APPLICATION OF WISDOM'S PERTURBATIVE METHOD TO THE 5:2 AND 7:3 RESONANCES

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Abstract. Wisdom's perturbative method is applied to the 5:2 and 7:3 resonances. Some comparisons with Yoshikawa's model are performed: for values of eccentricity up to about 0.3–0.4, agreement exists and it is better for 5:2 resonance. A clear difference between the cases 5:2 and 7:3 is observed : the former one, like in the case 3:1, can show significant variations of eccentricity, even starting from very small values, close to zero, while the latter seems to undergo such variations, but with initial eccentricity not less than a value near 0.1.

1. Introduction

Although a complete understanding of the Kirkwood gaps is still far, we can see that some new ideas developed by Wisdom have brought very useful information and new motivation for the problem. In the study of the 3:1 resonance, Wisdom (1985) used a truncated model of second order in eccentricity (e), for the disturbing function, and developed an interesting perturbative treatment which explains the high excursions of e he had found in a previous work . At a first sight, the study of other resonances could be done in a similar way, provided the truncated disturbing function is able to well represent the real dynamics of the motion. However, according to Henrard and Lemaître (1987) and Lemaître and Henrard (1988), truncation effects are very serious. For example, for 2:1 and 3:2 commensurabilities, the large variations of e and the area of phase space covered by chaotic solutions are very sensible to the order of truncation. In fact, the ratio of convergence of the series in e, for resonances of first order, is smaller than in the 3:1 case. In addition, the dominant part of resonant terms is of order one, while secular and long-period parts are of order two. For higher-order resonances like 5:2 or 7:3, the ratio of the semi-major axes is a little better, but now, the resonant part is weaker than secular and long-period parts. In this work, using Wisdom's perturbative method, we briefly present some preliminary results (full results will be reported elsewhere) for 5:2 and 7:3 resonances. To see the performance of the method, some comparisons with Yoshikawa's (1989,1990) and Šidlichovsky's (1986) results are made. Due to the truncated model used for the disturbing function, only small or moderate values of e can be considered.

2. Averaged System

Like in Wisdom (1985) and Šidlichovský (1987), for the cases 5:2 or 7:3, the dominant critical terms of the disturbing function can be grouped in one single cosine. Then, neglecting short-period terms and after a proper expansion in the neighbourhood of exact resonant point, the Hamiltonian can be written in the general form:
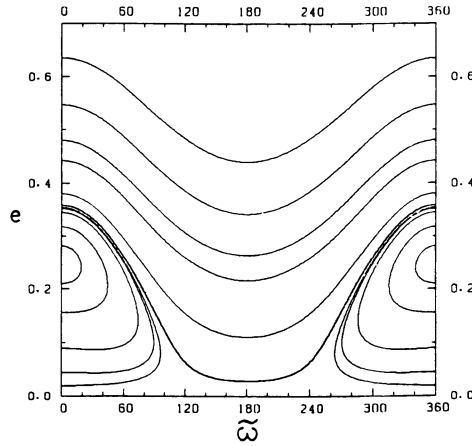
$$H = H'(x, \theta) + F_s(x_2, x_4) \tag{1}$$

Fig. 1. Level curves in (e, ϖ)-plane for $I_L = 0$ in eqn.2 (case 5:2).
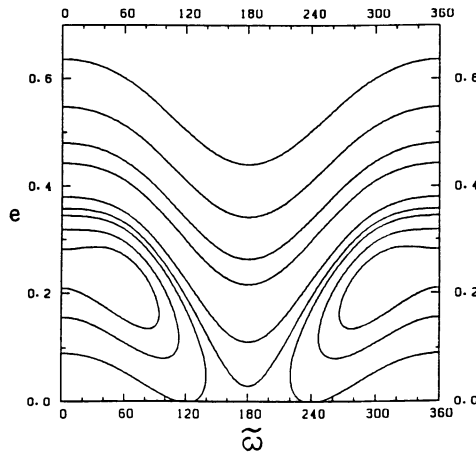

