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PROBABILITY DENSITY FUNCTION OF THE PRODUCT AND QUOTIENT OF TWO CORRELATED EXPONENTIAL RANDOM VARIABLES

BY

HENRICK J. MALIK AND ROGER TRUDEL

ABSTRACT. This article deals with the distributions of the product and the quotient of two correlated exponential random variables. We consider here three types of bivariate exponential distributions: Marshall–Olkin's bivariate exponential distribution, Gumbel's Type I bivariate exponential distribution, and Gumbel's Type II bivariate exponential distribution.

1. **Introduction**. Multivariate exponential distributions play an important role in studying the reliability of complicated systems since the failure of various components may be correlated. Many workers have investigated various exponential distributions. For a review of the literature on bivariate and multivariate exponential distributions the reader is referred to Johnson and Kotz (1972) and Basu and Block (1975). On setting the index parameters p_1, p_2 and p_3 to unity in Cherian's (1941) bivariate gamma distribution, and $p = 1$ in Kibble's (1941) bivariate gamma distribution, we obtain two corresponding bivariate exponential distributions. Constructing a model where successive damage leads to ultimate failure, Downton (1970) extensively studies in a reliability context a special case of this latter distribution where the correlation must be positive. Marshall and Olkin (1967) develop a bivariate exponential distribution using a two-component system subjected to "shocks" and a suitably defined two-dimensional Poisson process. Gumbel (1960) introduced two types of exponential distributions.

The distributions of the products and quotients of random variables are widely used in many areas of statistical and system analysis. They are met in problems of selection and ranking rules which are described and exhibited by Gulati (1970) and Gupta (1965). They are also found in the context of life testing and in the closely related problem of reliability. They also can occur when the random variables have dimensions of a ratio such as cost of a structure per pound of payload or fuel consumption per mile. Some engineering applications involving products and quotients of random variables are examined by Donahue (1964). In the area of system reliability the products and quotients of random variables are used in determining the system's availability and system's effectiveness. A comprehensive treatment and bibliography of products and quotients of random variables is given by Springer (1979).

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We consider here three bivariate exponential distributions, one introduced by Marshall and Olkin (1967) and two studied by Gumbel (1960). These last two, type I and type II, respectively, belong to Pearson's (1970) and Morgenstern's (1956) families of distributions. Since the exponential distribution is a special case of the generalized gamma distribution, we find the distribution of the product and quotient of two independent variates distributed exponentially as particular cases of those found for the generalized gamma variates by Malik (1967, 1968).

2. **Marshall–Olkin's Bivariate Exponential Distribution**. In this paper we find the distribution of the product and the quotient of two correlated exponential random variables. Consider a two-component system where failure is caused by three types of Poisson "shocks". These mutually independent "shocks' occur at rates λ_1, λ_2 and λ_3 causing failure in the first, the second and both components, respectively. Then Marshall and Olkin (1967) give their bivariate exponential distribution as the joint distribution function of the lives of the two components. The authors also give its multivariate analogue as well as a corresponding multivariate Weibull distribution by performing a simple change of variables. In particular, on transforming the random variable (X, Y) to $(X^{1/\beta}, Y^{1/\gamma})$ in their bivariate exponential density function, we have a bivariate Weibull distribution, namely,

$$H(x, y) = \exp.\ [-\{\lambda_1 x^\beta - \lambda_2 y^\gamma - \lambda_3 \max(x^\beta, y^\gamma)\}] \tag{2.1}$$

where $\lambda_1, \lambda_2, \lambda_3 > 0$ and $x, y > 0$. The Weibull distribution has found wide applications in the industrial field, mainly in life testing problems.

The distribution function of Marshall–Olkin's bivariate exponential distribution is given by

$$F(x, y) = 1 - e^{-\{(\lambda_1+\lambda_3)x\}} - e^{-\{(\lambda_2+\lambda_3)y\}} + e^{-\{\lambda_1 x + \lambda_2 y + \lambda_3 \max(x,y)\}} \tag{2.2}$$

where $x, y > 0$ and $\lambda_1, \lambda_2, \lambda_3 > 0$, or equivalently,

$$\begin{aligned} H(x, y) &= P(X > x, Y > y) \\ &= 1 - F(x) - F(y) + F(x, y) \end{aligned}$$

$$H(x, y) = e^{-\{\lambda_1 x + \lambda_2 y + \lambda_3 \max(x,y)\}} \tag{2.3}$$

with $x, y, \lambda_1, \lambda_2, \lambda_3 > 0$.

