GEOMETRY OF GROUP REPRESENTATIONS

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To the memory of TADASI NAKAYAMA

The many unanswerable questions (1) which arise in the study of finite groups have lead to a review of fundamental ideas, e.g. the Theorem of Burnside (3, p. 299 ; 2, 6) that *if λ be any faithful irreducible representation of G over a field K, then every irreducible representation of G over K is contained in some tensor power of λ.*

If we take K to be the complex field and write the inner tensor product in question $\lambda\times\lambda\times\cdots$ (n factors) as $\lambda\times^n$, we recall Schur's result that this representation of G splits according to the formula (7, p. 129)

1.1 $$\lambda\times^n=\sum f_\nu\lambda\otimes[\nu]$$

where $\lambda\otimes[\nu]$ is the *symmetrized* inner product associated with the irreducible representation $[\nu]$ of degree f_ν of the symmetric group S_n. For a finite group G, $\lambda\otimes[\nu]$ is in general reducible, while for the full linear group and certain of its subgroups this representation is irreducible.

These symmetrized tensor products are hard to handle, though their degrees δ^ν are given by the formula (5, p. 60)

1.2 $$\delta^\nu(f_\lambda)=G^\nu(f_\lambda)/H^\nu$$

where f_λ is the degree of λ. If we denote the Young diagram associated with the irreducible representation ν of S_n by $[\nu]$, then H^ν is the product of hook length of $[\nu]$ and $G^\nu(f_\lambda)=\prod_{i,j}(f_\lambda+j-i)$, taken over $[\nu]$. It follows immediately that for $n\le f_\lambda$, all these symmearized products are defined.

It would be interesting if Burnside's theorem could be refined so as to relate the apperances of the different irreducible representations of G to these symmetrized components of $\lambda\times^n$, but the difficulties seem insurmountable at present.

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2. Another application of these tensor products is of interest. In Chapter XII of (3) Burnside studies at some length the permutation representation g_i of G induced by the identity representation of a subgroup H_i $(i=1, 2, \ldots, r)$ of orders h_i. It is natural to arrange the H_i so that $H_1=I$ and g_1 is the regular representation of G, $h_i \leq h_{i+1} \leq h_r$ with $H_r=G$ so that g_r is the identity representation of G. If we suppose g_i to be represented on the variables x_u and g_j on the variables y_v, the tensor product $g_i \times g_j$ is represented on the variables $x_u y_v$ and

2.1 $$g_i \times g_j = \sum a_{ijk} g_k.$$

If $j=i$, we obtain the symmetrized components for $n=2$ on the variables (5, p. 57).

$$x_1 y_1,\ x_2 y_2,\ \ldots,\ \frac{1}{2}(x_u y_v + x_v y_u)\ ;\ \ldots\ \frac{1}{2}(x_u y_v - x_v y_u)$$

by setting $y_u = x_u$. It follows, as in the case of $g_i \times g_j$, that $g_i \otimes [2]$ *is also a permutation representation of* G, while $g_i \otimes [1^2]$ is not. The argument is quite general so that 2.1 becomes

2.2 $$g_i \times {}^n = \sum_j a^n_{ij} g_j,$$

and we have

2.3 $$g_i \otimes [n] = \sum_j b^n_{ij} g_j,$$

where the a^n_{ij}, b^n_{ij} are rational integers.

3. What is of interest here is that 2.1-2.3 can be interpreted in a natural way relative to the geometry of the irreducible representations λ of G. A start was made on this many years ago (4). For purposes of illustration, we reproduce two tables which set the stage for this interpretation in the case of S_4. Here we write

$$g_i = \sum_\nu m^\nu_i [\nu]$$

and Table 2 gives the values of the m^ν_i. For completeness, it would have been desirable to list all the solutions of 2.1, but this has been omitted in favour of Table 3 which gives the solutions of 2.2 and 2.3 for $n=2, 3$. Since there are *five* irreducible representations of S_4, we have the following linear relations between the g_i:

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TABLE 1

H	sub-group	h
H_1	1	1
H_2	1, (12)	2
H_3	1, (12)(34)	2
H_4	1, (123), (132)	3
H_5	1, (1234), (13)(24), (1432)	4
H_6	1, (12)(34), (14)(23), (13)(24)	4
H_7	1, (12), (34), (12)(34)	4
H_8	1, (12), (13), (23), (123), (132)	6
H_9	1, (12), (34), (12)(34), (14)(23), (13)(24), (1324), (1423)	8
H_{10}	A_4	12
H_{11}	S_4	24

TABLE 2

	$[1^4]$	$[2, 1^2]$	$[2^2]$	[3.1]	[4]
g_1	1	3	2	3	1
g_2	•	1	1	2	1
g_3	1	1	2	1	1
g_4	1	1	•	1	1
g_5	•	1	1	•	1
g_6	1	•	2	•	1
g_7	•	•	1	1	1
g_8	•	•	•	1	1
g_9	•	•	1	•	1
g_{10}	1	•	•	•	1
g_{11}	•	•	•	•	1
			m_i^ν		