Fig. 2. Level curves in (e, ϖ) plane for $5\lambda' - 2\lambda - 3\varpi = 0$ and semi-major axis at exact resonant point in the truncated Hamiltonian for 5:2 resonance.

where x_2, x_4 are long-period variables and H' is a pendulum-like system, whose angular variable θ depends on the resonant combination of mean longitudes and also on x_2, x_4 (Wisdom 1985). Defining action-angle variables (I, ϕ) for the pendulum H' and taking a canonical transformation from (x, θ, x_2, x_4) to a new set $(I, \phi, \tilde{x}, \tilde{y})$, the new Hamiltonian (neglecting higher-order terms) can be written:

$$H = H'(I, A(\tilde{x}, \tilde{y})) + F_s(\tilde{x}, \tilde{y}). \tag{2}$$

In this approximation, I is a constant. Then, level curves for eqn. 2 can be drawn fixing either I or H. In case of separatrix of the pendulum, the above canonical

transformation is not defined. The curve where this occurs is called critical curve (Wisdom, 1985):

$$H = A(\tilde{x}, \tilde{y}) + F_s(\tilde{x}, \tilde{y}) \tag{3}$$

Curves of eqn. 2 parametrized by H are closely related to (x_2, x_4)-curves obtained from eqn. 1 via surfaces of section (Guckenheimer-Holmes 1983). Then, it is useful to see one special situation: depending on H the whole plane x_2, x_4 is not accessible in the surface of section. For example, for $\theta = 0$, any x_2, x_4 should satisfy:

$$H + A(x_2, x_4) - F_s(x_2, x_4) > 0 \tag{4}$$

For the liomit case ($=0$) of eqn. 4 we define a new curve, the zero velocity curve (ZVC). When searching an analytical solution for the critical curve (CC), it is convenient to square eqn. 3. In this case, the squared equation may contain ZVC as a solution, which should not be confused with CC.

3. Some Comparisons and Brief Results

Starting with 5:2 resonance let's call I_L the action of pendulum in the libration case. We are deep in libration region when I_L is small (Henrard and Lemaître 1987). Fig.1 shows (e, ϖ)-curves obtained from eqn. 2 for $I_L = 0$. Comparing with Yoshikawa's case (fig.4 in his article of 1989),we see that our curves are flattened and squeezed in the vertical direction. The averaging methods are differents: Yoshikawa's curves are obtained fixing $5\lambda' - 2\lambda - 3\varpi = 0$ and taking the semi-major axis at the exact resonant point. If these conditions are taken in our case, we get fig.2. Then, a rough idea of the limit of validity of truncated model can be obtained: comparing fig.2 with Yoshikawa's fig.4, agreement seems to be possible at most up to $e = 0.4$.

Fig.1 shows an interesting feature : values of e near 0.35 can be attained from very small values of e, even close to zero. Now, let's take some asteroids studied by Yoshikawa (1989). For asteroid B-1, energy is $H = -1.933472 \times 10^6$. Fixing this H and taking several values of I,guiding curves are drawn (fig.3) using eqn. 2. the large dot is B-1, the innermost curve is ZVC and inside it no motion is allowed. Equation 3 does not have solution and, in the absence of CC, no chaos appeared in the surface section for this H (fig.4). Although not shown here, fig.3, plotted in the variables (e, ϖ), gives almost the same as fig.4. The smooth variation of e in Yoshikawa's fig.5 is, then, predicted by Wisdom method. Figs. 5 and 6 correspond to $H = 4.384025 \times 10^6$, asteroid B-3. CC is marked with thick line. It is clear that B-3 will collide with CC, but before, it will spend some short time in initial (low) eccentricity mode. After collision, it is reasonable to occur the jump in e and chaotic motion (see Yoshikawa's fig.5). Another typical asteroid is B-5. It lies inside CC, so that almost no big variation is expected as it is shown in Yoshikawa's numerical integration. Following this kind of analysis, many useful features of B-asteroids, given by Yoshikawa, can be predicted.

