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Abstract

We prove that for any prime number $p \geqslant 3$, there exists a positive number κ_p such that $\chi(\mathcal{O}_X) \geqslant \kappa_p c_1^2$ holds true for all algebraic surfaces X of general type in characteristic p. In particular, $\chi(\mathcal{O}_X) > 0$. This answers a question of Shepherd-Barron when $p \geqslant 3$.

1. Introduction

The Enriques–Kodaira classification of algebraic surfaces divides proper smooth algebraic surfaces into four classes according to their Kodaira dimension $-\infty, 0, 1, 2$. A lot of problems remain unsolved for the last class, the so-called *surfaces of general type*. One of the leading problems among these is the following so-called geography problem of minimal surfaces of general type (see [Per87]).

Question 1.1. Which values of $(a, b) \in \mathbb{Z}^2$ are the Chern invariants (c_1^2, c_2) of a minimal surface of general type?

Over the complex numbers, though not yet settled completely, much is known about this problem. Here we collect some classical relations between c_1^2 and c_2 of a minimal surface X of general type:

$$c_1^2 > 0;$$
$$c_1^2 + c_2 \equiv 0 \mod 12;$$
$$5c_1^2 - c_2 + 36 \geqslant 0; \tag{N}$$
$$3c_2 \geqslant c_1^2. \tag{BMY}$$

The first inequality is from the definition of a minimal surface of general type, the second condition is from Noether's formula

$$12\chi(\mathcal{O}_X) = c_1^2 + c_2; \tag{1.1}$$

the inequality (N) is derived from the following so-called Noether inequality:

$$K_X^2 \geqslant 2p_g - 4; \tag{1.2}$$

here $p_g := h^0(X, K_X)$. The last inequality (BMY) is called the Bogomolov–Miyaoka–Yau inequality. Due to (1.1), the inequality (BMY) can also be interpreted as below:

$$9\chi(\mathcal{O}_X) \geqslant c_1^2. \tag{BMY'}$$

It is known that most of the numbers (a, b) satisfying the above relations are the Chern numbers of a surface of general type over $\mathbb{C}$. For more details and backgrounds on these inequalities, please refer to [Miy77], [Yau77], [BHPV04, ch. 7] and [IS96, chs 8 and 9].

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Then we turn to the geography problem in the positive-characteristic case. Noether's inequality (1.2) (see [Lie08a]) and Noether's formula (1.1) (see [Bad01, ch. 5]) remain true, while the Bogomolov–Miyaoka–Yau inequality (BMY) as stated no longer holds [Szp79, § 3.4]. In fact, even the following weaker inequality (CdF) due to Castelnuovo and de Franchis fails:

$$c_2 \geqslant 0 \tag{CdF}$$

(see, e.g., § 4 of this paper). So, it is natural to formulate an inequality in positive characteristic bounding c_2 from below by c_1^2. Using Noether's formula, it is the same as bounding χ from below. In fact, Shepherd-Barron has already considered a similar question and proved that $\chi > 0$ (equivalently, $c_2 > -c_1^2$) with a few possible exceptional cases when $p \leqslant 7$ [She91b, Theorem 8]. Here we generalise it to the following question.

Question 1.2. What is the optimal number κ_p such that $\chi \geqslant \kappa_p c_1^2$ holds for all surfaces of general type defined over a field of characteristic p?

By definition, we have

$$\kappa_p = \inf\{\chi/c_1^2 \mid \text{minimal algebraic surface of general type defined over an algebraically closed field of characteristic } p\}.$$

In particular, $\kappa_p > 0$ implies $\chi > 0$.

The purpose of this paper is an investigation of κ_p. Of course, we will be in the best situation if we can work out κ_p for each p; however, this looks difficult and, instead, we try to find some interesting bounds of κ_p, say, to show that $\kappa_p > 0$ for all $p > 2$. The main result of this paper is the following theorem.

THEOREM 1.3 (Main theorem). *Let κ_p be defined as above; then:*

(1) *if $p > 2$, then $\kappa_p > 0$;*
(2) *if $p \geqslant 7$, then $\kappa_p > (p-7)/12(p-3)$;*
(3) *$\lim_{p\to\infty} \kappa_p = 1/12$;*
(4) *$\kappa_5 = 1/32$.*

This theorem is a simple summation of Corollary 3.13, Theorem 3.26 and Corollary 5.12 in the context.

Moreover, we have a conjecture on the values of κ_p.

CONJECTURE 1.4 (Conjecture 5.13). If $p \geqslant 5$, then $\kappa_p = (p^2 - 4p - 1)/4(3p^2 - 8p - 3)$.

Note that if $p = 5$, then

$$(p^2 - 4p - 1)/4(3p^2 - 8p - 3) = 1/32$$

and, if $p \geqslant 7$, then

$$(p^2 - 4p - 1)/4(3p^2 - 8p - 3) > (p-7)/12(p-3).$$

One inequality of this conjecture ($\kappa_p \leqslant (p^2 - 4p - 1)/4(3p^2 - 8p - 3)$) is Raynaud's example (see § 4). For the other inequality, we shall provide some evidence in favour of it in § 5.

In [She91b], remark after Lemma 9, Shepherd-Barron raised the question of whether any minimal surface of general type X satisfies $\chi(\mathcal{O}_X) > 0$. Our Theorem 1.3 implies that the answer is yes if $p \neq 2$.

Corollary 1.5. *If $p \neq 2$, then $\chi > 0$ holds for all surfaces of general type.*

This corollary can help to improve and to better understand several other results (e.g. [BBT95, Proposition 2.2] and [She91a, Theorems 25–27]), where the authors need to take care of the possibility of $\chi \leqslant 0$.

As another application of Theorem 1.3, we give following theorem concerning the canonical map of surfaces of general type, which can be seen as an analogue of Beauville's relevant result over $\mathbb{C}$ [Bea79, Propositions 4.1 and 9.1].

Theorem 1.6. *Let S be a proper smooth surface of general type over an algebraically closed field of characteristic $p > 0$ with $p_g(S) \geqslant 2$; then:*

(1) *if $p \geqslant 3$ and $|K_S|$ is composed with a pencil of curves of genus g, then we have*

$$g \leqslant 1 + \frac{p_g + 2}{2\kappa_p(p_g - 1)};$$

(2) *if $p \geqslant 3$ and the canonical map is a generically finite morphism of degree d, then we have*

$$d \leqslant \frac{p_g + 1}{\kappa_p(p_g - 2)}.$$

The proof of this theorem is a simple copy of Beauville's, replacing simply the inequality (BMY)' there by $\chi \geqslant \kappa_p c_1^2$; the only thing essential is that $\kappa_p > 0$ in these cases. Hence, we will not include its proof in this paper. The interesting part of this theorem is the following remark.

Remark 1.7. If we bound $\chi(\mathcal{O}_S)$ from below (hence it bounds $p_g \geqslant \chi(\mathcal{O}_S) + 1$ from below) as Beauville did in [Bea79] and substitute κ_p by our lower bounds given in Theorem 1.3, we can bound g and d from above as in [Bea79]. As far as we know, whether Beauville's bounds on g and d are optimal is not yet solved (recently, there seems to be an improvement on these bounds; see [Che14]), not to mention ours.

We shall also give a new characterisation of algebraic surfaces with negative χ directly after Theorem 1.3 and [Lie13, Proposition 8.5].

Theorem 1.8 (After [Lie13, Proposition 8.5]). *Let S be an algebraic surface over a field of characteristic $p > 0$ with $\chi(\mathcal{O}_S) < 0$. Then:*

(1) *S is birationally ruled over a curve of genus $1 - \chi(\mathcal{O}_S)$; or*
(2) *S is quasi-elliptic of Kodaira dimension 1 and $p \leqslant 3$; or*
(3) *S is of general type and $p = 2$.*

This paper is organised as follows.

In § 2, we give some necessary preliminaries. We rewrite Tate's formula on genus change to obtain some intermediate results which are more or less implicit in both Tate's original paper [Tat52] and Schröer's paper [Sch09]. Then we recall the theory of flat double covers, a Bertini-type theorem and some other supplements.

In § 3, we study the numerical properties of surfaces of general type with negative c_2, and prove our Theorem 1.3 except for the equation $\kappa_5 = 1/32$.

In § 4, we give some examples of algebraic surfaces of general type with negative c_2 and compute some of their numerical invariants.

In § 5, we carry out a calculation of a special kind of algebraic surfaces of general type with negative c_2, namely those X whose Albanese fibration is hyperelliptic and has the smallest possible genus. We show that our conjectural κ_p (Conjecture 1.4) are the best bounds for these surfaces. This also completes the proof of our main theorem. During the calculation, a lemma on a special kind of singularities is used; as the proof is a bit long, we put it in an appendix.

Remark 1.9. In this paper we shall use the following notation.

(1) For any invertible sheaf $\mathcal{E}$ over a scheme, $\mathbb{P}(\mathcal{E}) := \operatorname{Proj}(\operatorname{Sym}(\mathcal{E}))$.

(2) If $S \to T$ is a morphism of schemes in characteristic p, we denote by $F_S : S \to S$ the absolute Frobenius morphism and by $F_{S/T} : S \to S^{(p)}$ the relative Frobenius morphism (where $S^{(p)} = S \times_T T$ is obtained by base changing $S \to T$ via $F_T : T \to T$). If $\pi : S \to Y$ is a morphism of T-schemes, we denote by $\pi^{(p)} : S^{(p)} \to Y^{(p)}$ the morphism of T-schemes induced by π.

(3) Given any closed point s of a scheme S, we will denote by $\kappa(s)$ the residue field of s.

2. Preliminaries

In this section, *fields are assumed to be of a positive characteristic* p unless otherwise stated.

2.1 Genus-change formula

Let S be a normal projective and geometrically integral curve over a field K (in particular, $H^0(S, \mathcal{O}_S) = K$), of arithmetic genus $g(S) := 1 - \chi(\mathcal{O}_S) = \dim H^1(S, \mathcal{O}_S)$. The latter is also called the genus of the function field $K(S)$. Let L/K be a finite extension and let $(S_L)'$ be the normalisation of $S_L := S \times_K L$. A theorem of Tate [Tat52] states that

$$(p-1) \mid 2(g((S_L)') - g(S)).$$

This is proved in the scheme-theoretical language in [Sch09]. Below we give a slightly different proof in the scheme-theoretical language (in some places close to Tate's original one) and some more precise intermediate results; in particular, we show that if $g(S)$ is small with respect to p, then the normalisation of $S^{(p)}$ is smooth (Proposition 2.8).

LEMMA 2.1. *Let S, Y be geometrically integral normal curves over a field K and let $\pi : S \to Y$ be a finite inseparable morphism of degree p. Then $\Omega_{S/Y}$ is invertible and we have an exact sequence*

$$0 \to F_S^*\Omega_{S/Y} \to \pi^*\Omega_{Y/K} \to \Omega_{S/K} \to \Omega_{S/Y} \to 0 \tag{2.1}$$

with $F_S^*\Omega_{S/Y} \simeq \Omega_{S/Y}^{\otimes p}$.

