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STRUCTURE INDUCED ON SUBSETS OF A LATTICE VIA INFORMATION GATHERED FROM TENSOR PRODUCTS

BRENT SMITH

ABSTRACT. Suppose that a subset of a finite dimensional lattice has the property that there are many orthogonal tensor products that are almost 1 on the set, then the set is forced to have unusual concentrations of points in small cartesian products.

Denote by $\mathcal{L}_s$ the integer lattice $[s, 2s)^2$. Let f, g be functions defined on the integer interval $[s, 2s)$. Define the tensor product $f \otimes g$ on $\mathcal{L}_s$ by $(f \otimes g)(x, y) = f(x) \cdot g(y)$.

Let R be a subset of $\mathcal{L}_s$ having r points on each row and column. We will also consider R to be a 0, 1 matrix indexed by $\{s, s+1, \ldots, 2s-1\}^2$.

We say R is β tensor empty under the following circumstances. Consider the vectors in $\mathbb{C}^s$ whose y coordinate for $y = s, \ldots, 2s-1$ is

$$\sigma_x(y) = \begin{cases} 1 & \text{if } (x, y) \in R \\ 0 & \text{otherwise.} \end{cases}$$

Put $\breve{\sigma}_x = \sigma_x - \frac{r}{s}(1, \ldots, 1)$; that is, σ_x without its component in the constant direction. R is "β tensor empty" if

$$\Big\| \sum_x \alpha(x) \breve{\sigma}_x \Big\|_2^2 \leq r^{1-\beta} \sum_x \|\breve{\sigma}_x\|_2^2$$

whenever $\|\alpha\|_2^2 = s$. Ivo Klemes showed that a similar concept was not empty in [2].

When we view R as 0, 1 matrix we will call it σ. The operator norm of σ from s dimensional complex space, $\mathbb{C}^s$, to $\mathbb{C}^s$ is exactly r.

$$\|\sigma\vec{\alpha}\|_2 \leq r\|\vec{\alpha}\|_2, \quad \vec{\alpha} \in \mathbb{C}^s$$

and

$$\sigma\vec{1} = r\vec{1},$$

where $\vec{1}$ is the constant vector having all entries equal to 1.

We use this refinement of the notion of tensor empty. We use "the singular value decomposition theorem," [1], from linear algebra explained in the appendix. We fix an eigentensor expansion. σ is

(1) β tensor empty with fewer than B flat exceptions

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if it satisfies the property that there are at most B eigentensors that have their corresponding singular value modulus larger than $r^{1-\beta}$, and that have the first component u of the eigentensor $u \otimes v$ satisfying the flatness condition:

$$||u||_\infty \leq 10\frac{1}{\sqrt{s}} \cdot ||u||_2,$$

where we are computing norms in $l^\infty(\{s, s+1, \ldots, 2s-1\})$, and $l^2(\{s, s+1, \ldots, 2s-1\})$.

The introduction of the concept of "exceptions" was motivated by examples of non-tensor empty sets that were still "evenly distributed" that were constructed by Ivo Klemes in 1989.

TENSOR STRUCTURE THEOREM. *For any $0 < \delta, \epsilon < 1$, there is some $\beta = \beta(\delta, \epsilon) > 0$ such that for a sufficiently large s and for any $1 \leq B \leq s$, σ is not β tensor empty with fewer than B flat exceptions implies that there must be some m by n submatrix X of σ (this means that X is the intersection of some m rows and some n columns of σ) such that:*

1. Each row of X has at least $r^{1-\epsilon}$ entries equal to 1,

2. We have

$$m \leq \frac{s}{B} r^{.16\epsilon} \text{ and } \frac{n}{m} = 10^4 r^{\frac{6\epsilon}{198}}.$$

We make some remarks on the theorem. This gives a higher density of points on each row of X than the original $\frac{r}{s}$ as soon as $B > r^{1.16 \cdot \epsilon}$. Although X is not a square matrix, it is not too degenerate a rectangle, in the sense that the row length n is at least $r^{1-\epsilon}$ because of 1., and the ratio $\frac{n}{m}$ is $10^4 r^{\frac{6\epsilon}{198}}$.

From σ we form a new object $\sigma^{(2)} = \sigma^* \sigma$ where * means conjugate transpose and we mean standard matrix composition. Hence, $\sigma^* \sigma$ is the matrix that puts the integer $\sigma_x \cdot \sigma_y$ at the position (x, y) where σ_x is used to denote a column vector of σ. σ_1 has l^1 norm r^2 for each row and column. Let us consider two s dimensional vectors u, v having 2-norm^2 $= s$. We will view σ both as a matrix and as a function of two variables defined on the lattice $\mathcal{L}_s$.

