta+n-2)(\beta+2n-2)(\beta+2n-1)} \neq 0,$$

at $n = 2$. Thus $\beta = 0$ or $\beta = -3$. In either case we get $\alpha + \beta + 1 =$ a negative integer, which contradicts Inequality (1.8). Thus $\alpha \neq -2$. A similar argument shows that $\beta \neq -2$.

In a similar manner, case (iv) $P_2^{\alpha-1,\beta-1}(b) = 0$ implies that (iv a) $\alpha = -1$, (iv b) $\beta = -1$, (iv c) $\alpha + \beta + 4 = 0$, or (iv d) $P_1^{\alpha-1,\beta-1}(b) = 0$. The argument to show that cases (iv a), (iv b) and (iv c) can't occur is the same as the one used above to show that cases (i), (ii) and (iii) can't occur.

Case (iv d) $P_1^{\alpha-1,\beta-1}(b) = 0$ implies that $\alpha = 0$ or $\beta = 0$ or $\alpha + \beta + 2 = 0$. The last case can't occur. Thus the only remaining cases to consider are $\alpha = 0$ or $\beta = 0$.

For $\alpha = 0$ we know that

$$P_n^{-1,\beta-1}(b) = (-1)^n(\beta + n - 1)_n \left(\frac{1 - b}{2}\right)^n = 0,$$

for $n = 3, 4, 5, \dots$. Thus $b = 1$. Similarly, for $\beta = 0$

$$P_n^{\alpha-1,-1}(b) = (-1)^n(\alpha + n - 1)_n \left(\frac{b + 1}{2}\right)^n = 0,$$

for $n = 3, 4, 5, \dots$. Thus, $b = -1$.

Q.E.D.

The following characterization theorem follows directly from Theorem (4.1) and Theorem (4.2).

THEOREM 4.3. $\{P_n^{\alpha,\beta}(ax + b) | n = 0, 1, 2, \dots\}$ is an orthogonal polynomial sequence with respect to convolution if and only if either (i) $b = 1, \alpha = 0$ and $\beta + 1$ is not equal to a negative integer or (ii) $b = -1, \beta = 0$ and $\alpha + 1$ is not equal to a negative integer.

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