Proof. The second part $\pi^*\Omega_{Y/K} \to \Omega_{S/K} \to \Omega_{S/Y} \to 0$ is canonical and always exact. Let us show the existence of a complex $0 \to F_S^*\Omega_{S/Y} \to \pi^*\Omega_{Y/K} \to \Omega_{S/K}$ and prove the exactness under the assumption of the lemma.

As π is purely inseparable of degree p, we have the inclusions of function fields $K(S)^p \subseteq K(Y) \subseteq K(S)$; hence, $F_{S/K} : S \to S^{(p)}$ factors into $\pi : S \to Y$ and some $f : Y \to S^{(p)}$ (which is in fact the normalisation map). We have a canonical commutative diagram

$$\begin{tikzcd} S \arrow[r, "F_{S/K}"'] \arrow[rr, bend left, "F_S"] \arrow[d, "\pi"'] & S^{(p)} \arrow[r, "q"] \arrow[d, "\pi^{(p)}"] & S \arrow[d, "\pi"] \\ Y \arrow[r] \arrow[ur, "f"'] & Y^{(p)} \arrow[r] & Y \end{tikzcd}$$

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where $q : S^{(p)} \to S$ is the projection map. We have $q^*\Omega_{S/Y} = \Omega_{S^{(p)}/Y^{(p)}}$, because the last square is *Cartesian*, and a canonical map $f^*\Omega_{S^{(p)}/Y^{(p)}} \to \Omega_{Y/Y^{(p)}}$ and hence a canonical map $F_S^*\Omega_{S/Y} = \pi^* f^*\Omega_{S^{(p)}/Y^{(p)}} \to \pi^*\Omega_{Y/Y^{(p)}}$. Note that the canonical map $F_{Y/K}^*\Omega_{Y^{(p)}/K} \to \Omega_{Y/K}$ is identically zero, so the canonical map $\Omega_{Y/Y^{(p)}} \to \Omega_{Y/K}$ is an isomorphism. Therefore, we have a map $\Phi : F_S^*\Omega_{S/Y} \to \pi^*\Omega_{Y/K}$. Its composition with $\pi^*\Omega_{Y/K} \to \Omega_{S/K}$ is zero because locally it maps a differential form db to $d(b^p) = 0$. So,

$$0 \to F_S^*\Omega_{S/Y} \to \pi^*\Omega_{Y/K} \to \Omega_{S/K}$$

is a complex.

Let $s \in S$ and let $y = \pi(s) \in Y$. Then $A := \mathcal{O}_{Y,y} \to B := \mathcal{O}_{S,s}$ is a finite extension of discrete valuation rings of degree p, so $B = A[T]/(T^p - a)$ for some $a \in A$ (the element $a \in A$ is either a uniformising element or a unit whose class in the residue field of A is not a pth power). The stalk of the complex (2.1) becomes

$$0 \to B \otimes_A A da \to \Omega_{A/K} \otimes_A B \to ((\Omega_{A/K} \otimes_A B) \oplus BdT)/Bda \to BdT \to 0, \tag{2.2}$$

which is clearly exact. This also shows that $\Omega_{S/Y}$ is locally free of rank 1. As a general fact, we then have $F_S^*\Omega_{S/Y} \simeq \Omega_{S/Y}^{\otimes p}$. □

Proposition 2.2. *Let S, Y be normal projective geometrically integral curves over K and let $\pi : S \to Y$ be a finite inseparable morphism of degree p. Let $\mathcal{A} = \mathrm{Ker}(\Omega_{S/K} \to \Omega_{S/Y})$. Then:*

(1) *$\mathcal{A} = \Omega_{S/K,\mathrm{tor}}$ is the torsion part of $\Omega_{S/K}$ and we have*

$$(p-1)\deg\det(\mathcal{A}) = 2p(g(S) - g(Y));$$

here one may refer to [LLR04, § 5] for the definition of the determinant sheaf of a coherent sheaf on a normal curve;

(2) *$\deg\det(\mathcal{A}) = \deg\mathcal{A} = \sum_{s\in S}(\mathrm{length}_{\mathcal{O}_{S,s}}\mathcal{A}_s)[\kappa(s) : K] = \dim_K H^0(S, \mathcal{A})$;*

(3) *if $g(S) = g(Y)$, then S is smooth over K.*

Proof. (1) Split the exact sequence (2.1) into

$$0 \to \Omega_{S/Y}^{\otimes p} \to \pi^*\Omega_{Y/S} \to \mathcal{A} \to 0 \tag{2.3}$$

and

$$0 \to \mathcal{A} \to \Omega_{S/K} \to \Omega_{S/Y} \to 0. \tag{2.4}$$

As $\Omega_{S/K}$ has rank 1 (because S is geometrically reduced) and $\Omega_{S/Y}$ is invertible, we have $\mathcal{A} = \Omega_{S/K,\mathrm{tor}}$.

Next note that we have $\det\Omega_{S/K} = \omega_{S/K}$ and similarly for $\Omega_{Y/K}$. In fact, take an arbitrary embedding $S \hookrightarrow \mathbb{P}_K^n$; since S is regular, this is a regular embedding and we have an exact sequence

$$0 \to \mathcal{I}_S/\mathcal{I}_S^2 \to \Omega_{\mathbb{P}_K^n/K}|_S \to \Omega_{S/K} \to 0.$$

Here $\mathcal{I}_S$ is the ideal sheaf of S and $\mathcal{I}_S/\mathcal{I}_S^2$ is locally free. Taking determinants, we then have

$$\det(\Omega_{S/K}) = \det(\Omega_{\mathbb{P}_K^n/K}) \otimes \det(\mathcal{I}_S/\mathcal{I}_S^2)^{-1} = \omega_{S/K}.$$

Now, by taking the determinants in the two exact sequences (2.3) and (2.4), we get

$$\pi^*\omega_{Y/K} = \det\mathcal{A} \otimes \omega_{S/Y}^{\otimes p}$$

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and

$$\omega_{S/K} = \det \mathcal{A} \otimes \omega_{S/Y}.$$

Hence,

$$(\det \mathcal{A})^{\otimes(p-1)} \simeq \omega_{S/K}^{\otimes p} \otimes \pi^* \omega_{Y/K}^{-1}.$$

By Riemann–Roch, $\deg \omega_{S/K} = 2g(S) - 2$ (and similarly for Y). Part (1) is then obtained by taking the degrees in the above isomorphism.

(2) This is well known and can be proved locally at every stalk of $\mathcal{A}$ (see, e.g., [LLR04, Lemma 5.3(b)]).

(3) Finally, if $g(S) = g(Y)$, then $\deg \mathcal{A} = 0$ and hence $\mathcal{A} = 0$. This implies that $\Omega_{S/K}$ is free of rank one and hence S is smooth over K. □

Remark 2.3. The support of $\mathcal{A}$ consists of non-smooth points (over K) of S, and it is well known that such points are inseparable over K [Liu02, Proposition 4.3.30]. In particular, $p \mid [\kappa(s) : K]$ for any $s \in \operatorname{Supp}(\mathcal{A})$.

COROLLARY 2.4 (Tate genus-change formula). *Let S be a normal projective geometrically integral curve over K. Let L be an algebraic extension of K and let Y be the normalisation of S_L (viewed as a curve over L). Then*

$$p - 1 \mid 2(g(S) - g(Y)).$$

Proof. The result will be derived from Proposition 2.2. We can suppose that L/K is purely inseparable. Let us first treat the case $L = K^{1/p}$. Decompose the absolute Frobenius $K \to K$, $x \mapsto x^p$ as

$$K \xrightarrow{i} K^{1/p} \xrightarrow{\rho} K,$$

where i is the canonical inclusion and ρ is an isomorphism. Let us extend Y to $Y_K := Y \otimes_L K$ using ρ. Then Y_K is a normal projective and geometrically integral curve over K, of arithmetic genus (over K) equal to that of Y over L. Moreover, Y_K is birational to $(S_L) \otimes_L K = S^{(p)}$. So, we have an inseparable finite morphism $S \to Y_K$ of degree p. By Proposition 2.2(1), we have $(p-1) \mid 2(g(S) - g(Y))$ and $g(S) > g(Y) \geqslant 0$ unless S is already smooth over K. Repeating the same argument, for any $n \geqslant 1$, if S_n denotes the normalisation of $S_{K^{1/p^n}}$, then $p-1$ divides $2(g(S) - g(S_n))$, and S_n is smooth over K^{1/p^n} if n is big enough.

Now let L/K be a finite purely inseparable extension. Then there exists $m \geqslant 1$ such that $L \subseteq K^{1/p^m} \subseteq L^{1/p^m}$ and S_m is smooth. This implies that the normalisation Y_m of $Y_{L^{1/p^m}}$ is $(S_m)_{L^{1/p^m}}$. On the other hand, applying the previous result to the L-curve Y instead of S, we see that $p-1$ divides $2(g(Y) - g(Y_m))$. As $g(Y_m) = g(S_m)$, we find that $p-1$ divides $2(g(S) - g(Y))$. The case of any algebraic extension follows immediately. □

LEMMA 2.5. *Let $\pi : S \to Y$ be as in Lemma 2.1.*

(1) *We have*

$$p \deg \Omega_{Y/K,\mathrm{tor}} \leqslant \deg \Omega_{S/K,\mathrm{tor}}.$$

(2) *Let $s \in S$ and let $y = \pi(s)$. Suppose that $\kappa(y) = \kappa(s)$; then*

$$\operatorname{length}_{\mathcal{O}_{S,s}}(\Omega_{S/K,\mathrm{tor}})_s \geqslant p \dim_{\kappa(s)} \Omega_{\kappa(s)/K}.$$

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Proof. (1) Denote $\mathcal{A} = \Omega_{S/K,\mathrm{tor}}$ and $\mathcal{B} = \Omega_{Y/K,\mathrm{tor}}$. Let $s \in S$ and $y = \pi(s)$. The canonical map $\mathcal{B}_y \otimes \mathcal{O}_{S,s} = \pi^*(\mathcal{B})_s \to \mathcal{A}_s$ is injective by the exact sequence (2.3), because $\Omega_{S/Y}^{\otimes p}$ is torsion-free. Therefore,

$$e_s \mathrm{length}_{\mathcal{O}_{Y,y}}(\mathcal{B}_y) = \mathrm{length}_{\mathcal{O}_{S,s}}(\mathcal{B}_y \otimes \mathcal{O}_{S,s}) \leqslant \mathrm{length}_{\mathcal{O}_{S,s}}(\mathcal{A}_s),$$

where e_s is the ramification index of $\mathcal{O}_{Y,y} \to \mathcal{O}_{S,s}$. The desired inequality holds because $p = e_s[\kappa(s) : \kappa(y)]$.