This means

$$\sum_{\mathcal{L}_s} (u \otimes \bar{v}) \cdot \sigma = \langle \sigma u, v \rangle.$$

We have that $|\sum (u \otimes \bar{v}) \cdot \sigma| \geq \kappa ||\sigma||_1$ implies $\sum (u \otimes \bar{u}) \cdot \sigma^{(2)} \geq \kappa^2 ||\sigma^{(2)}||_1$. This is true because

$$\sum (u \otimes \bar{u}) \cdot \sigma^{(2)} = ||\sigma u||_2^2.$$

Suppose α satisfies $||\alpha||_2^2 = s$ and

$$||\sigma \alpha||^2 \geq \kappa^2 r \sum ||\sigma_x||_2^2.$$

Then there is an s dimensional vector β with $||\beta||_2^2 = s$ such that

$$\left| \sum (\alpha \otimes \bar{\beta}) \cdot \sigma \right| \geq \kappa ||\sigma||_1.$$

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simply put

$$\beta = \sqrt{s}\frac{\sigma\alpha}{\|\sigma\alpha\|_2}.$$

Then

$$\begin{aligned}\Big|\sum_{x,y}(\alpha\otimes\bar{\beta})(x,y)\cdot\sigma(x,y)\Big| &= |\langle\sigma\alpha,\beta\rangle| \\ &= \sqrt{s}\frac{\langle\sigma\alpha,\sigma\alpha\rangle}{\|\sigma\alpha\|_2} \\ &= \sqrt{s}\|\sigma\alpha\|_2 \geq \kappa(sr)^{\frac{1}{2}}\Big(\sum\|\sigma_x\|_2^2\Big)^{\frac{1}{2}} \\ &= \kappa sr = \kappa\|\sigma\|_1.\end{aligned}$$

And repeating this argument

$$\Big(\sum u\otimes\bar{u}\Big)\cdot\sigma^{(2)} \geq \kappa^{2^2}\|\sigma^{(2)}\|_1 \text{ implies } \Big(\sum u\otimes\bar{u}\Big)\cdot\sigma^{(3)} \geq \kappa^{2^3}\|\sigma^{(3)}\|_1 \tag{2}$$

where $\sigma^{(3)} = \sigma^{(2)*}\sigma^{(2)}$. The power κ^{2^2} in (2) is called "the correctness" for $\sigma^{(2)}$; the power κ^{2^3} in (2) is called the correctness for $\sigma^{(3)}$.

We continue this process k steps to arrive at r^{2^k} l^1 norm on each row and column with κ^{2^k} correctness in regards to $u\otimes\bar{u}$. Suppose $r = s^\delta$ and $\kappa = r^{-\frac{\epsilon}{100}\delta^{\frac{100}{\delta\epsilon}}}$. If we put $2^k = (\frac{1}{\delta})^{\frac{100}{\epsilon}}$, then

$$\kappa^{2^k} \geq r^{-\frac{\epsilon}{100}}.$$

At the k-th step

$$\sum(u\otimes\bar{u})\cdot\sigma^{(k)} \geq \kappa^{2^k}\cdot\|\sigma^{(k)}\|_1; \tag{3}$$

we call this κ^{2^k} correctness. And there is r^{2^k} mass on each row and column.

Let ${}_k\sigma_x$ denote the x-th column vector of $\sigma^{(k)}$. And we use this normalization $r^{-2^k}{}_k\sigma_x = {}_kc_x$. A crucial observation is that $\|{}_kc_{x1}+\cdots+{}_kc_{xJ}\|_2^2$ is a decreasing function of k. We see this by looking at the situation from k to $k+1$. We may think of ${}_kc_{x1}+\cdots+{}_kc_{xJ} = {}_k\Theta_0$ as a sum of 1-step step functions with decreasing supports under inclusion written $C_1+\cdots+C_l$. In other words upon putting $m = \min\{{}_k\Theta_0(i) : {}_k\Theta_0(i)\neq 0\}$

$$C_1(j) = \begin{cases} m & \text{if } {}_k\Theta_0(j)\neq 0 \\ 0 & \text{otherwise.}\end{cases}$$

Put ${}_k\Theta_1 = {}_k\Theta_0 - C_1$ and repeat the above in regards to the new Θ to get C_2.