2.1 **Distribution of the Product**. If we now make the transformation $U = XY$, $V = Y$ in (2.3) we have the joint distribution of U and V as

$$H(u, v) = \frac{1}{v} e^{-\{\lambda_1 \frac{u}{v} + \lambda_2 v + \lambda_3 \max(\frac{u}{v}, v)\}}. \tag{2.4}$$

When $\max(X, Y) = X$ or $\max(U/V, V) = U/V$, the above equation (2.4) reduces to

$$H(u, v) = \frac{1}{v} e^{-\{(\lambda_1+\lambda_3)\frac{u}{v} + \lambda_2 v\}} \tag{2.5}$$

where $0 < v < \sqrt{u}$, so that the distribution of the product of the variates in (2.3) is given by

$$H_1(u) = \int_0^{\sqrt{u}} \frac{1}{v} e^{-\frac{(\lambda_1 + \lambda_3)u}{v} - \lambda_2 v} \, dv \tag{2.6}$$

with $\lambda_1, \lambda_2, \lambda_3, u > 0$.

Now when $\max(X, Y) = Y$, that is, $\max(U/V, V) = V$, (2.4) becomes

$$H(u, v) = \frac{1}{v} e^{-\{\lambda_1 \frac{u}{v} + (\lambda_2 + \lambda_3)v\}} \tag{2.7}$$

where $0 < \sqrt{u} < v$, and so that the distribution of the product of our variates in (2.3) is obtained as

$$H_2(u) = \int_{\sqrt{u}}^{\infty} \frac{1}{v} e^{-\frac{\lambda_1 u}{v} - (\lambda_2 + \lambda_3)v} \, dv \tag{2.8}$$

with $\lambda_1, \lambda_2, \lambda_3, v > 0$.

Both (2.6) and (2.8) can be evaluated to any desired degree of accuracy from numerical integration techniques once the parameters λ_i ($i = 1, 2, 3$) are specified.

2.2 **Distribution of the Quotient**. Applying the transformation $U = X/Y$, $V = Y$ to (2.3), we get

$$H(u, v) = v \, e^{-\{\lambda_1 uv + \lambda_2 v + \lambda_3 \max(uv, v)\}} \tag{2.9}$$

If $\max(X, Y) = X$ or $\max(UV, V) = UV$, then (2.9) reads

$$H(u, v) = v \, e^{-\{(\lambda_1 + \lambda_3)uv + \lambda_2 v\}} \tag{2.10}$$

with $0 < v < \infty$ and $1 < u < \infty$, so that the distribution of the quotient of the variates in (2.3) using Gradshteyn and Ryzhik (1965; 3.351(3) p. 310) is given as

$$H_1(u) = \frac{1}{[(\lambda_1 + \lambda_3)u + \lambda_2]^2} \tag{2.11}$$

where $u > 1$ and $\lambda_1, \lambda_2, \lambda_3 > 0$.

If now $\max(X, Y) = Y$, that is, $\max(UV, V) = V$, then (2.9) simplifies to

$$H(u, v) = v \, e^{-\{\lambda_1 uv + (\lambda_2 + \lambda_3)v\}} \tag{2.12}$$

with $0 < v < \infty$ and $0 < u < 1$. Using Gradshteyn and Ryzhik (1965; 3.351(3) p. 310), we can now find the respective distribution of the quotient of our variates in (2.3) as

$$H_2(u) = \frac{1}{[\lambda_2 + \lambda_3 + \lambda_1 u]^2} \tag{2.13}$$

where $0 < u < 1$ and $\lambda_1, \lambda_2, \lambda_3 > 0$.

The distribution of the ratio of the variates in our bivariate exponential distribution (2.3) is therefore given by

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(2.14) $$H(u) = \begin{cases} [(\lambda_1 + \lambda_3)u + \lambda_2]^{-2} \text{ if } u > 1 \\ [(\lambda_1 u + \lambda_2 + \lambda_3]^{-2} \text{ if } 0 < u < 1 \end{cases}$$

where $\lambda_1, \lambda_2, \lambda_3 > 0$.