(2) Let $A = \mathcal{O}_{Y,y}$ and $B = \mathcal{O}_{S,s}$. As $\kappa(y) = \kappa(s)$, $A \to B$ has ramification index p. So, $B = A[T]/(T^p - t)$ for some uniformising element t of A. The exact sequence (2.2) gives the exact sequence

$$0 \to (\Omega_{A/K}/Adt) \otimes_A B \to \Omega_{B/K} = ((\Omega_{A/K}/Adt) \otimes_A B) \oplus BdT.$$

In particular,

$$\mathcal{A}_s = (\Omega_{A/K}/Adt) \otimes_A B. \tag{2.5}$$

The usual exact sequence

$$tA/t^2A \to \Omega_{A/K} \otimes_A \kappa(y) \to \Omega_{\kappa(y)/K} \to 0$$

implies that we have a surjective map

$$\mathcal{A}_s \twoheadrightarrow \Omega_{\kappa(y)/K} \otimes_A B = \Omega_{\kappa(y)/K} \otimes_{\kappa(y)} B/tB.$$

So,

$$\mathrm{length}_B \mathcal{A}_s \geqslant p \dim_{\kappa(y)} \Omega_{\kappa(y)/K} = p \dim_{\kappa(s)} \Omega_{\kappa(s)/K}.$$ □

Remark 2.6. Let $S = S_0 \to S_1 \to \cdots \to S_n$ be a tower of inseparable covers of degree p of geometrically integral normal projective curves over K. Let $g_i := p_a(S_i)$. Then, by the first statement of the above lemma, we have $g_{i+1} - g_i \leqslant (g_i - g_{i-1})/p$. In particular, $\deg \Omega_{S/K,\mathrm{tor}} = 2p(g_0 - g_1)/(p-1) > 2(g_0 - g_n)$ by Proposition 2.2.

Lemma 2.5(2) is not used in the sequel. But we think that it can be of some interest in the understanding of genus changes. It implies immediately [Sal11, Corollary 3.3].

DEFINITION 2.7. We call a curve S over K *geometrically rational* if $S_{\bar{K}}$ is integral with normalisation isomorphic to $\mathbb{P}^1_{\bar{K}}$.

A slightly weaker version of the next proposition can also be found in [Sal11, Corollary 3.2].

PROPOSITION 2.8. *Let S be a projective normal and geometrically rational curve over K of (arithmetic) genus g. Suppose that S is not smooth. Let Y be the normalisation of $S^{(p)}$.*

(1) *We have $2g \geqslant (p-1)$. If $2g = p-1$, then Y is a smooth conic. Moreover, S has exactly one non-smooth point, the latter being of degree p over K.*

(2) *If $g < (p^2-1)/2$, then Y is a smooth conic over K. In particular, we have $\deg \Omega_{S/K,\mathrm{tor}} = 2pg/(p-1)$.*

Proof. (1) This is an immediate consequence of Proposition 2.2.

(2) If Y is not smooth, as non-smooth points have inseparable residue fields (see, e.g., [Liu02, Proposition 4.3.30]), we have

$$\deg \Omega_{S/K,\mathrm{tor}} \geqslant p \deg \Omega_{Y/K,\mathrm{tor}} \geqslant p^2$$

by Lemma 2.5(1). So, $g \geqslant g(Y) + p(p-1)/2 \geqslant (p^2-1)/2$ since $g(Y) \geqslant (p-1)/2$, which is a contradiction. So, Y is smooth. In particular, Y is a smooth conic because S is assumed to be geometrically rational. □

2.2 Flat double covers

In this subsection, we will furthermore *assume that* $p > 2$. We recall some basic facts on flat double covers over schemes where 2 is invertible. One can also consult [CD89, ch. 0] or [BHPV04, ch. III, § 6–7] for a standard introduction.

DEFINITION 2.9. A finite morphism between noetherian schemes $f : S \to Y$ is called a *flat double cover* if $f_*\mathcal{O}_S$ is locally free of rank 2 over $\mathcal{O}_Y$.

For our purpose, we suppose that Y is an *integral noetherian* scheme defined over a field K, as assumed, of characteristic $p > 2$.

Construction 2.10. All the flat double covers of Y are constructed as follows. Choose an invertible sheaf $\mathcal{L}$ on Y, and $s \in H^0(Y, \mathcal{L}^{\otimes 2}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}^{-2}, \mathcal{O}_Y)$. Endow the $\mathcal{O}_Y$-module $\mathcal{O}_Y \oplus \mathcal{L}^{-1}$ with the $\mathcal{O}_Y$-algebra structure by

$$\mathcal{L}^{-1} \times \mathcal{L}^{-1} \to \mathcal{L}^{\otimes(-2)} \xrightarrow{s} \mathcal{O}_Y.$$

Then $S := \mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{L}^{-1})$ is a flat double cover of Y. Note that if we replace s with $a^2 s$ for some $a \in H^0(Y, \mathcal{O}_Y)^*$, then we get a flat double cover isomorphic to the initial one. We call the invertible sheaf $\mathcal{L}$ above *the associated invertible sheaf* of f.

For the cover $S \to Y$ as above, if $s \neq 0$, S is reduced and $S \to Y$ is generically étale, the *branch divisor* is equal to $B := \mathrm{div}(s)$.

With the help of the Riemann–Roch theorem, we can compute some numerical invariants of flat double covers.

PROPOSITION 2.11. (1) *If Y is a geometrically connected smooth projective curve over K and $S \to Y$ is a flat double cover with branch divisor B, then*

$$p_a(S) = 2p_a(Y) - 1 + \deg(B)/2.$$

(2) *If Y is a geometrically connected smooth projective surface over K, then*

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_Y) + \chi(\mathcal{L}^{-1}) = 2\chi(\mathcal{O}_Y) + (B^2 + 2B \cdot K_Y)/8,$$

where K_Y is the canonical divisor of Y. □

Next we recall the following proposition.

PROPOSITION 2.12. *Let $f : S \to Y$ be a flat double cover over Y with branch divisor B.*

(1) *If Y is normal, then S is normal if and only if B is reduced.*
(2) *If Y is regular, then S is regular if and only if B is regular.*
(3) *If Y is smooth over K, then S is smooth over K if and only if B is smooth over K.*

Proof. See [CD89, ch. 0]. □

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Now suppose that Y is regular. Let $f : S \to Y$ be a flat double cover given by data $\mathcal{L}$ and $0 \neq s \in H^0(Y, \mathcal{L}^{\otimes 2})$ as in Construction 2.10 with $B = \mathrm{div}(s)$ being the branch divisor of f. Then B can be uniquely written as sum of effective divisors: $B = B_1 + 2B_0$ such that B_1 is reduced. Then $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}_Y(-B_0)$ and $s \in H^0(Y, \mathcal{L}'^{\otimes 2})$ (here we use $\mathcal{L}^{\otimes 2} = s\mathcal{O}_Y(B) \supseteq s\mathcal{O}_Y(B_1) = \mathcal{L}'^{\otimes 2}$, and thus s is also a global section in $\mathcal{L}'^{\otimes 2}$) also defines a flat double cover $f' : S' \to Y$. This scheme S' is nothing but the normalisation of S by Proposition 2.12. Note here that B_1 is the branch divisor of $S' \to Y$.

Now we recall the following process of resolution of singularities from a flat double cover. One may also refer to [BHPV04, ch. III, § 6].

Definition 2.13 (Canonical resolution). Let Y_0 be a non-singular algebraic surface over an algebraically closed field k. Let $f_0 : S_0 \to Y_0$ be a flat double cover given by data $\{\mathcal{L}_0, 0 \neq s \in H^0(Y_0, \mathcal{L}_0^{\otimes 2})\}$ and assume that the branch locus $B := \mathrm{div}(s)$ is reduced (i.e., S_0 is normal by Proposition 2.12). Then the canonical resolution of singularities of S_0 is the following process.

If B_0 is not regular, choose a non-regular closed point $y_0 \in B_0$; denote $m_0 := \mathrm{mult}_{y_0} B_0$ and $l_0 := \lfloor m_0/2 \rfloor$. Blowing up y_0, we obtain a morphism $\sigma_0 : Y_1 \to Y_0$, and denote by $\widetilde{B}_0$ and E the strict transform of B_0 and the exceptional divisor, respectively. Let S_1 be the normalisation of $S_0 \times_{Y_0} Y_1$. Then $f_1 : S_1 \to Y_1$ is a flat double cover with associated invertible sheaf $\mathcal{L}_1 = \sigma^*\mathcal{L}_0 \otimes \mathcal{O}_{Y_1}(-l_0 E)$ and branch divisor $B_1 = \sigma^*(B_0) - 2l_0 E$. If B_1 is not regular, continue this process, until we reach some n such that B_n is regular. We draw the following diagram as a picture of this process. □

$$\begin{tikzcd} S_0 \arrow[d, "f_0"'] & S_1 \arrow[l, "g_0"] \arrow[d, "f_1"'] & \cdots \arrow[l, "g_1"] & S_n \arrow[l, "g_{n-1}"] \arrow[d, "f_n"] \arrow[lll, bend right, "g"'] \\ Y_0 & Y_1 \arrow[l, "\sigma_0"] & \cdots \arrow[l, "\sigma_1"] & Y_n \arrow[l, "\sigma_{n-1}"] \end{tikzcd}$$

We will denote by $y_i \in B_i$ the centre of the blowing up morphism $\sigma_i : Y_{i+1} \to Y_i$, E_i the exceptional locus, $m_i := \mathrm{mult}_{y_i} B_i$ and $l_i := \lfloor m_i/2 \rfloor$. Then it follows that

$$\chi(R^1 g_{i*}\mathcal{O}_{S_{i+1}}) = (l_i^2 - l_i)/2. \tag{2.6}$$

In particular, if Y is proper, then

$$\chi(\mathcal{O}_{S_n}) - \chi(\mathcal{O}_{S_0}) = -\sum_{0 \leqslant i < n} (l_i^2 - l_i)/2. \tag{2.7}$$

Definition 2.14. Given a flat double cover $f_0 : S_0 \to Y_0$ as above, suppose that y is a closed point of B; then there is a unique $s \in S_o$ lying above y and we define $\xi_y := \dim_k R^1 g'_*(\mathcal{O}_{X'})_s$, where $g' : S' \to S$ is an arbitrary resolution of singularities.

Keep the notation we introduced for the canonical resolution; then, by formula (2.6), we can compute ξ_y:

$$\xi_y := \sum_{i \leqslant n-1} \delta_i(y)(l_i^2 - l_i)/2, \tag{2.8}$$

where

$$\delta_i(y) = \begin{cases} 1 & \text{if } y_i \text{ is mapped to } y \text{ by } Y_i \to Y, \\ 0 & \text{otherwise.} \end{cases}$$

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By definition, in case Y is projective, we have

$$\chi(\mathcal{O}_{S_0}) - \chi(\mathcal{O}_{S_n}) = \chi(R^1 g_* \mathcal{O}_{S_n}) = \sum_{y \in B} \xi_y. \tag{2.9}$$

DEFINITION 2.15. (1) A point $y \in B$ as above is called a negligible singularity of the first kind if B is locally the union of two non-singular divisors.

(2) A point $y \in B$ as above is called a negligible singularity of the second kind if B is locally the union of three non-singular divisors such that at least two of them meet properly at y.