For 7:3-commensurability, truncated model (4th order) is rather restricted. With $I_L = 0$ in eqn. 2 we get fig.9.

Now let's see figs 7 and 8 (Yoshikawa's model with $e' = 0.027$ and 0.048 respectively). In the former, there is an equilibrium point for $\varpi - \varpi' = 0$, while for the
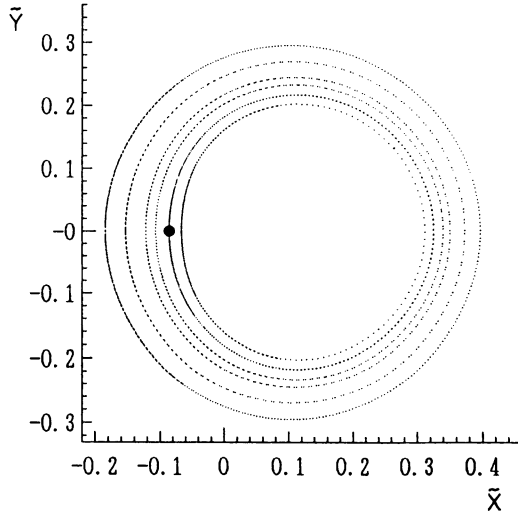
Fig. 3. Guiding trajectories for $H = -1.933472 \times 10^{-6}$ (asteroid B-1) in $(\tilde{x}, \tilde{y})$-plane. Empty central area is the forbidden region. Innermost curve is the ZVC. The dot is the initial position of the asteroid
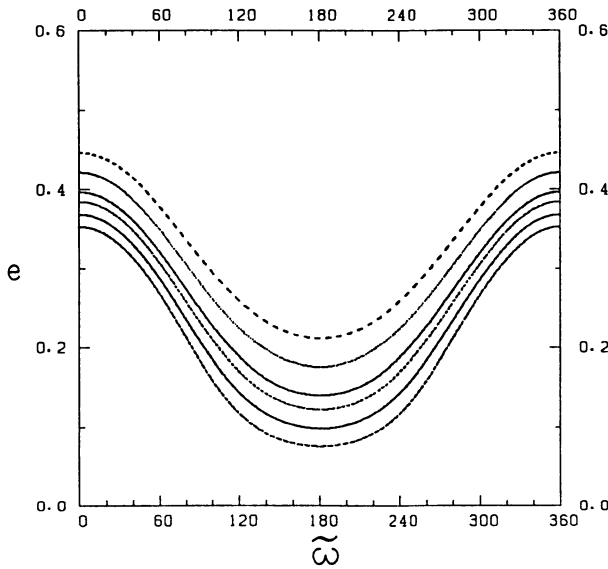
Fig. 4. Surface of section for $H = -1.933472 \times 10^{-6}$ in (e, ϖ) plane, integrating Hamiltonian 1. Only regular motions were observed.
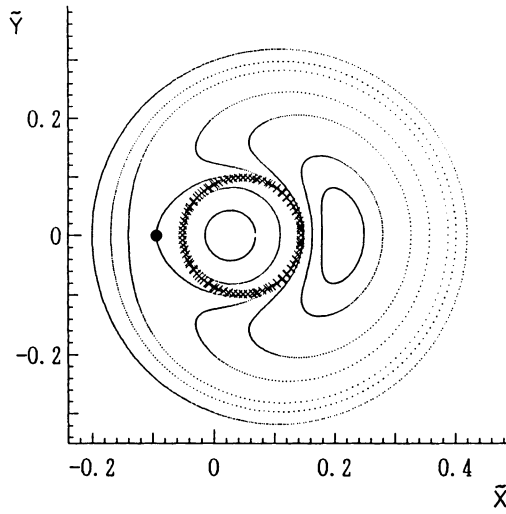
Fig. 5. Guiding trajectories for $H = 4.384025 \times 10^{-6}$ (asteroid B-3). The shaded area corresponds to the neighbourhood of CC.
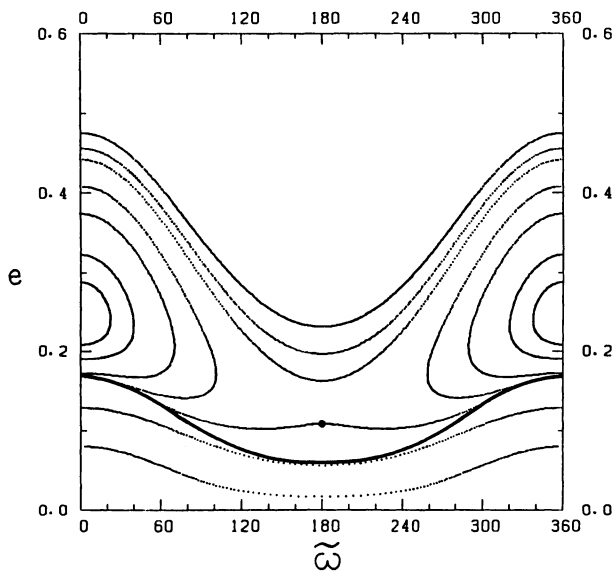