3. **Gumbel's Type I Bivariate Exponential Distribution**. The probability density function of Gumbel's Type I (1960) bivariate exponential distribution is given by

(3.1) $$f(x, y) = \{(1 + ax)(1 + ay) - a\} e^{-x-y-axy}$$

where $x, y > 0$ and $0 \leq a \leq 1$. Its distribution function is then found as

(3.2) $$F(x, y) = 1 - e^{-x} - e^{-y} + e^{-x-y-axy}$$

with $x, y > 0$ and $0 \leq a \leq 1$. If $a = 0$, (3.2) becomes

(3.3) $$\begin{aligned} F(x, y) &= 1 - e^{-x} - e^{-y} + e^{-x-y} \\ &= (1 - e^{-x})(1 - e^{-y}) \end{aligned}$$

and so our random variables X and Y become stochastically independent.

3.1 **Distribution of the product**. In (3.1) we make the transformation $U = XY$, $V = Y$ to obtain the joint distribution of U and V as

(3.4) $$f(u, v) = \frac{1}{v^2}\{(au + v)(av + 1) - av\} e^{-\frac{u}{v} - v - au}$$

With the assistance of Gradshteyn and Ryzhik [(1965); 3.41(9) p. 340 and 8.486(16) p. 970)], we now get the distribution of the product of our exponential variates in (3.1) as the marginal density

(3.5) $$\begin{aligned} h(u) &= e^{-au} \int_0^\infty \left\{a + \frac{a^2 u - a + 1}{v} + \frac{au}{v^2}\right\} e^{-(\frac{u}{v} + v)} \, dv \\ &= 2e^{-au} \{a^2 u - a + 1) K_0(2\sqrt{u}) + 2a\sqrt{u} K_1(2\sqrt{u})] \end{aligned}$$

where $u > 0$ and $0 \leq a \leq 1$ and $K_\nu(z)$ is the modified Bessel function of the second kind.

3.2 **Distribution of the quotient**. Let us apply the transformation $U = X/Y$, $V = Y$ to (3.1), getting

(3.6) $$f(u, v) = v\{(1 + auv)(1 + av) - a\} e^{-uv-v-auv^2}.$$

Using Gradshteyn and Ryzhik [1965; 3.462(1) p. 337], we then find the distribution of the quotient of the variates in Gumbel's Type I bivariate exponential distribution as the marginal of U

$$h(u) = \int_0^\infty v\{(1 - a) + v(au + a) + a^2 uv^2\} e^{-(1+u)v-auv^2} \, dv$$

$$(3.7) \qquad = \frac{(1+u)^2 e^{-8au}}{2au} \left\{ (1-a) D_{-2}\left(\frac{1+u}{\sqrt{2au}}\right) + (1+u)\frac{2a}{\sqrt{u}} D_{-3}\left(\frac{1+u}{\sqrt{2au}}\right) + 3a\, D_{-4}\left(\frac{1+u}{\sqrt{2au}}\right) \right\}$$

where $u > 0$ and $0 \leq a \leq 1$ and $D_p(z)$ is parabolic cylinder function.

4. **Gumbel's Type II Bivariate Exponential Distribution**. Gumbel's Type II (1966) bivariate exponential distribution has the probability density function

$$(4.1) \qquad f(x, y) = e^{-x-y}\{1 + \alpha(2e^{-x} - 1)(2e^{-y} - 1)\}$$

where $x, y > 0$ and $|\alpha| \leq 1$, with the distribution function

$$(4.2) \qquad F(x, y) = (1 - e^{-x})(1 - e^{-y})[1 + \alpha e^{-(x+y)}].$$

Clearly, X and Y are stochastically independent random variables if and only if $\alpha = 0$.

4.1 **Distribution of the Product**. We effect the regular transformation $U = XY$, $V = Y$ in (4.1) to obtain

$$(4.3) \qquad f(u, v) = \frac{1}{v} e^{-\frac{u}{v} - v}\{1 + \alpha(2e^{-\frac{u}{v}} - 1)(2e^{-v} - 1)\}.$$

From Gradshteyn and Ryzhik [(1965); 3.471(9) p. 340], we can now find the distribution of the product of the variates in Gumbel's Type II bivariate exponential as the marginal

$$h(u) = \int_0^\infty \frac{1}{v} e^{-(v+\frac{u}{v})}\{1 + 4\alpha e^{-(v+\frac{u}{v})} - 2\alpha e^{-v} - 2\alpha e^{-\frac{u}{v}} + \alpha\}\, dv$$

$$(4.4) \qquad = 2(1+\alpha) K_0(2\sqrt{u}) + 8\alpha[K_0(4\sqrt{u}) - K_0(2\sqrt{2u})]$$

where $u > 0$ and $|\alpha| \leq 1$ and $K_\nu(z)$ is the modified Bessel function of the second kind.