It is evident from (2.8) that both kinds of negligible singularities have $\xi_y = 0$. So, we are allowed to neglect them in the computation of $\chi(\mathcal{O}_{X_n})$.

Finally, we give an application of the theory of flat double covers.

DEFINITION 2.16. In this paper we call a projective curve E over a field K *hyperelliptic* (respectively *pseudo-hyperelliptic*) if it is geometrically integral and admits a flat double cover over $\mathbb{P}^1_K$ (respectively a smooth plane conic).

PROPOSITION 2.17. *Let E be a normal projective geometrically rational curve (see Definition 2.7) over a field K. If E is pseudo-hyperelliptic, then $p_a(E) = (p^i + p^j - 2)/2$ for some non-negative integer i, j.*

Proof. We can extend K to its separable closure and suppose that K is separably closed. We have a flat double cover $E \to \mathbb{P}^1_K$ with reduced branch divisor B (E is normal). Write $B = b_1 + \cdots + b_n$. Let $d_i := [k(b_i) : K]$; this is a power of p by assumption. The flat double cover $E_{\bar{K}} \to \mathbb{P}^1_{\bar{K}}$ has its branch divisor $B_{\bar{K}}$ supported in n points, with multiplicities powers of p. So, the normalisation of $E_{\bar{K}}$ is a flat double cover of $\mathbb{P}^1_{\bar{K}}$ branched at these n (reduced) points. However, by assumption, this normalisation being a smooth rational curve, we find that $n = 2$ by Proposition 2.11(2). So, $\deg(B) = d_1 + d_2$ is of the form $p^i + p^j$ and $p_a(E) = \deg(B)/2 - 1$ is of the form $(p^i + p^j - 2)/2$ by Proposition 2.11(2). $\square$

2.3 On a Bertini-type theorem

Let S be a proper scheme over a field k and let $\mathcal{L} = \mathcal{O}_S(D)$ be an invertible sheaf on S. By $|D|$, we denote the set of the effective divisors linearly equivalent to D. Let $H^0(S, \mathcal{O}_S(D))^\vee$ be the dual of the k-vector space $H^0(S, \mathcal{O}_S(D))$. We have a bijection

$$(H^0(S, \mathcal{O}_S(D)) \backslash \{0\})/k^* = \mathbb{P}(H^0(S, \mathcal{O}_S(D))^\vee)(k) \to |D|,$$

which maps $s \in H^0(S, \mathcal{O}_S(D)) \backslash \{0\}$ to $D + \mathrm{div}(s)$.

A linear system V of $|D|$ is, by definition, the set $D + \mathrm{div}(s)$, $s \in \widetilde{V} \backslash \{0\}$, where $\widetilde{V}$ is a linear subspace of $H^0(S, \mathcal{O}_S(D))$; we call this linear system the associated linear system of V. The above bijection establishes a bijection between V and the rational points $\mathbb{P}(\widetilde{V}^\vee)(k)$.

Let $f : X \to C$ be a flat fibration between proper integral varieties over an infinite field k. Let K be the function field of C and let X_η/K denote the generic fibre of f. Let $\mathcal{L} = \mathcal{O}_X(D)$ be an invertible sheaf on X and let $V \subseteq |D|$ be a linear system. Denote by D_η the restriction of the divisor D to X_η and by V_K the linear system of $|D_\eta|$ generated by the effective divisors D'_η, $D' \in |D|$. Namely, the vector space $\widetilde{V}_K$ associated to V_K is exactly $K(i(\widetilde{V})) \subseteq H^0(X_\eta, \mathcal{O}_{X_\eta}(D_\eta))$, where $i : H^0(X, \mathcal{O}_X(D)) \hookrightarrow H^0(X_\eta, \mathcal{O}_{X_\eta}(D_\eta))$ is the canonical restriction map.

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Lemma 2.18. *Consider the map*

$$r : V = \mathbb{P}(\widetilde{V}^\vee)(k) \to V_K = \mathbb{P}((\widetilde{V}_K)^\vee)(K)$$

defined by $D' \mapsto D'_\eta$. Then r is continuous for the Zariski topology. Moreover, for any Zariski non-empty open subset U of V_K, $r^{-1}(U)$ is a non-empty Zariski open subset of V.

Proof. Let $(\widetilde{V}_K)^\vee \hookrightarrow \widetilde{V}^\vee \otimes_k K$ be the dual map of the surjective map $\widetilde{V} \otimes_k K \to \widetilde{V}_K$. It induces a dominant rational map $j : \mathbb{P}(\widetilde{V}^\vee \otimes_k K) \dashrightarrow \mathbb{P}((\widetilde{V}_K)^\vee)$. Let Ω be the domain of definition of this rational map. Then we see easily that the canonical map $\mathbb{P}(\widetilde{V}^\vee)(k) \to \mathbb{P}(\widetilde{V}^\vee \otimes_k K)(K)$ is continuous for the Zariski topology and has image in $\Omega(K)$. Our map r is nothing but the composition of this canonical map and $\Omega(K) \to \mathbb{P}(\widetilde{V}_K^\vee)(K)$.

So, r is continuous for the Zariski topology. In particular, $r^{-1}(U)$ is open. As k and K are infinite, it is well known that $\mathbb{P}(\widetilde{V}^\vee)(k) \hookrightarrow \mathbb{P}(\widetilde{V}^\vee \otimes_k K)(K)$ has dense image in $\mathbb{P}(\widetilde{V}^\vee \otimes_k K)$; in particular, it is dense in Ω. Since Ω dominates $\mathbb{P}(\widetilde{V}_K)$, $r^{-1}(U)$ is non-empty. □

We say that a general member of V has a certain property (P) if there is a non-empty (Zariski) open subset of $\mathbb{P}(\widetilde{V})(k)$ such that each member in this subset satisfies the property (P). This lemma then shows that if a general member of V_K has property (P), so does $D'_\eta := D'|_\eta$, where $D' \in V$ is a general member.

Proposition 2.19. *Assume that $f : X \to C$ is a fibration from a smooth proper surface to a smooth curve over an algebraically closed field; if the generic fibre $X_\eta/K(C)$ is geometrically integral and V is a movable linear system on X, let $D \in V$ be a general member: then its horizontal part D_h is reduced and separable over C if the morphism $\phi : X_\eta \to \mathbb{P}(\widetilde{V}_K)$ defined by V_K is separable.*

Proof. Note that D_h is reduced and separable over C if and only if D_η is étale over K. By Lemma 2.18, it then suffices to prove that a general member of V_K is étale over K. As V is movable, so is V_K. Therefore, a general member of V_K equals

$$\phi^*(\text{a general hyperplane in } \mathbb{P}(\widetilde{V}_K)).$$

Now, since $\phi(X_\eta)$ is geometrically integral (hence only has finitely many non-smooth points over K) and ϕ is separable (hence étale outside finitely many points), a general member of V_K will evidently be étale over K. □

Remark 2.20. Let V, D be as above.

(1) If $p \nmid D \cdot F$ (F is a fibre of X/C), then ϕ is automatically separable.
(2) If V is not composed with pencils, then D is furthermore irreducible by [Jou83, Theorem 6.11].

3. Surfaces of general type with negative c_2

In this section, we first recall some basic background on algebraic surfaces with negative c_2 and then proceed to prove our Theorem 1.3, except for the last statement, which will be done in the section below.

We shall briefly explain our idea. Note that once we know that the inequality (CdF) fails in positive characteristic (see § 4), we immediately obtain $\kappa_p < 1/12$ from (1.1) and, moreover, in order to study κ_p, we only have to consider those surfaces of general type with negative c_2. In this section, we will elaborately study the numerical invariants of algebraic surfaces of general type with negative c_2 after [She91b].

Let k be any algebraically closed field of characteristic $p > 0$ and let X be a minimal surface of general type with negative $c_2(X)$. We first recall a theorem of Shepherd-Barron on the structure of the Albanese morphism of X.

Theorem 3.1 [She91b, Theorem 6]. *The Albanese morphism of X factors through a fibration $f : X \to C$ such that:*

(1) *C is a non-singular projective curve of genus $q := g(C) \geqslant 2$, $f_*\mathcal{O}_X \simeq \mathcal{O}_C$ and $\mathrm{Alb}_X \simeq \mathrm{Alb}_C$;*
(2) *the geometric generic fibre of f is an integral singular rational curve with unibranch singularities only.*

We then introduce the following notation according to this theorem:

(a) $K := K(C)$ (respectively $\overline{K}$; η; $\overline{\eta}$) is the function field of C (respectively a fixed algebraic closure of K; the generic point of C; a fixed geometric generic point of C);
(b) F : is a general fibre of f;
(c) $g := p_a(X_\eta)$ is the arithmetic genus of the generic fibre of f;
(d) $p_g := h^2(X, \mathcal{O}_X)$ is the geometric genus of X;
(e) $q(X) := h^1(X, \mathcal{O}_X)$ is the irregularity of X. Since $\mathrm{Alb}_X \simeq \mathrm{Alb}_C$, we have the following inequality due to Igusa [Igu60]:

$$q(X) \geqslant \dim \mathrm{Alb}_X = \dim \mathrm{Alb}_C = q; \tag{3.1}$$

(f) denote by Z the fixed part of $|K_X|$, Z_h the horizontal part of Z and $Z_0 := (Z_h)_{\mathrm{red}}$;
(g) let $f^*(\Omega_{C/k})(\Delta)$ be the saturation of the injection $f^*\Omega_{C/k} \to \Omega_{X/k}$. Define $N := f^*K_C + \Delta$ to be the divisor class of $f^*(\Omega_{C/k})(\Delta)$. It is well known that Δ is supported on the non-smooth locus of f; in particular, each irreducible horizontal component of Δ is inseparable over C;
(h) for any effective divisor D on X, we will use both D_η and $D|_{X_\eta}$ to denote its restriction to the generic fibre of f and we use D_h, D_v to denote its horizontal and vertical parts.

If we let $S := X_\eta/K$, then by our construction we have $\mathcal{O}_{\Delta_\eta} \simeq \mathcal{A} := (\Omega_{X_\eta/K})_{\mathrm{tor}}$ and $\mathcal{O}_X(\Delta)|_{X_\eta} \simeq \det \mathcal{A}$. Therefore, Proposition 2.8 implies the following lemma.

Lemma 3.2. *We have:*

(1) *$(p-1) \mid 2g$;*
(2) *if $g < (p^2-1)/2$, then $\deg_K(\Delta_\eta) = 2pg/(p-1)$; In particular, if $g = (p-1)/2$, then Δ_h is integral.*

We also have an elementary lemma on p_g, which is useful later on.