Let $\tilde{C}_1$ be the image of C_1 in ${}_{k+1}c_{x1}+\cdots+{}_{k+1}c_{xJ}$. That is,

$$\tilde{C}_1(x) = C_1\cdot{}_kc_x. \tag{4}$$

We view $\tilde{}$ as a map from $\mathbb{C}^s\longrightarrow\mathbb{C}^s$, and the dependence of $\tilde{}$ on k will be implicit and understood by context. We first check that $\max\tilde{C}_1 \leq \max C_1$, because this implies that for all i,j we have $\tilde{C}_i\cdot\tilde{C}_j \leq C_i\cdot C_j$. Recall that there is exactly r^{2^k} mass on each row and column of R_k. A fixed column ${}_k\sigma_x$ inner producted against C_1 can give $\max C_1$ r^{2^k}

if and only if its support is completely contained within that of C_1. This is the only way possible for $\tilde{C}_1$ to take on the value $\max C$. Recall that $\operatorname{supp} C_i \supset \operatorname{supp} C_j$ for $i \leq j$. And so

$$\begin{aligned} C_i \cdot C_j &= \max C_i \cdot \|C_j\|_1 \geq \max \tilde{C}_i \cdot \|C_j\|_1 \\ &= \max \tilde{C}_i \cdot \|\tilde{C}_j\|_1 \geq \tilde{C}_i \cdot \tilde{C}_j. \end{aligned}$$

And this proves that the 2 norms are decreasing.

Now let us restrict our interest to ${}_k c_{x_0}$ where x_0 is fixed. When $k = H \log_2 \log_r s$ in $\sigma^{(k)}$ we can still maintain correctness $r^{-\frac{\epsilon}{100}}$ if the initial correctness was chosen to be $\kappa = r^{-\frac{\epsilon}{100} \cdot \delta^H}$. This means simply that

$$\kappa^{2^k} \geq r^{\frac{-\epsilon}{100}}.$$

We refer to $[s^{-1}, r^{-1}]$ as "the possibility interval" for $\|{}_k c_{x_0}\|_2^2$ because it is the case that

$$\|{}_k c_{x_0}\|_2^2 \in [s^{-1}, r^{-1}].$$

We explain the left hand end point s^{-1}.

$$\|{}_k c_{x_0}\|_2 = \sup_{t, \|t\|_2 = 1} \langle {}_k c_{x_0}, t \rangle \geq \left\langle {}_k c_{x_0}, \left(\frac{1}{\sqrt{s}}, \ldots, \frac{1}{\sqrt{s}} \right) \right\rangle$$

and this is $\|{}_k c_{x_0}\|_1 \cdot \frac{1}{\sqrt{s}} = \frac{1}{\sqrt{s}}$. The right hand end point is due to the decreasing 2-norm[2] and to the initial calculation

$$\|{}_0 c_{x_0}\|_2^2 = r^{-1}.$$

We make two claims.

CLAIM 1. Let $h_0 = \log_2 \log_r s$. and

$$h = H \cdot h_0, \tag{5}$$

where $H = \frac{100}{\delta \epsilon}$.

We show there is $k \leq h$ such that

$$\|{}_{k+1} c_{x_0}\|_2^2 \geq r^{-\frac{\epsilon}{100}} \|{}_k c_{x_0}\|_2^2.$$

CLAIM 2. There is $x_0 \in \{s, \ldots, 2s-1\}$ such that

$$\|{}_h c_{x_0}\|_2^2 \geq \frac{B}{s} r^{-\frac{8\epsilon}{100}}.$$

We begin with the first claim. Put $y_k = \log_2(-\log_r \|{}_k c_{x_0}\|_2^2)$. We know y_k is increasing because $\|{}_k c_{x_0}\|_2^2$ is decreasing. Because

$$\|{}_k c_{x_0}\|_2^2 \in [s^{-1}, r^{-1}].$$

we also know $0 \leq y_k \leq h_0$. We claim there is y_k such that $y_{k+1} - y_k \leq \frac{1}{H}$. Suppose not, then for all $k = 0, 1, \ldots, h$, we have $y_{k+1} - y_k > \frac{1}{H}$, and this implies

$$\sum_{i=0}^{h} y_i - y_{i-1} > \frac{h}{H} = \frac{h_0 H}{H} = h_0,$$

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and this contradicts the lower bound estimate for these 2-norm^2. Let $\tilde{H} \geq H$ satisfy $y_{k+1} - y_k = \frac{1}{\tilde{H}}$. Hence

$$\log_2(-\log_r ||_{k+1}c_{x_0}||_2^2) = \log_2(-\log_r ||_k c_{x_0}||_2^2) + \frac{1}{\tilde{H}}.$$

Exponientiating twice then gives

$$||_{k+1}c_{x_0}||_2^2 = (||_k c_{x_0}||_2^2)^{2^{\frac{1}{\tilde{H}}}}.$$

This is then the proof of Claim 1. Because $H = \frac{100}{\delta\epsilon}$ and

$$||_k c_{x_0}||_2^2 \geq s^{-1},$$

we have

$$||_{k+1}c_{x_0}||_2^2 \geq r^{-\frac{\epsilon}{100}}(||_k c_{x_0}||_2^2).$$

Thus, the 2-norm^2 of column x normalized has remained almost constant here.