Fig. 6. The same of fig.5 but in e, ϖ variables. Thick curve is CC.
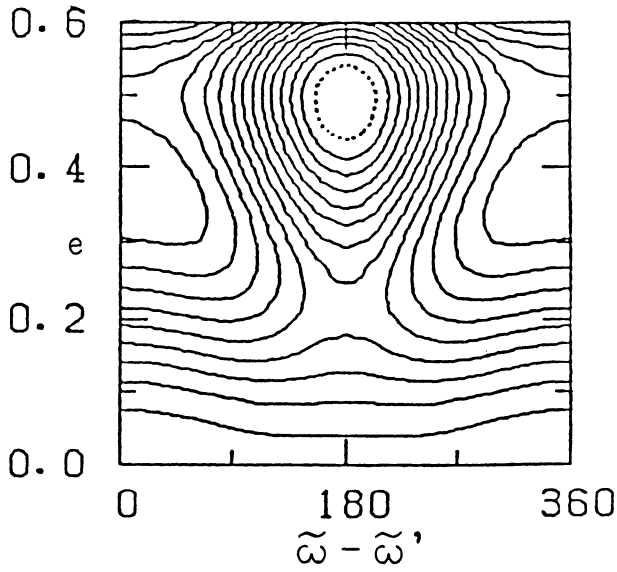
Fig. 7. Yoshikawa's level curves in $(e, \varpi - \varpi')$ plane for 7:3 resonance, for $e' = 0.027$.

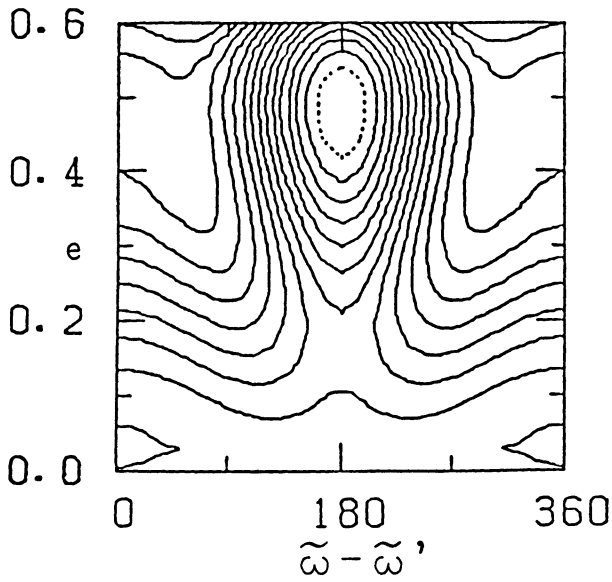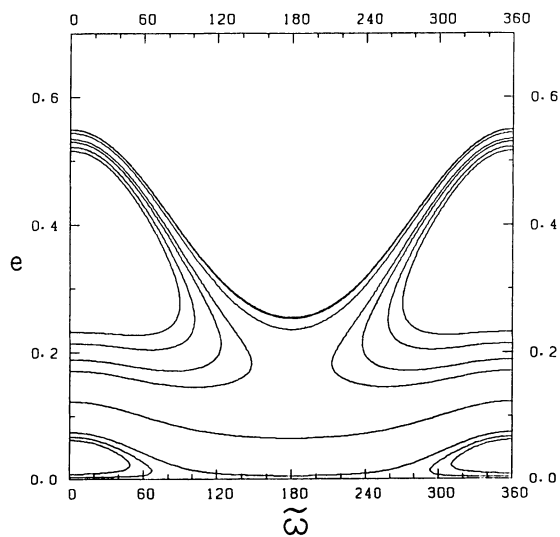Fig. 8. The same of fig. 7 for $e' = 0.048$.