Lemma 3.3. *We have $p_g > 2(q-1)/3$.*

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Proof. We have

$$\begin{aligned} p_g - 1 &= \chi(\mathcal{O}_X) - 1 + (q(X) - 1) \\ &\geqslant \frac{K_X^2 - 4(q-1)}{12} - 1 + (q-1) \\ &= \frac{K_X^2 + 8(q-1) - 12}{12} \end{aligned} \tag{3.2}$$

and hence $p_g > 2(q-1)/3$. □

From Noether's formula (1.1), to bound κ_p from below, we only have to bound $\lambda(X) := K_X^2/(q-1)$ and $\gamma(X) := c_2(X)/(q-1)$. One lower bound of $\gamma(X)$ comes naturally once we apply the Grothendieck–Ogg–Shafarevich formula [SGA5, Exposé X] to X to obtain the following formula:

$$c_2(X) = -4(q-1) + \sum_{c \in |C|} (b_2(X_c) - 1) \geqslant -4(q-1). \tag{3.3}$$

Here we note that $H^1_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Q}_l) = 0$, as $X_{\overline{\eta}}$ is a rational curve with unibranch singularities only; hence, the Swan conductor and $b_1(X_c)$ both vanish. By the way, this formula also shows that X is supersingular in the sense of Shioda.

Proposition 3.4. *The surface X is supersingular in the sense that $b_2(X) = \varrho(X)$; here $\varrho(X)$ is the Picard number of X.*

Proof. Using the fibration $f : X \to C$, we have

$$\varrho(X) \geqslant 2 + \sum_{c \in |C|} (^\sharp\{\text{irreducible components of } X_c\} - 1) = 2 + \sum_{c \in |C|} (b_2(X_c) - 1).$$

On the other hand, since $b_1(X) = 2q$ and $c_2(X) = 2 - 2b_1 + b_2$, we get

$$b_2 = 2 + \sum_{c \in |C|} (b_2(X_c) - 1) \leqslant \varrho(X)$$

from (3.3). Hence, $b_2 = \varrho(X)$ and X is supersingular. □

Remark 3.5. Since X is dominated by a ruled surface, Proposition 3.4 can also be derived from [Shi74, § 2, Lemma].

It remains to find lower bounds of $\lambda(X) = K_X^2/(q-1)$. Note that pulling back by an étale cover of C, $\lambda(X)$ is invariant while $q-1$ and K_X^2 are multiplied by the degree of the cover; thus, we can assume that

$$q \gg \lambda(X) > 0, \quad K_X^2 \gg 0.$$

We shall first go through Shepherd-Barron's method in [She91b] quickly, based on which we will give an improvement.

Lemma 3.6. *Assume that H is a reduced horizontal divisor on X such that any of its irreducible components is separable over C; then we have*

$$N \cdot H \leqslant (H + K_X) \cdot H.$$

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Proof. We consider the morphism $\mathcal{O}_X(N)|_H \to \omega_{H/k}$ given by the composition $\mathcal{O}_X(N)|_H = f^*(\Omega_{C/k})(\Delta)|_H \to \Omega_{X/k}|_H \to \Omega_{H/k} \to \omega_{H/k}$. We show that under our assumption this morphism is injective. Taking the degrees in $\mathcal{O}_X(N)|_H \hookrightarrow \omega_{H/k}$ will then imply that $N \cdot H \leqslant \deg(\omega_{H/k}) = (K_X + H) \cdot H$.

Let $\xi_i \in X_\eta$ be the generic point of an irreducible component H_i of H. Then ξ_i belongs to the smooth locus of X_η/K, so $(\mathcal{O}_X(N)|_H)_{\xi_i} \to \omega_{H/k,\xi_i}$ coincides with $(f^*\Omega_{C/k})_{\xi_i} \to \Omega_{H/k,\xi_i}$ and the latter is injective because $H_i \to C$ is separable. So, the kernel of $\mathcal{O}_X(N)|_H \to \omega_{H/k}$ is a skyscraper sheaf. As $\mathcal{O}_X(N)$ is an invertible sheaf and H has no embedded points (it is locally a complete intersection), the kernel is trivial and $\mathcal{O}_X(N)|_H \to \omega_{H/k}$ is injective. □

COROLLARY 3.7. *If the complete linear system $|H|$ is movable and defines a separable generically finite map, then $N \cdot H \leqslant (H + K_X) \cdot H$.*

Proof. It suffices to show that a general member of $|H|$ is integral and separable over C, but this follows immediately from Proposition 2.19 and its remark. □

With the help of [She91a, Theorems 24, 25 and 27] and under our assumption $q \gg \lambda(X) > 0$, $K_X^2 \gg 0$, we then see that the linear systems:

(1) $|2K_X|$ for $p > 2, g > 2$;
(2) $|3K_X|$ for $p = 2, g > 2$

are base-point free and define birational morphisms.

Applying Lemma 3.6 to the above linear systems, we then obtain the following result.

COROLLARY 3.8. (1) *If $p \geqslant 3, g \geqslant 3$, then*

$$4(g-1)(q-1) + K_X \cdot \Delta_h \leqslant 3K_X^2. \tag{3.4}$$

(2) *If $p = 2, g \geqslant 3$, then*

$$4(g-1)(q-1) + K_X \cdot \Delta_h \leqslant 4K_X^2. \tag{3.5}$$

Since $K_X \cdot \Delta_h > 0$, we obtain:

COROLLARY 3.9 (Shepherd-Barron). (1) *if $p \geqslant 3, g \geqslant 3$, then $K_X^2 > 4(g-1)(q-1)/3$;*
(2) *if $p = 2, g \geqslant 3$, then $K_X^2 > (g-1)(q-1)$.*

We now begin to improve this estimation of $\lambda = K_X^2/(q-1)$ by considering its canonical system $|K_X|$.

LEMMA 3.10. *If $|K_X|$ is composed with a pencil, then $|K_X| = Z + f^*|M|$, where M is a divisor on C such that $h^0(C, M) = p_g$, and*

$$K_X^2 \geqslant \min\{4(p_g - 1)(g-1), 2(p_g + q - 1)(g-1)\}.$$

Proof. Assume that $|K_X|$ is composed with a pencil. If the pencil is not C, then $K_X \sim_{alg} Z + aV$, with $a \geqslant p_g - 1$, and V is an integral divisor dominating C. So, by [Eke88, Proposition 1.3], we have either

$$K_X^2 \geqslant 2a(p_a(V) - 1) \geqslant 2(p_g - 1)(q-1) > \lambda(X)(q-1) = K_X^2$$

or

$$K_X^2 \geqslant a^2 \geqslant (2(q-1)/3-1)^2 > \lambda(X)(q-1) = K_X^2,$$

which is a contradiction. Here we have used our assumption $q-1 \gg \lambda(X)$ and Lemma 3.3. So, the pencil is C and therefore $|K_X| = Z + f^*|M|$ and $h^0(C, M) = p_g$. The inequality

$$K_X^2 \geqslant K_X \cdot f^*M = (2g-2)\deg M \geqslant \min\{4(p_g-1)(g-1), 2(p_g+q-1)(g-1)\}$$

follows from the Clifford theorem below. □

Lemma 3.11 (Clifford theorem; see, e.g., [Bea79], Lemme 1.3). *Let C be a smooth projective curve of genus $q := g(C)$ and let $D \geqslant 0$ be an effective divisor; then either:*

(1) *$\deg D > 2(q-1)$ and $\deg D = h^0(\mathcal{O}_C(D)) + q - 1$; or*
(2) *$2(h^0(\mathcal{O}_C(D)) - 1) \leqslant \deg D \leqslant 2(q-1)$. In particular, this time we have $h^0(\mathcal{O}_C(D)) \leqslant q$.*

Theorem 3.12. *If $p \geqslant 5$, then there is a positive number ϵ (depending on p only) such that $K_X^2 \geqslant (p-3+\epsilon)(q-1)$.*

From this theorem, the inequality $c_2 \geqslant 4(q-1)$ and Noether's formula (1.1), we will immediately obtains a lower bound of κ_p.

Corollary 3.13. *If $p \geqslant 7$, then $\kappa_p > (p-7)/12(p-3)$.*

Remark 3.14. For $p = 5$, we will give the precise value of κ_5 later on.

Proof of Theorem 3.12. Since $(p-1) \mid 2g$, we have either $g \geqslant (p-1)$ or $g = (p-1)/2$. When $g \geqslant p-1$, it follows from Corollary 3.9 (note that this fails if $p = 2, 3$) that $K_X^2 > 4(g-1)(q-1)/3 \geqslant 4(p-2)(q-1)/3$. So, we are left to deal with $g = (p-1)/2$.

Assume that $g = (p-1)/2$. If $|K_X|$ is composed with a pencil, then $K_X^2 \geqslant \min\{2(p_g-1)(p-3), (p_g+q-1)(p-3)\} > (p-3+\epsilon)(q-1)$ for some $\epsilon > 0$ by Lemmas 3.3 and 3.10.

Now we assume that $|K_X|$ is not composed with pencils. Choose a general member $D' \in |K_X - Z|$. Since $D' \cdot F \leqslant K_X \cdot F = 2g-2 = p-3 < p$, D' is integral and separable over C by Proposition 2.19 and its remark. Note that Z_0 is also separable over C and has no common components with D'; since D' is general, we can apply Lemma 3.6 to $H = D' + Z_0$ to obtain

$$(K_X + H) \cdot H \geqslant H \cdot N.$$

Write $Z_h = \sum_i r_i E_i$ into sums of irreducible components and let $G := \sum_i (r_i - 1)E_i$; then $Z_0 = Z_h - G = \sum_i E_i$. As $K_X = D' + Z = H + G + Z_v$, we have

$$\begin{aligned} H \cdot N &= H \cdot f^*K_C + H \cdot \Delta \\ &= \deg_K(H)_\eta \cdot 2(q-1) + H \cdot \Delta \\ &= 2(2g-2)(q-1) - 2\deg_K(G)_\eta \cdot (q-1) + H \cdot \Delta \\ &= 2(p-3)(q-1) - 2\sum_i (r_i-1)(q-1)[K(E_i):K] + H \cdot \Delta \\ &\geqslant 2(p-3)(q-1) - 2\sum_i (r_i-1)(q-1)[K(E_i):K] + H \cdot \Delta_h. \end{aligned}$$

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On the other hand,

$$\begin{aligned}(K_X+H)\cdot H &= (2K_X-G-Z_v)\cdot(K_X-G-Z_v)\\ &= 2K_X^2-2K_X\cdot(G+Z_v)-H\cdot(G+Z_v)\\ &\leqslant 2K_X^2-2K_X\cdot G-\sum_i(r_i-1)E_i^2\\ &= 2K_X^2-K_X\cdot G-\left(\sum_i\right)(r_i-1)[K_X\cdot E_i+E_i^2]\\ &= 2K_X^2-\sum_i 2(r_i-1)(p_a(E_i)-1)-K_X\cdot G\\ &\leqslant 2K_X^2-2\sum_i(r_i-1)(q-1)[K(E_i):K]-K_X\cdot G.\end{aligned}$$

Here we note that since E_i is separable over C, $2p_a(E_i)-2\geqslant 2[K(E_i):K](q-1)$. Combining the two inequalities, we get

$$K_X^2\geqslant(p-3)(q-1)+H\cdot\Delta_h/2+K_X\cdot G/2. \tag{3.6}$$

If $G\neq 0$, then $K_X\cdot G/(q-1)$ will be bounded from below by a positive number depending only on p (see Lemma 3.16 below), so, by (3.6), $K_X^2/(q-1)-p+3$ will be bounded from below by a positive number ϵ depending on p.