We want to compute correctness at stage h. We want to see if $\kappa^2 = r^{-\epsilon'}$, then

$$\kappa^{2^h} = r^{-\frac{\epsilon}{100}}.$$

Now from (5) we know $h = \frac{100}{\delta\epsilon}\log_2\log_r s$. We want

$$\kappa^{2^{\log_2 \delta^{\frac{-100}{\delta\epsilon}}}} = r^{-\frac{\epsilon}{100}}.$$

That is

$$\kappa^{\delta^{-\frac{100}{\delta\epsilon}}} = r^{-\frac{\epsilon}{100}}.$$

The conclusion is that

$$\kappa = r^{-\frac{\epsilon}{100}\cdot\delta^{\frac{100}{\delta\epsilon}}}.$$

And we know

$$r^{-\epsilon'} = \kappa^2 = r^{-2\frac{\epsilon}{100}\cdot\delta^{\frac{100}{\delta\epsilon}}}.$$

And this demonstrates the choice of

$$\epsilon' = 2\frac{\epsilon}{100}\cdot\delta^{\frac{100}{\delta\epsilon}}.$$

Now we prove Claim 2. We list the first components of the B exceptional eigentensors given in the definition (1) as $u_1, u_2, \ldots, u_B$.

A first remark is that it must be the case that there exists $x \in \{s, \ldots, 2s-1\}$ such that

$$|\langle \bar{u}_1(x)u_1 + \cdots + \bar{u}_B(x)u_B, {}_h\sigma_x\rangle| \geq r^{-\frac{\epsilon}{100}}Br^{2^h}.$$

This remark is a consequence of our iteration of (1) which stated that we had these B flat exceptions, and so we know exactly that

$$|\langle \sigma^{(h)}u_1, u_1\rangle + \langle \sigma^{(h)}u_2, u_2\rangle + \cdots + \langle \sigma^{(h)}u_B, u_B\rangle| \geq s\cdot r^{-\frac{\epsilon}{100}}Br^{2^h}.$$

If we write our matrix composition as a sum of column applications we get that our previous statement must happen on average in x.

We can use the linear combination $\bar{u}_1(x)u_1 + \cdots + \bar{u}_B(x)u_B$ of these vectors to show for the chosen x there is a fixed set of cardinality at most $\frac{s}{B}r^{4\frac{\epsilon}{100}}$ that must contain at least $r^{-2\frac{\epsilon}{100}}$ of the l^1 norm of ${}_hc_x$ where $h = H\log_2\log_r s$; this is done via an application of Chebyshev:

$$\text{Cardinality}\Big\{y : \Big|\sum_{i=1,2,\ldots,B} \bar{u}_i(x)u_i\Big|(y) > Br^{-2\frac{\epsilon}{100}}\Big\}\cdot B^2r^{-4\frac{\epsilon}{100}} \leq \Big\|\sum \bar{u}_i(x)u_i\Big\|_2^2 = s\cdot\sum |u_i(x)|^2.$$

And because of the flatness assumption in (1), the last expression is bounded by $10Bs$. Put $F = \{y : |\sum_{i=1,2,\ldots,B}\bar{u}_i(x)u_i|(y) > Br^{-2\frac{\epsilon}{100}}\}$. Put $U = \bar{u}_1(x)u_1 + \cdots + \bar{u}_B(x)u_B$. We wish to see that

$$\|{}_hc_x|_F\|_1 > r^{-2\frac{\epsilon}{100}}.$$

We suppose not and obtain a contradiction.

$$\begin{aligned} r^{-\frac{\epsilon}{100}}B \leq |\langle U, {}_hc_x\rangle| &= |\langle U|_F, {}_hc_x|_F\rangle + \langle U|_{F^c}, {}_hc_x|_{F^c}\rangle| \\ &\leq B\cdot \|{}_hc_x|_F\|_1 + Br^{-2\frac{\epsilon}{100}}\cdot 1 \leq Br^{-2\frac{\epsilon}{100}} + Br^{-2\frac{\epsilon}{100}}. \end{aligned}$$

We estimated the two inner products above using supremum norm times 1-norm. Now we want to test the 2-norm of ${}_hc_x$ with χ_F.

$$\|{}_hc_x\|_2 \geq \left\langle \frac{\chi_F}{|F|^{1/2}}, {}_hc_x\right\rangle \geq r^{-2\frac{\epsilon}{100}}\frac{1}{|F|^{1/2}}.$$

We then square both sides to get Claim 2.