4.2 **Distribution of the Quotient**. By performing the change of variables $U = X/Y$, $V = Y$ in (4.1), we have

$$(4.5) \qquad f(u, v) = ve^{-uv-v}\{1 + \alpha(2e^{-uv} - 1)(2e^{-v} - 1)\}.$$

In finding the marginal of U, we obtain the distribution of the quotient of the exponential variates in (4.1) as

$$h(u) = \int_0^\infty v\, e^{-v(1+u)}\{1 + \alpha + 4\alpha e^{-v(1+u)} - 2\alpha e^{-v} - 2\alpha e^{-uv}\}\, dv$$

$$= \frac{1+2\alpha}{(1+u)^2} - 2\alpha\left[\frac{1}{(2+u)^2} + \frac{1}{(1+2u)^2}\right]$$

$$(4.6) \qquad = \frac{(1+2u)^2(2+u)^2 + 2\alpha(2u + 11u^2 + 2u^3 - u^4 - 1)}{(1+u)^2(2+u)^2(1+2u)^2}$$

where $u > 0$ and $|\alpha| \leq 1$.

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REFERENCES

1. A. P. Basu and H. W. Block (1975). *On characterizing univariate and multivariate exponential distributions with applications. In a modern course on statistical distribution in scientific work* (G. P. Patil, et al. editors). Reidel Publishing Company, Holland.

2. K. C. Cherian (1941). *A bivariate correlated gamma type distribution function*. Journal of the Indian Mathematical Society, **5**, pp. 133–144.

3. J. D. Donahue (1964). *Products and quotients of random variables and their applications*. Office of Aerospace Research, U.S.A.F.

4. F. Downton (1970). *Bivariate exponential distributions in reliability theory*. Journal of the Royal Statistical Society, Series B, **32**, pp. 408–417.

5. I. S. Gradshteyn and I. M. Ryzhik (1965). *Table of integrals, series, and products, fourth edition*. Academic Press, Inc.: New York, London.

6. L. Gulati (1970). *On the distributions of the products of order statistics*. (Abstract) Annals of mathematical statistics, **41**, p. 1156.

7. E. J. Gumbel (1958). *Statistics of Extremes*. Columbia University Press: New York, London.

8. E. J. Gumbel (1960). *Bivariate exponential distributions*. Journal of the American Statistical Association, **55**, pp. 698–707.

9. S. S. Gupta (1965). *On some multiple decision (selection and ranking) rules*. Technometrics, **7**, pp. 225–245.

10. N. L. Johnson and S. Kotz (1972). *Distributions in statistics: Continuous multivariate distributions*. John Wiley & Sons, New York.

11. W. F. Kibble (1941). *A two-variate gamma type distribution*. Sankhya, **5**, pp. 137–150.

12. Z. A. Lomnicki (1967). *On the distribution of products of random variables*. Journal of the Royal Statistical Society, Series B, **29**, pp. 513–524.

13. H. J. Malik (1967). *Exact distribution of the quotient of independent generalized gamma random variates*. Canadian Mathematical Bulletin, **10**, pp. 463–465.

14. H. J. Malik (1968). *Exact distribution of the product of two generalized gamma variates*. Annals of Mathematical Statistics, **39**, pp. 1751–1752.

15. K. V. Mardia (1970). *Families of Bivariate Distributions*. Hafner Publishing Company: Darien, Connecticut.

16. A. W. Marshall and I. Olkin (1967). *A multivariate exponential distribution*. Journal of the American Statistical Association, **62**, pp. 30–44.

17. D. Morgenstern (1956). *Einfache Beispiele zweidimensionaler Vertelungen*. Miteilungsblatt Fur Mathematische Statistik, **8**, pp. 234–235.

18. M. D. Springer and W. E. Thompson (1964). *The distribution of products of independent random variables*. General Motors Defense Research Laboratories, Mathematics and Evaluation Studies Department: Santa Barbara, California.

19. M. D. Springer and W. E. Thompson (1966). *The distribution of product of independent random variables*. S.I.A.M. Journal of Applied Mathematics, **14**, pp. 511–526.

20. M. D. Springer (1979). *The Algebra of Random Variables*. John Wiley and Sons, New York.

21. E. W. Stacy and G. A. Mihram (1965). *Parameter estimation for a generalized gamma distribution*. Technometrics, **7**, pp. 349–358.

UNIVERSITY OF GUELPH
GUELPH, ONTARIO

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