Now we only have to deal with the case where $G=0$. In this case H is nothing but the horizontal part of a general member of $|K_X|$. So, we shall apply Lemma 3.15 below with $\Pi=\Delta_h$, so $d=p, g=(p-1)/2$, and we obtain

$$K_X^2\geqslant\frac{6p^2-22p+12}{6p^2-30p+27}(p-3)(q-1)>(p-3+\epsilon)(q-1). \qquad \square$$

LEMMA 3.15. (1) *Let H be the horizontal part of a divisor in $|K_X|$ and assume that H is reduced and separable over C. Let Π be a reduced subdivisor of Δ_h. Assume that $d:=\Pi\cdot F\leqslant 4(g-1)$ and a general member of $|2K_X|$ is an integral divisor that is separable over C. Then we have*

$$\left(2+\frac{3(g-1)}{d}-\frac{d}{4(g-1)}\right)K_X^2\geqslant\frac{4d+4g-2}{d}(g-1)(q-1).$$

(2) *The same as above, but assume furthermore that $d\leqslant 2(g-1)$ and $2\Pi\leqslant\Delta_h$. Then*

$$\left(2+\frac{3(g-1)}{d}-\frac{d}{2(g-1)}\right)K_X^2\geqslant\frac{4(g+d)}{d}(g-1)(q-1).$$

Proof. (1) By our assumptions, we have the following inequalities by Lemma 3.6:

$$\begin{aligned}(K_X+H)\cdot H &\geqslant N\cdot H,\\ &= f^*K_C\cdot H+H\cdot\Delta,\\ &\geqslant 4(g-1)(q-1)+H\cdot\Delta_h,\\ &\geqslant 4(g-1)(q-1)+H\cdot\Pi,\end{aligned} \tag{3.7}$$

$$3K_X^2\geqslant 4(q-1)(g-1)+K_X\cdot\Delta_h\geqslant 4(q-1)(g-1)+K_X\cdot\Pi. \tag{3.8}$$

Note that since $(2(g-1)\Pi - dH)\cdot F = 0$, by the Hodge index theorem, we have

$$(2(g-1)\Pi - dH)^2 \leqslant 0$$

or, equivalently,

$$H\cdot\Pi \geqslant \frac{(g-1)}{d}\Pi^2 + \frac{d}{4(g-1)}H^2. \tag{3.9}$$

So, we have

$$\begin{aligned}\left(1+\frac{4(g-1)-d}{4(g-1)}\right)K_X^2 &\geqslant K_X\cdot H + \frac{4(g-1)-d}{4(g-1)}H^2\\ &= (K_X+H)\cdot H - \frac{d}{4(g-1)}H^2\\ &\geqslant 4(q-1)(g-1) + H\cdot\Pi - \frac{d}{4(g-1)}H^2 \quad (3.7)\\ &\geqslant 4(q-1)(g-1) + \frac{g-1}{d}\Pi^2 \quad (3.9).\end{aligned} \tag{3.10}$$

On the other hand, we have

$$K_X\cdot\Pi + \Pi^2 = 2p_a(\Pi) - 2 \geqslant 2(q-1) \tag{3.11}$$

and hence

$$\begin{aligned}\left(2+\frac{3(g-1)}{d}-\frac{d}{4(g-1)}\right)K_X^2 &= \left(1+\frac{4(g-1)-d}{4(g-1)}\right)K_X^2 + \frac{3(g-1)}{d}K_X^2\\ &\geqslant 4(q-1)(g-1) + \frac{g-1}{d}\Pi^2 + \frac{3(g-1)}{d}K_X^2 \quad (3.10)\\ &\geqslant 4(q-1)(g-1) + \frac{g-1}{d}\Pi^2 + \frac{4(q-1)(g-1)^2}{d}\\ &\quad + \frac{g-1}{d}K_X\cdot\Pi \quad (3.8)\\ &\geqslant \frac{4d+4g-4}{d}(q-1)(g-1) + \frac{2(g-1)(q-1)}{d} \quad (3.11)\\ &= \frac{4d+4g-2}{d}(g-1)(q-1).\end{aligned}$$

(2) Similar to (1). Just note this time that $H\cdot\Delta \geqslant 2H\cdot\Pi$. □

Lemma 3.16. *Let B be a horizontal integral divisor with $r := [K(B):K]_{\mathrm{sep}}$; then $K_X\cdot B + B^2 \geqslant 2r(q-1)$. In particular,*

$$K_X\cdot B \geqslant \sqrt{2r(q-1)K_X^2 + (K_X^2)^2/4} - K_X^2/2 = (\sqrt{\lambda^2+8r\lambda}-\lambda)(q-1)/2;$$

here $\lambda = \lambda(X)$.

Proof. It is well known that $2(p_a(B)-1) \geqslant 2(p_a(B')-1) \geqslant 2r(p_a(C)-1)$, where B' is the normalisation of B (see [Liu02, pp. 289–291]). So, $K_X\cdot B + B^2 = 2(p_a(B)-1) \geqslant 2r(q-1)$.

(i) If $B^2 \leqslant 0$, clearly $K_X\cdot B \geqslant 2r(q-1) > \sqrt{2r(q-1)K_X^2 + (K_X^2)^2/4} - K_X^2/2$.

(ii) If $B^2 > 0$, B is nef and $(K_X\cdot B)^2 \geqslant B^2K_X^2$; hence,

$$(K_X\cdot B)^2/K_X^2 + (K_X\cdot B)^2 \geqslant 2r(q-1),$$

so $K_X\cdot B > \sqrt{2r(q-1)K_X^2 + (K_X^2)^2/4} - K_X^2/2$. □

Next we apply this method to the cases $p = 3, 5$.

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3.1 Case $p = 5$

When $p = 5$, we aim to work out κ_5 explicitly. The main reason why we can do this is that the smallest possible value of $g = (p-1)/2$ is equal to 2, in which case X_η will automatically be hyperelliptic. We will carry out a calculation of $\chi(\mathcal{O}_X)$ in the case f is hyperelliptic in a later section, which provides a more precise lower bound of $\chi(\mathcal{O}_X)/(q-1)$, and consequently gives the precise value of $\chi/c_1^2 \geqslant 1/32$ when $g = 2$. In this subsection, as a preparation of the proof of $\kappa_5 = 1/32$, we need to show when $g > 2$ that the bound $\chi/c_1^2 \geqslant 1/32$ also holds.

Notice that following Noether's formula and (3.3), in order to prove that $\chi/c_1^2 \geqslant 1/32$, it suffices to show that $K_X^2 \geqslant 32(q-1)/5$. When $g \geqslant 6$, this inequality follows immediately from (3.4). So, we are left to deal with the case $g = 4$ only. So, we assume that $g = 4$ in the rest of this subsection.

Let

$$i : H^0(X, K_X) \hookrightarrow H^0(X_\eta, K_X|_{X_\eta}) \simeq H^0(X_\eta, \omega_{X\eta/K})$$

be the canonical restriction map, $V := |K_X|$ and V_K be its restriction, i.e., $V_K \subset |\omega_{X_\eta/K}|$ is the linear system associated to the K-subspace spanned by $\mathrm{Im}(i)$ (see § 2.3).

LEMMA 3.17. *If $|K_X|$ is not composed with a pencil, and $D' \in |K_X - Z|$ is a general member, then either:*

(1) *D' is integral and separable over C; or*
(2) *Z_h is a section of f and $D'^2 \geqslant 5(p_g - 2)$.*

Proof. We consider the morphism $\phi : X_\eta \to \mathrm{P}_K^{r-1}$ defined by V_K; here r is the dimension of the K-subspace spanned by $\mathrm{Im}(i)$ (note that $r = 1$ will imply that $|K_X|$ is composed with pencils). Note that, by construction, we have a formula

$$\deg\phi \deg(\phi(X_\eta)) + \deg_K Z_\eta = \deg_K \omega_{X_\eta/K} = 6.$$

(1) If ϕ is separable, then D' is integral and separable over C by Proposition 2.19 and its remark.

(2) If ϕ is not separable, then $\deg\phi = 5$ and Z_η is therefore a rational point; hence, Z_h must be a section. On the other hand, since we have $\deg(\phi)|\deg(\phi_{|K_X-Z|})$ (here $\phi_{|K_X-Z|}$ is the canonical map of X), $\deg(\phi_{|K_X-Z|}) \geqslant 5$ and hence $D'^2 \geqslant 5(p_g-2)$ by [Eke88, Proposition 1.3(ii)]. □

THEOREM 3.18. *Under the hypothesis $g = 4$, $K_X^2 \geqslant 32(q-1)/5$.*

Proof. (1) If $|K_X|$ is composed with a pencil, then $|K_X| = Z + f^*|M|$, since this time by Corollary 3.9 we have $K_X^2 > 4(q-1)$; hence, $\chi(\mathcal{O}_X) > 1$ and therefore $\deg M = p_g + q - 1 > 2(q-1)$ by Lemma 3.11. So, we have

$$K_X^2 \geqslant K_X \cdot f^*M = 6\deg M = 6(p_g + q - 1) > 12(q-1).$$

(2) Suppose that $|K_X|$ is not composed with any pencil and a general member $D' \in |K_X - Z|$ is integral and separable over C. Then $D' + Z_0$ is the sum of reduced divisors separable over C. We can apply Lemma 3.6 to the divisor $H := D' + Z_0$. Assume that $Z_h = \sum_i r_i Z_i$ and let $G := Z_h - Z_0 = \sum_i (r_i - 1)Z_i$; then in a similar way for the inequality (3.6), we can obtain

$$2K_X^2 \geqslant 12(q-1) + \sum_i (r_i - 1)K_X \cdot Z_i + H \cdot \Delta_h.$$

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Note that $H \cdot \Delta_h \geqslant 0$ as no component of Z_0 could be inseparable over C. In particular, $K_X^2/(q-1) \geqslant 6$ and consequently

$$K_X \cdot Z_i > (\sqrt{21} - 3)(q-1) > 3(q-1)/2$$

by Lemma 3.16. So, if $K_X^2 \leqslant 32(q-1)/5$, we must have $r_i = 1$ for all i, namely $G = 0$. This means that $H = D' + Z_0 = D' + Z_h$ is the horizontal part of an element in $|K_X|$. Next, by Lemma 3.2, we can find a reduced subdivisor Π of Δ_h such that either:

(1) $d = \Pi \cdot F = 10$; or
(2) $d = \Pi \cdot F = 5$ and $2\Pi \leqslant \Delta_h$.

Now apply this to Lemma 3.15 with the above H, Π; we then immediately obtain $K_X^2 \geqslant 32(q-1)/5$ by an easy calculation.