Now we want to find E with $|E| < \frac{s}{B}\cdot r^{20\frac{\epsilon}{100}}$ and such that $\|{}_kc_x|_{E^c}\|_2^2$ is small. If we think of our column vector ${}_kc_x$ as a function of the row index y, then it has a finite range because the domain of rows is finite. Write ${}_kc_x = T_1 + \cdots + T_{\tau-1} + T_\tau + \cdots + T_t$ as a decomposition into 1-step step functions with increasing supports. Thus T_1 is supported where ${}_kc_x$ takes on its maximal value and to each member of its support assigns this maximal value. Then T_2 is defined in the analogous manner for ${}_kc_x - T_1$. Suppose that T_τ is the first step function with support cardinality $> \frac{s}{B}\cdot r^{20\frac{\epsilon}{100}}$, and put $E = \mathrm{supp}(T_{\tau-1})$. We will prove that E works by using:

$$\|{}_kc_x|_{E^c}\|_2^2 \leq \|T_\tau + \cdots + T_t\|_2^2.$$

We note that $\max T_\tau \leq \frac{B}{s}\cdot r^{-20\frac{\epsilon}{100}}$, because if $|\mathrm{supp}(T_\tau)| = n$, and $\max T_\tau = m$. Then $n\cdot m \leq 1$, and we know $n > \frac{s}{B}\cdot r^{20\frac{\epsilon}{100}}$.

$$\langle T_\tau + \cdots + T_t, T_\tau + \cdots + T_t\rangle \leq \|\cdot\|_\infty\|\cdot\|_1.$$

And

$$\begin{aligned} \max(T_\tau + \cdots + T_t) &= \max(T_\tau) + \cdots + \max(T_t) \\ &= \frac{\|T_\tau\|_1}{|\mathrm{supp}(T_\tau)|} + \cdots + \frac{\|T_t\|_1}{|\mathrm{supp}(T_t)|} \\ &\leq \frac{1}{\frac{s}{B}\cdot r^{20\frac{\epsilon}{100}}}. \end{aligned}$$

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Hence,

$$\|{}_kc_x|_{E^c}\|_2^2 \le \frac{B}{s} \cdot r^{-20\frac{\epsilon}{100}}.$$

Therefore,

$$\|{}_kc_x|_E\|_2 \ge \|{}_kc_x\|_2 - \|{}_kc_x|_{E^c}\|_2 \ge \sqrt{\frac{B}{s}} \cdot r^{-4\frac{\epsilon}{100}} - \sqrt{\frac{B}{s}} \cdot r^{-10\frac{\epsilon}{100}} \ge r^{-5\frac{\epsilon}{100}} \cdot \sqrt{\frac{B}{s}}.$$

So

$$\|{}_kc_x|_{E^c}\|_2 \le r^{-5\frac{\epsilon}{100}} \cdot \|{}_kc_x|_E\|_2.$$

We wish to verify a Claim 1 analogous statement involving a restriction to E. Note that because

$$\|({}_kc_x|_{E^c})\tilde{}\|_2^2 \le \|{}_kc_x|_{E^c}\|_2^2,$$

we have also that

$$\|({}_kc_x|_{E^c})\tilde{}\|_2 \le r^{-5\frac{\epsilon}{100}} \cdot \|{}_kc_x|_E\|_2.$$

We now prove the analogue.

$$\begin{aligned}
\|({}_kc_x|_E)\tilde{}\|_2 &= \|({}_kc_x|_E)\tilde{} + ({}_kc_x|_{E^c})\tilde{} - ({}_kc_x|_{E^c})\tilde{}\|_2 \ge \|({}_kc_x)\tilde{}\|_2 - \|({}_kc_x|_{E^c})\tilde{}\|_2 \\
&\ge r^{-\frac{\epsilon}{200}} \|({}_kc_x)\|_2 - r^{-5\frac{\epsilon}{100}} \cdot \|{}_kc_x|_E\|_2 \\
&\ge r^{-\frac{\epsilon}{200}} (1 - r^{-5\frac{\epsilon}{100}}) \cdot \|{}_kc_x|_E\|_2 - r^{-5\frac{\epsilon}{100}} \cdot \|{}_kc_x|_E\|_2 \\
&= \left(r^{-\frac{\epsilon}{200}} (1 - r^{-5\frac{\epsilon}{100}}) - r^{-5\frac{\epsilon}{100}}\right) \cdot \|{}_kc_x|_E\|_2 \ge r^{-\frac{\epsilon}{199}} \|{}_kc_x|_E\|_2.
\end{aligned}$$

Our next objective will be to make another analogous statement that replaces ${}_kc_x|_E$ by a characteristic function that is supported within E.