Fig. 9. The same of fig. 1 for 7:3 resonance.

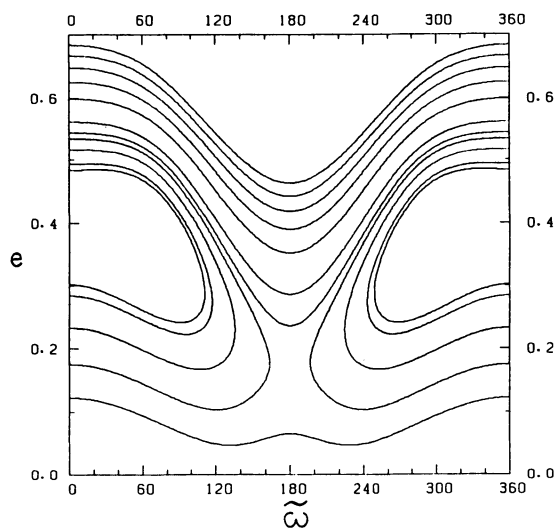
Fig. 10. The same of fig. 2 for 7:3 resonance and taking $7\lambda' - 3\lambda - 4\varpi = \pi$.

latter, it is not so clear, but if it exists, it occurs for $e > 0.4$. However, for $e \approx 0.4$, the truncated model is already not reliable, neither is the equilibrium point in fig.8. Using Yoshikawa's hypothesis $7\lambda' - 3\lambda - 4\varpi = \pi$ and taking semi-major axis at exact resonant point, we get fig.10. Compared with fig.9 we can roughly estimate the limit of validity of the truncated model: agreement seems to be possible for $e < 0.3$. In spite of this, if CC or ZVC lies in a reliable part of (e, ϖ)-plane, useful informations can still be obtained. Indeed, like in 5:2-case, we tested some C-asteroids given by Yoshikawa and agreement was observed in the expected regions.

4. Conclusion

Fig.1 shows that even starting from very small values, e can reach values near to 0.4. Indeed, in Yoshikawa's numerical integration, eccentricity of B-3 varies from 0.04 to 0.76. However for 7:3 case, fig.7 reveals that, if initial e is below a value around 0.1, no large variation is possible. Also, for this resonance, some calculations via surface of section showed the existence of chaotic zone trapped in regions of very small eccentricity. Details and other additional calculations should be reported elsewhere. Despite its limitations, the truncated model still gives useful informations and many qualitative features can be predicted.

Acknowledgements

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Discussion

G.Quinlan – You mentioned that Wisdom's perturbative method had some trouble at small eccentricities. Please explain this further.

T.Yokoyama – When the eccentricity is very small, the basic assumption that two different time scales exist, is no more valid. On the other hand, when eccentricity is 0.064 (5:2 commensurability) and 0.0569 (7:3 commensurability), the canonical transformation is singular.