(3) Suppose that $|K_X|$ is not composed with any pencil, Z_h is a section and $D'^2 \geqslant 5(p_g - 2)$. Then

$$K_X^2 \geqslant D'^2 + K_X \cdot Z_h \geqslant 5(p_g - 2) + K_X \cdot Z_h. \tag{3.12}$$

In particular, by (3.2),

$$K_X^2 \geqslant 5(p_g - 2) \geqslant 5(K_X^2 + 8(q-1) - 24)/12$$

and hence

$$K_X^2 \geqslant (40(q-1) - 120)/7 \geqslant 39(q-1)/7,$$

as $q \gg 0$ by assumption. Combining this with Lemma 3.16, we obtain

$$K_X \cdot Z_h \geqslant (\sqrt{3705} - 39)(q-1)/14 \geqslant 3(q-1)/2.$$

Returning to (3.12) and using (3.2) again, we have

$$K_X^2 \geqslant 5(K_X^2 + 8(q-1) - 24)/12 + 3(q-1)/2,$$

which implies that $K_X^2 > 32(q-1)/5$ as $q \gg 0$ by assumption.

Lemma 3.17 shows that the three cases above are exhaustive. □

Proposition 3.19. *If $p = 5, g \geqslant 4$, then $\chi(\mathcal{O}_X)/c_1^2(X) \geqslant 1/32$.*

Proof. By the above theorem and Corollary 3.9, whenever $g \geqslant 4$, we will have $K_X^2 \geqslant 32(q-1)/5$. As $c_2(X) \geqslant -4(q-1)$, one immediately obtains $\chi(\mathcal{O}_X)/c_1^2(X) \geqslant 1/32$ by Noether's formula (1.1). □

3.2 Case $p = 3$

As another application of our method, we show that $\kappa_3 > 0$ in this subsection. It suffices to prove that there is some positive number ϵ_0 independent of X such that $K_X^2 \geqslant (4 + \epsilon_0)(q-1)$ holds. Following Corollary 3.9, this inequality holds automatically if $g \geqslant 4$. So, we divide our discussion into cases $g = 2$ and 3.

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3.2.1 *Case $g = 3$.*

Lemma 3.20. *One of the following properties is true:*

(1) *$|K_X|$ is composed with a pencil;*
(2) *$|K_X|$ is not composed with a pencil, Z_h is reduced and a general member $D' \in |K_X - Z|$ is integral and separable over C;*
(3) *Z_h is a section and $(K_X - Z)^2 \geqslant 3(p_g - 2)$.*

Proof. Assume that $|K_X|$ is not composed with a pencil. Let $V = |K_X|$ and V_K be its restriction to the generic fibre. Then $B := Z_\eta$ is the fixed part of V_K. Let $\phi : X_\eta \to \mathbb{P}_K^{r-1}$ be the morphism defined by $V_K - B$. Note that as in case $p = 5$, we have a formula

$$\deg\phi \deg(\phi(X_\eta)) + \deg_K B = \deg_K \omega_{X_\eta/K} = 4.$$

Hence, if $\deg_K B \geqslant 2$, we must have either $\deg\phi = 2$, $\deg(\phi(X_\eta)) = 1$ or $\deg\phi = 1$, $\deg(\phi(X_\eta)) = 2$. This first case implies that X_η is pseudo-hyperelliptic, in contradiction to Proposition 2.17; the second implies that $\phi(X_\eta)$ is a plane conic, which is indeed smooth since it is geometrically integral, and X_η is birational to this plane conic, which is a contradiction. So, $\deg_K B \leqslant 1$ and hence Z_h is reduced.

If ϕ is separable, then a general member $D' \in |K_X - Z|$ is as stated in part (2) of our lemma by Proposition 2.19.

If ϕ is inseparable, then $\deg\phi = 3$, $\deg B = 1$. So, Z_h is a section. Note that in this case the canonical map $\phi_{|K_X|} = \phi_{|K_X - Z|}$ of X is also inseparable and hence its degree is at least 3; therefore, $(K_X - Z)^2 \geqslant 3(p_g - 2)$ by [Eke88, Proposition 1.3]. □

Theorem 3.21. *Under the assumption of this subsection* ($g = 3$), *there is some positive constant ϵ_0 independent of X such that $K_X^2 > (4 + \epsilon_0)(q - 1)$.*

Proof. There are three possibilities as below by the previous lemma.

(1) The canonical system $|K_X|$ is composed with a pencil. Then it follows from Lemma 3.10 that

$$K_X^2 \geqslant 4\min\{2p_g - 2, p_g + q - 1\}.$$

Combining this inequality with (3.2), we have:

(1) $K_X^2 \geqslant 2(K_X^2 + 8(q-1) - 12)/3$; or
(2) $K_X^2 \geqslant (K_X^2 + 20(q-1))3$.

Both conditions imply that $K_X^2 \geqslant (4 + \epsilon_0)(q - 1)$ for some constant $\epsilon_0 > 0$ independent of X as $q \gg 0$.

(2) The canonical system $|K_X|$ is not composed with a pencil, Z_h is reduced and a general member $D' \in |K_X - Z|$ is integral and separable over C. So, $D = D' + Z \in |K_X|$ and $D' + Z_h = D_h$ is also a reduced divisor that is separable over C. Let $H = D_h$.

It is clear by Lemma 3.2 that we can find a subdivisor Π of Δ_h such that either:

- $d = \Pi \cdot F = 6$ or 9; or
- $d = \Pi \cdot F = 3$ and $2\Pi \leqslant \Delta_h$.

Now we can apply Lemma 3.15 to the above H and Π, and an easy calculation gives $K_X^2 \geqslant (4 + \epsilon_0)(q - 1)$.

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(3) The canonical system $|K_X|$ is not composed with a pencil, Z_h is a section and $(K_X-Z)^2 \geqslant 3(p_g-2)$. Then we have

$$K_X^2 \geqslant (K_X-Z)^2 + K_X\cdot Z_h \geqslant 3(p_g-2) + K_X\cdot Z_h.$$

Note that (3.4) implies that $K_X^2 \geqslant 8(q-1)/3$ and hence Lemma 3.16 implies that $K_X\cdot Z_h > 4(q-1)/3$, so we get

$$K_X^2 \geqslant 3(p_g-2)+4(q-1)/3.$$

After combining with (3.2) and an easy computation, this inequality will soon imply that $K_X^2 \geqslant (4+\epsilon_0)(q-1)$ for some constant $\epsilon_0>0$ independent of X. □

3.2.2 *Case $g=2$.*

Lemma 3.22. *If $g=2$, then Δ_h is reduced and $\deg_K(\Delta_\eta)=6$.*

Proof. The canonical morphism of X_η/K here is automatically a flat double cover of $\mathbb{P}(H^0(X_\eta,\omega_{X_\eta/K}))$. Let $B\subseteq \mathbb{P}(H^0(X_\eta,\omega_{X_\eta/K}))$ be the branch divisor associated to this double cover; then $\deg B=6$ by Proposition 2.11. Note that X_η/K is geometrically rational, so $\deg_{\overline{K}}(B_{\overline{K}})_{\mathrm{red}}=2$. Hence, B is either an inseparable point of degree 6 or the sum of two inseparable points of degree 3. Now, since Δ_η dominates B and has the same degree over K, Δ_η must be reduced. □

Lemma 3.23. *The bi-canonical system $|2K_X|$ is base-point free and a general member of $|2K_X|$ is integral and separable over C.*

Proof. First, by [She91a, Theorem 25] and our assumption that $K_X^2 \gg 0$, we see that $|2K_X|$ is free of base points. Everything then follows from Proposition 2.19 and its remark. □

From this lemma, we shall apply Lemma 3.6 to $H=2K_X$; hence,

$$3K_X^2 \geqslant 4(q-1)+K_X\cdot\Delta_h. \tag{3.13}$$

Lemma 3.24. *Either:*

(1) *$|K_X|$ is composed with a pencil; or*

(2) *$|K_X|$ is not composed with a pencil, Z is vertical and a general member $D\in|K_X-Z|$ is an integral horizontal divisor such that $D^2\geqslant 2(p_g-2)$.*

Proof. Suppose that $|K_X|$ is not composed with a pencil. Let $V:=|K_X-Z|$; since V has a horizontal part, $1<F\cdot(K_X-Z)\leqslant F\cdot K_X=2$ and hence Z is vertical. It then follows from Proposition 2.19 and its remark that a general member $D\in V$ is integral and separable over C. Finally, [Eke88, Proposition 1.3] shows that $D^2\geqslant 2(p_g-2)$ as the canonical map has degree at least 2 in this case. □

Theorem 3.25. *Under the assumption of this section ($g=2$), we have $K_X^2>(4+\epsilon_0)(q-1)$ for some positive constant ϵ_0 independent of X.*

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Proof. (1) If $|K_X|$ is composed with a pencil, then $|K_X| = Z + f^*|M|$, and $\deg M \geqslant \min\{2p_g - 2, p_g + q - 1\}$ (Lemma 3.10). Note in this case that the components Δ_h are different from any component of Z for sake of degree over C, so

$$K_X \cdot \Delta \geqslant K_X \cdot \Delta_h = Z \cdot \Delta_h + 6 \deg M \geqslant 6 \deg M.$$

Hence, (3.13) shows that

$$3K_X^2 \geqslant 4(q-1) + 6 \deg M.$$

After combining this with (3.2) and an easy computation, we obtain

$$K_X^2 \geqslant (4 + \epsilon_0)(q-1).$$

(2) Suppose that $|K_X|$ is not composed with a pencil. Let $D \in |K_X - Z|$ be a general member. By Lemma 3.6, we have

$$(K_X + D) \cdot D \geqslant N \cdot D \geqslant 4(q-1) + D \cdot \Delta. \tag{3.14}$$

Since by construction $(3D - \Delta_h) \cdot F = 0$, we have $(3D - \Delta_h)^2 \leqslant 0$, i.e.

$$D \cdot \Delta_h \geqslant 3D^2/2 + \Delta_h^2/6.$$

Combining this with (3.14) and Lemma 3.24, we see that

$$\begin{aligned} K_X^2 &\geqslant (K_X + D) \cdot D - D^2 \geqslant 4(q-1) + D \cdot \Delta_h - D^2 \\ &\geqslant 4(q-1) + D^2/2 + \Delta_h^2/6 \geqslant 4(q-1) + p_g - 2 + \Delta_h^2/6. \end{aligned}$$

Combining with (3.13) and (3.2), we obtain

$$\begin{aligned} 3K_X^2/2 &= (3K_X^2)/6 + K_X^2 \\ &\geqslant (4 + 4/6)(q-1) + p_g - 2 + (\Delta_h^2 + K_X \cdot \Delta_h)/6 \\ &\geqslant 16(q-1)/3 + p_g - 2 \\ &\geqslant 16(q-1)/3 + (K_X^2 + 8(q-1))/12 - 2. \end{aligned}$$

Hence, $K_X^2 \geqslant 72(q-1)/17 - 24/17 \geqslant (4 + \epsilon_0)(q-1)$ as $q \gg 0$. □

As a consequence of both Theorem 3.21 and the previous theorem, we have the following theorem.