Recall the definition of $\tilde{}$ from (4). If we fix a function P and $1 \ge \theta > 0$ satisfies

$$\|\tilde{P}\|_2^2 \ge \theta \cdot \|P\|_2^2,$$

then we say that P was θ preserved; we use θ generically in what follows and note that there will be slight decreases in its value at each stage. The starting choice will be

$$\theta_1 = r^{-\frac{2\epsilon}{199}}. \tag{6}$$

We argue the θ preservation of a characteristic function supported within E. Write ${}_kc_x|_E = d_1 + d_2 + \cdots + d_D + g$ where each d_i is a dyadic step function meaning that it is a sum of 1-step step functions having heights exactly equal to a power of 2, and where

$$\|d_{i+1}\|_2^2 \le 2^{-2}\|d_i\|_2^2.$$

Put

$$F_1 = \operatorname{supp}_k c_x|_E - \{y :_k c_x|_E(y) \le s^{-2}\}$$

define d_1 as follows: For $y \in F_1$

$$d_1(y) = \max\{2^i : 2^i \le_k c_x|_E(y)\}.$$

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And for $y \notin F_1$ put

$$d_1(y) = 0.$$

Replacing ${}_k c_x|_E$ by ${}_k c_x|_E - d_1(y)$ we repeat the same procedure to get $d_2(y)$. In particular the new F_2 would be

$$F_2 = \operatorname{supp}\big({}_k c_x|_E - d_1(y)\big) - \{y :_k c_x|_E(y) - d_1(y) \leq s^{-2}\}.$$

And where the garbage g is built from the smaller than s^{-2} values. The g has 2-norm smaller than $s^{-\frac{3}{2}}$, because we are working in $l^2(\{s, s+1, \ldots, 2s-1\})$, and allows us to assume that the number of 1-step step functions involved in a dyadic building block d_i has cardinality smaller than $2 \cdot (\log_2 s)$. And that $D \leq 2 \cdot (\log_2 s)$. We claim one of the d_i is "θ preserved". d_i is "θ-preserved" means that

$$\|\tilde{d}_i\|_2^2 \geq \theta \|d_i\|_2^2.$$

We are using θ in a generic sense to avoid notational complexity; it becomes slightly smaller at each step of the argument. Suppose not. Then

$$\begin{aligned}
\|({}_k c_{x_0}|_E)\tilde{}\|_2 &\leq \sum_{i=1,2,\ldots,D} \|\tilde{d}_i\|_2 + \|\tilde{g}\|_2 \leq \sum_{i=1,2,\ldots,D} \sqrt{\theta} \cdot \|d_i\|_2 + \|g\|_2 \\
&\leq \sqrt{\theta} \sum \Big(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\Big) \|d_1\|_2 + s^{-\frac{3}{2}} \\
&\leq \sqrt{\theta} \cdot 3 \cdot \|{}_k c_{x_0}|_E\|_2.
\end{aligned}$$

So we can assume that d_1, say, is

$$\theta_2 = \frac{\theta_1}{9} \tag{7}$$

preserved. Write $d_1 = P_1 + \cdots + P_q$ where each P is a 1-step step function and we know $q \leq 2 \cdot (\log_2 s)$, and where each P is supported within E. Recall that: For $y \in F_1$

$$d_1(y) = \max\{2^i : 2^i \leq_k c_x|_E(y)\}.$$

And for $y \notin F_1$ put

$$d_1(y) = 0.$$

Let

$$\{2^{i_1}, 2^{i_2}, \ldots, 2^{i_n}\}$$

be the list of powers of 2 used in this expression to define d_1, and let

$$Z_1, Z_2, \ldots, Z_n$$

be the sets on which d_1 takes on the given power of 2. Then

$$P_j = 2^{i_j} \chi_{Z_j}.$$

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We want to show that one of these $P's$ is at least θ preserved. Suppose not.

$$\begin{aligned}\|\tilde{d}_1\|_2 &\leq \|\tilde{P}_1\|_2 + \cdots + \|\tilde{P}_q\|_2 \\ &\leq \sqrt{\theta}\|P_1\|_2 + \cdots + \sqrt{\theta}\|P_q\|_2 \\ &\leq \sqrt{\theta} \cdot q \cdot \max \|P_q\|_2 \leq 2\sqrt{\theta} \log s \cdot \|d_1\|_2.\end{aligned}$$

Hence, we get

$$4\theta_3(\log s)^2 \geq \theta_2,$$

and so we take

$$\theta_3 = \frac{\theta_2}{4(\log s)^2} = \frac{\theta_1}{36(\log s)^2}. \tag{8}$$

We have now reached our major objective that there is a characteristic function χ with $\operatorname{supp} \chi \subset E$ where

$$|E| \leq \frac{s}{B} r^{20\frac{\epsilon}{100}} \tag{9}$$

and

$$\|\tilde{\chi}\|_2^2 \geq \theta_3 \|\chi\|_2^2 = \frac{1}{36 \cdot (\log_2 s)^2} \theta_1 \|\chi\|_2^2.$$

Thus,

$$\theta_3 \geq r^{-\frac{2\epsilon}{198}} \tag{10}$$

because of (6) when the dimensions of the lattice become sufficiently large. Now to arrive at the theorem statement we need to unravel the information contained in the above equation.