TABLE 3

	$\times^2$	$\times^3$	$\otimes[2]$	$\otimes[3]$
g_1	$24\ g_1$	$576\ g_1$	$8\ g_1+6\ g_2+3\ g_3$	$17\ g_1+4\ g_4$
g_2	$5\ g_1+2\ g_2$	$70\ g_1+4\ g_2$	$g_1+4\ g_2+g_6$	$11\ g_1+7\ g_2+2\ g_4$
g_3	$4\ g_1+4\ g_3$	$64\ g_1+16\ g_3$	$3\ g_2+3\ g_3+g_5$	$10\ g_1+9\ g_3+2\ g_4$
g_4	$2\ g_1+2\ g_4$	$20\ g_1+4\ g_4$	$g_2+g_3+g_4+g_8$	$4\ g_1+3\ g_4$
g_5	$g_1+2\ g_5$	$8\ g_1+4\ g_5$	$g_2+g_5+g_9$	$g_1+g_3+g_4+g_5$
g_6	$6\ g_6$	$36\ g_6$	$3\ g_6+g_9$	$9\ g_6+g_{10}$
g_7	$g_1+2\ g_7$	$8\ g_1+4\ g_7$	$g_2+g_7+g_9$	$g_1+g_3+2\ g_7+2\ g_8$
g_8	g_2+g_8	$g_1+3\ g_2+g_8$	g_7+g_8	$g_2+2\ g_8$
g_9	g_6+g_9	$4\ g_6+g_9$	$2\ g_9$	$g_6+g_9+g_{11}$
g_{10}	$2\ g_{10}$	$4\ g_{10}$	$g_{10}+g_{11}$	
g_{11}	g_{11}	g_{11}		

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$$2\,g_6+g_1=3\,g_3 \qquad 2\,g_9+g_1=g_2+g_3+g_5$$
$$2\,g_7+g_1=2\,g_2+g_3 \qquad 2\,g_{10}+g_1=g_3+2\,g_4$$
$$2\,g_8+g_1=2\,g_2+g_4 \qquad 2\,g_{11}+g_1=g_2+g_4+g_5$$

Consider, in particular the irreducible representation [3, 1] whose invariant configuration is a regular tetrahedron. Since $H_4 \subset H_3$, the groups of stability of the vertices are H_3 and its conjugates. Taking the bi-vector defined by two such vertices, we have from Table 3,

$$g_3 \times {}^2 = g_3 + g_2$$

which indicates that the group of stability of the corresponding edge is H_2 with $m_2^{[3,1]}=2$. However, this does not take into account the extra symmetry arising by interchanging the two vertices. For this we go to

$$g_3 \otimes [2] = g_3 + g_7,$$

and the group of stability of the mid-edge point is H_7. As already mentioned, the component

$$g_3 \otimes [1^2] = [3,\ 1] + [2,\ 1^2]$$

has no geometrical significance.

We may study the geometry of the representation $[2,\ 1^2]$ in a similar fashion, noting from Table 2 that only the vertices of the fundamental region are well defined; since $H_3 \subset H_5$, the groups of stability are H_2, H_4 and H_5 and their conjugates. It may be verified that

$$g_2 \times g_4 = 4\,g_1,\ g_2 \times g_5 = 3\,g_1,\ g_4 \times g_5 = 2\,g_1$$

and from Table 3

$$g_2 \times {}^2 = 5\,g_1 + 2\,g_2,\ g_4 \times {}^2 = 2\,g_1 + g_4,\ g_5 \times {}^2 = g_1 + 2\,g_5.$$

Moreover, these inner products and the $g_i \otimes [2]$ and $g_i \otimes [3]$ $(i=2,\ 4,\ 5)$ interpreted relative to $[2,\ 1^2]$, describe the familiar arrangement of the vertices, mid-edge and mid-face points, of the octahedron, since the rotation group of the octahedron is isomorphic to the representation $[2,\ 1^2]$ of S_4.

4. Thus it appears that the geometry of the fundamental region of a real irreducible λ can be completely described in terms of $g_i \times g_j$ and $g_i \otimes [n]$. In order to clarify further these ideas, consider the relation

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$$g_7 \otimes [2] = g_7 + g_9 + g_2$$

which is more interesting than $g_3 \otimes [2] = g_3 + g_7$, since the octahedron is centrally symmetrical. Denoting the mid-point of the edge ij of the tetrahedron by P_{ij}, we have three possibilities: i) pairing P_{12} with P_{12} yields g_7; ii) pairing P_{12} with P_{34} allows an extra symmetry, since H_7 is invariant under (1324), which yields g_9; iii) pairing P_{12} with P_{13} yields a point on the edge of the fundamental region and so g_2. Since no point is invariant under H_7 and also (1324), g_9 does not register in either [3, 1] or [2, 1^2].

In particular, if H_i is a group of stability with $m_i^\lambda = 1$, considerations of linear dependence imply that

4.1 *$g_i \otimes [n]$ yields every g_j with $m_j^\lambda = 1$, for n sufficiently large.*

The geometry of the octahedron suggests immediately that $g_5 \otimes [3]$ yields g_4 but we must go to $g_2 \otimes [4]$ and $g_4 \otimes [4]$ to obtain g_5, as may readily be verified.

These ideas may be extended to apply to complex λ but we shall not consider such a genralization here.

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