Put $|\operatorname{supp} \chi| = l$. From (9) we know

$$l \leq \frac{s}{B} r^{20\frac{\epsilon}{100}}. \tag{11}$$

We claim there is an indexing set D of $l\frac{\theta_3}{10}$ column vectors satisfing

$$\langle {}_k c_z, \chi \rangle \geq \frac{\theta_3}{10}$$

for $z \in D$. D will be the choice of our some n columns. We suppose not and obtain a contradiction. Let

$$D = \left\{ z : \langle {}_k c_z, \chi \rangle \geq \frac{\theta_3}{10} \right\},$$

and so we are assuming that $|D| < \frac{\theta_3}{10} \cdot l$. Then

$$\|\tilde{\chi}\|_2^2 = \|\tilde{\chi}|_D\|_2^2 + \|\tilde{\chi}|_{D^c}\|_2^2 \leq |D| + \frac{\theta_3}{10} \cdot l < \frac{\theta_3}{10} \cdot l + \frac{\theta_3}{10} \cdot l,$$

which is a contradiction. The estimates come from bounding the inner product by supremum norm times 1-norm.

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Next we would like to estimate $\|\sum_{z\in D} {}_kc_z\|_2$.

$$\Big\langle \sum_{z\in D} {}_kc_z, \frac{\chi}{l^{1/2}}\Big\rangle \geq |D|\cdot\frac{\theta_3}{10}\cdot\frac{1}{l^{1/2}} = \frac{\theta_3^2}{10^2}\cdot l^{1/2}.$$

And hence

$$\Big\|\sum_{z\in D} {}_kc_z\Big\|_2^2 \geq l\frac{\theta_3^4}{10^4}.$$

Now because $\|\cdot\|_2^2$ decreases we obtain in regard to the original R:

$$\Big\|\sum_{z\in D} {}_0c_z\Big\|_2^2 \geq l\frac{\theta_3^4}{10^4}.$$

Put

$$D_1 = \Big\{y : \sum_{z\in D} {}_0c_z(y) \geq \frac{\theta_3^4}{10^4}\frac{1}{10}\Big\}. \tag{12}$$

D_1 will be the indexing set of our m rows. We claim that $|D_1| > \frac{\theta_3^4}{10^4}\frac{1}{10}l$. Again we argue the contrapositive by assuming this is not true.

$$\Big\|\sum_{z\in D} {}_0c_z\Big\|_2^2 = \Big\|\sum_{z\in D} {}_0c_z|_{D_1}\Big\|_2^2 + \Big\|\sum_{z\in D} {}_0c_z|_{D_1^c}\Big\|_2^2 \leq \frac{\theta_3^4}{10^4}\frac{l}{10} + \frac{\theta_3^4}{10^4}\frac{|D|}{10}.$$

And this last expression computes to

$$\frac{\theta_3^4}{10^4}\frac{l}{10} + \frac{\theta_3^4}{10^5}\frac{\theta_3}{10}\cdot l.$$

Now we can get the theorem. Choose D_1' to be a subset of D_1 having cardinality $\frac{\theta_3^4}{10^4}\frac{1}{10}\cdot l$. Then put $X = D\times D_1'$. The number of points in X after we convert back to the original $\sigma's$ is $|D_1|\cdot\frac{\theta_3^4}{10^4}\frac{1}{10}\cdot r$ which is

$$\Big(\frac{\theta_3^4}{10^4}\frac{1}{10}\Big)^2\cdot r\cdot l \text{ points in a } \frac{\theta_3}{10}\cdot l \text{ columns by } \frac{\theta_3^4}{10^4}\frac{1}{10}\cdot l \text{ rows rectangle.} \tag{13}$$

Recalling from (10) that θ_3 can be taken as

$$r^{-\frac{2\epsilon}{198}}$$

we compute the numbers of (13) as follows:

$$\Big(\frac{r^{-\frac{8\epsilon}{198}}}{10^4}\frac{1}{10}\Big)^2\cdot r\cdot l \text{ points in a } \frac{r^{-\frac{2\epsilon}{198}}}{10}\cdot l \text{ columns by } \frac{r^{-\frac{8\epsilon}{198}}}{10^4}\frac{1}{10}\cdot l \text{ rows rectangle.} \tag{14}$$

And from (11)

$$l \leq \frac{s}{B}r^{20\frac{\epsilon}{100}}.$$

We want to see that we have satisfied the 2 statements of the theorem for large s.

1. Each row of X has at least $r^{1-\epsilon}$ entries equal to 1,
2. We have

$$m \leq \frac{s}{B} r^{.16\epsilon} \text{ and } \frac{n}{m} = 10^4 r^{\frac{6\epsilon}{198}}.$$

To check 1 we note from (12) that each row has at least

$$\frac{\theta_3^4}{10^4} \frac{1}{10} r = \frac{r^{-\frac{8\epsilon}{198}}}{10^4} \frac{1}{10} r$$

points. And this last is at least $r^{1-\epsilon}$ for large r.

To check 2a. we see that

$$m \leq \frac{r^{-\frac{8\epsilon}{198}}}{10^4} \frac{1}{10} \cdot \frac{s}{B} r^{20\frac{\epsilon}{100}} \leq \frac{s}{B} r^{(.2-.04)\epsilon} \leq \frac{s}{B} r^{.16\epsilon}$$

for large s.

And finally checking 2b. we get

$$\frac{n}{m} = \frac{10^4}{\theta_3^3} = 10^4 r^{\frac{6\epsilon}{198}}.$$

Roozbeh Vakil is responsible for eliminating many extraneous parameters that were in an earlier version of this paper.

APPENDIX: EIGENTENSORS. We present some useful linear algebra that can be found in [1] under the topic of the singular value decomposition.

Suppose we have a matrix, σ, that has dimension C. Let $r_1, r_2, \ldots, r_C$ be a collection of orthonormal functions: $\{1, 2, \ldots, C\} \longrightarrow \mathbb{C}$. Then there exist normal (meaning they have 2-norm^2 = 1) vectors $s_1, s_2, \ldots, s_C$ such that

$$\sigma = a_1(r_1 \otimes \bar{s}_1) + \cdots + a_C(r_C \otimes \bar{s}_C)$$

is an orthonormal expansion. For example, if $a_1 \neq 0$, s_1 must be (modulo a signum choice) the normalization (this means 2-norm$^2 = 1$) of the vector

$$\sigma r_1.$$

This is because the C orthogonal vectors span the C dimensional row vectors of σ, and so we are simply taking orthogonal expansions. If we choose a_1, r_1 so that $|a_1|$ is maximal as r_1 ranges over the complete Hilbert space, and then chose a_2, r_2 so that $|a_2|$ is maximal as r_2 ranges over the complementary space to that generated by r_1, and so on, then the generated tensor products are called "eigentensors".

The point being that in this case the s's must also be orthogonal. Let us suppose this is not the case and obtain a contradiction. We would have normal vectors: u, v, w, x with $\langle u, v\rangle = \langle w, x\rangle = 0$ that upon putting

$$\sigma u = aw; \quad \sigma v = bw + cx$$
$$\sigma\left(\frac{u + tv}{\sqrt{1+t^2}}\right) = \frac{aw + tbw + tcx}{\sqrt{1+t^2}}$$

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satisfy:

$$(a + tb)^2 + t^2c^2 \leq (1 + t^2)a^2. \tag{15}$$

This is because we could take

$$u = r_i, v = r_j$$

with $i < j$ providing a contradiction about the pairwise orthogonality of s_i, s_j, and the fact that tensor value a for s_i was maximal. But inequality (15) is impossible if t is sufficiently small.

Eigentensors are "biorthogonal"; the first factors are pairwise orthogonal and the second factors are pairwise orthogonal.

Then because the $s's$ must be orthogonal

$$\sigma^* \cdot \sigma = |a_1|^2(r_1 \otimes \bar{r}_1) + |a_2|^2(r_2 \otimes \bar{r}_2) + \cdots + |a_C|^2(r_C \otimes \bar{r}_C).$$

We identify r_k with the vector $\big(r_k(1), \ldots, r_k(C)\big)$. Thus $r_1, \ldots, r_C$ are the eigenvectors for this hermitian matrix, and $s_1, \ldots, s_C$ are the eigenvectors for the hermitian matrix $\sigma \cdot \sigma^*$. If σ is normal then $r_i = s_i$. We call $\{r_i \otimes s_i\}$ the eigentensor system for σ. And we call the a's the singular values.

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Mathematics Department
Kansas State University
Manhattan, Kansas 66506-2602
U.S.A.