THEOREM 3.26. *We have $\kappa_3 > 0$.*

Proof. In fact, suppose that X is a surface of general type in characteristic 3 with $c_2(X) < 0$, and let $f : X \to C$ be its Albanese fibration as before; then Corollary 3.9 ($g \geqslant 4$), Theorem 3.21 ($g = 3$) and Theorem 3.25 ($g = 2$) show in any case that we have $K_X^2 \geqslant (4 + \epsilon_0)(q-1)$ with a positive constant ϵ_0; q is the genus of C here. On the other hand, since $c_2(X) \geqslant -4(q-1)$, we obtain that $\chi(\mathcal{O}_X)/K_X^2 \geqslant \epsilon_0/(12(4+\epsilon_0)) > 0$ by (1.1), namely $\kappa_3 > 0$. □

4. Examples

In this section, we will present some examples of surfaces of general type with negative c_2 and calculate some of their numerical invariants.

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4.1 Examples of Raynaud

Let us briefly recall the examples of Raynaud [Ray78].

Let k be an algebraically closed field of characteristic $p > 2$, and assume that C is a smooth projective curve of genus $q \geqslant 2$ such that there is an $f \in K(C)$ satisfying $(df) = pD$ for some divisor D. Let $\mathcal{L} = \mathcal{O}_C(D)$, $l = \deg D$ and $\mathcal{M}$ be any invertible sheaf on C such that $\mathcal{M}^{\otimes 2} \simeq \mathcal{L}$. We have $m := \deg \mathcal{M} = l/2$ and $2q - 2 = pl = 2pm$.

By [Ray78, Proposition 1], we can find a rank-2 locally free sheaf $\mathcal{E}$ and its associated ruled surface $\rho : Z := \mathbb{P}(\mathcal{E}) \to C$ such that:

(1) $\det(\mathcal{E}) \simeq \mathcal{L}$; in particular, $\mathcal{O}(1)^2 = l$;
(2) there is a section $\Sigma_1 \in |\mathcal{O}(1)|$;
(3) there is a multi-section Σ_2 such that the canonical morphism $\rho : \Sigma_2 \to C$ is isomorphic to the Frobenius morphism;
(4) $\Sigma_1 \cap \Sigma_2 = \emptyset$;
(5) $\mathcal{O}_Z(\Sigma_2) = \mathcal{O}(p) \otimes \rho^*(\mathcal{L}^{\otimes -p})$.

Let $\Sigma := \Sigma_1 + \Sigma_2$; then Σ is a non-singular divisor of Z, and

$$\mathcal{O}_Z(\Sigma) = \mathcal{O}(p+1) \otimes \rho^*(\mathcal{L}^{\otimes -p}) = \left(\mathcal{O}\left(\frac{p+1}{2}\right) \otimes \rho^*(\mathcal{M}^{\otimes -p})\right)^{\otimes 2};$$

hence, the data $\{\mathcal{O}((p+1)/2) \otimes \rho^*(\mathcal{M}^{\otimes -p}), \Sigma \in |(\mathcal{O}((p+1)/2) \otimes \rho^*(\mathcal{M}^{\otimes -p}))^{\otimes 2}|\}$ defines a flat double cover $\pi : S \to Z$ by Construction 2.10.

Proposition 4.1. *We have:*

(1) $\omega_{Z/k} = \mathcal{O}(-2) \otimes (\rho^*\mathcal{L}^{\otimes p+1})$ *and* $\omega_{S/k} = \pi^*(\mathcal{O}((p-3)/2) \otimes \rho^*\mathcal{M}^{\otimes p+2})$;
(2) $\chi(\mathcal{O}_S) = (p^2 - 4p - 1)l/8$, $K_S^2 = (3p^2 - 8p - 3)l/2$ *and* $c_2(S) = -4(q-1)$;
(3) *S is a minimal surface of general type if* $p \geqslant 5$.

Proof. By Proposition 2.12, S is regular.

(1) Since $\det \mathcal{E} = \mathcal{L}$, $\Omega_{C/k} \simeq \mathcal{L}^{\otimes p}$, we immediately get

$$\omega_{Z/k} = \mathcal{O}(-2) \otimes \rho^*\mathcal{L}^{\otimes p+1}$$

and

$$\omega_{S/k} = \pi^*\left(\mathcal{O}\left(\frac{p-3}{2}\right) \otimes \rho^*\mathcal{M}^{\otimes p+2}\right).$$

(2) By Proposition 2.11, we have

$$\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_Z) + \frac{\Sigma^2 + 2\Sigma \cdot K_Z}{8} = \frac{p^2 - 4p - 1}{8} l$$

and

$$K_Z^2 = \pi^*\left(\mathcal{O}\left(\frac{p-3}{2}\right) \otimes \rho^*\mathcal{M}^{\otimes p+2}\right)^2 = 2\left(\mathcal{O}\left(\frac{p-3}{2}\right) \otimes \rho^*\mathcal{M}^{\otimes p+2}\right)^2 = \frac{3p^2 - 8p - 3}{2} l;$$

therefore, $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = -2pl = -4(q-1)$.

(3) If $p \geqslant 5$, then any closed fibre of $S \to C$ is irreducible and has arithmetic genus $(p-1)/2$; hence, S is a minimal surface of general type. □

Remark 4.2. (1) Note that the fibration $S \to C$ is uniruled. In this case we do not have the positivity of the dualising sheaf $\omega_{S/C}$ (compare with [Szp81, § 2]). We shall point out that $\omega_{S/C}$ here is *not nef*. In fact, $\omega_{S/C} = \omega_{S/Z} \otimes \pi^*\omega_{Z/C} = \pi^*(\mathcal{O}((p-3)/2) \otimes \rho^*\mathcal{M}^{\otimes 2-p})$; however,

$$\Sigma_1 \cdot \left(\mathcal{O}\left(\frac{p-3}{2}\right) \otimes \rho^*\mathcal{M}^{\otimes 2-p}\right) = -l/2 < 0.$$

(2) If $p=3$, Raynaud's example is a quasi-elliptic surface and hence it is not of general type. This is one of the reasons why we can find κ_5 but not κ_3 in this paper.

(3) We have $\chi(\mathcal{O}_S)/K_S^2 = (p^2-4p-1)/(4(3p^2-8p-3))$.

4.2 Examples in characteristic 2, 3

First we give an example of surfaces with negative c_2 over a field k of characteristic 3. Choose $m = 3^n - 1$ points, say $t_1, \ldots, t_m$, on $\mathbb{A}_k^1 = \mathbb{P}_k^1 \backslash \{\infty\}$; we can construct a cyclic cover $C \to \mathbb{P}_k^1$ of degree m such that the branch locus equals $B := \sum_i t_i$ canonically as we did before for flat double covers (see Construction 2.10). In particular, by Hurwitz's formula, $q - 1 := g(C) - 1 = (3^n-1)(3^n-4)/2$.

Let $Y := \mathbb{P}_C^1$, $p_1 : Y \to C$ and $p_2 : Y \to \mathbb{P}_k^1$ be the canonical projections. Let Π_1 be the divisor $C \times_k \{\infty\}$, and Π_2 be the divisor which is the image of $C \xrightarrow{F^n \times h} C \times_k \mathbb{P}_k^1 = Y$; here F^n is the nth Frobenius morphism. Then $\Pi := \Pi_1 + \Pi_2$ is an even divisor (i.e., $\Pi = 2D$ for some divisor D); in particular, we can define a flat double cover $\pi : S' \to Y$ whose branch locus equals Π.

PROPOSITION 4.3. *Let S be the minimal model of S'; when $n \geqslant 2$, S is of general type and $c_2(S) \leqslant -4(q-1) + 3m$.*

Sketch of the proof. We consider the canonical resolution of S. We have that Π_1 and Π_2 intersect properly, and the singularities of Π_2 are the pre-images of B. Blowing up these points ($2m$ points in total), we get the desingularisation of Π. Consequently, we get a desingularisation $S_1 \to S'$. It is clear that $S_1 \to C$ has $2m$ non-irreducible fibres (each has two components); therefore, we have

$$c_2(S) \leqslant c_2(S_1) = -4(q-1) + 3m$$

by the Grothendieck–Ogg–Shafarevich formula (see formula (3.3) below). □

Remark 4.4. When $n \to +\infty$, we see that $c_2(S)/(q-1) \to -4$.

We mention that in characteristic 2 there are also surfaces of general type with negative c_2. One example is [Lie08b, Theorem 7.1], where $c_1^2 = 14, \chi = 1$ and $c_2 = -2$.

5. Case of hyperelliptic fibration

We keep the notation of § 4(a)–(h). In this section, we calculate $\chi(\mathcal{O}_X)$ directly under the assumption that $p \geqslant 5, g = (p-1)/2$ and X_η is pseudo-hyperelliptic. Our calculation will show that our conjectural κ_p is indeed the best bound of χ/c_1^2 for these surfaces. It is natural to believe that those surfaces whose $\chi/(c_1^2)$ approaches κ_p should appear in the case $g = (p-1)/2$, the smallest possible value of g, so somehow we have proven our conjecture for the 'hyperelliptic part'.

From now on we assume that X_η is pseudo-hyperelliptic and $g = (p-1)/2$.

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By our assumption, X_η is a flat double cover of a smooth plane conic P; the smooth conic is automatically equal to the strict line $\mathbb{P}^1$ by Tsen's theorem. Let $B \subset P$ be the branch divisor of this flat double cover; then $\deg B = p+1$ by Proposition 2.11. Since X_η/K is normal but not geometrically normal by assumption, B/K is reduced but not geometrically reduced (Proposition 2.12). Therefore, B contains at least one inseparable point. Consequently, B is the sum of a rational point and an inseparable point of degree p.

We then identify P with the generic fibre of $p_1 : Z = \mathbb{P}^1_C \to C$ in a way such that the rational point contained in B is the infinity point. Here we denote by U, V the two homogeneous coordinates of $\mathbb{P}^1$, and ∞ is defined by $V = 0$. Denote by Θ_K the inseparable point contained in B, so Θ_K is defined by $U^p - hV^p$ for a certain element $h \in K\backslash K^p$.

Let X_0 be the normalisation of Z in $K(X)$, and let Π be the branch divisor associated to this flat double cover $X_0 \to Z$; then $B = \Pi|_{\mathbb{P}^1_K}$. Define Π_1 (respectively Π_2) to be the closure of $\infty \in B$ (respectively $\Theta_K \in B$) in Z and Π_3 to be the remaining vertical branch divisors.

Here by abuse of language we denote by h not only the element of K mentioned above to define Θ_K but also the unique morphism $h : C \to \mathbb{P}^1_k$ that maps $u = U/V$ to h in function fields. Define $\alpha := \deg(h)$ and $A := h^*(\infty)$; it is clear that $\deg A = \alpha$.

With some local computations, we immediately obtain the next proposition on the configuration of Π.

Proposition 5.1. *We have:*

(1) $\Pi_1 = C \times_k \infty$, *and* $\Pi_1 \cap \Pi_2$ *equals* $A \times \infty$;

(2) $\mathcal{O}_Z(\Pi_2) = \mathcal{O}(p) \otimes \mathcal{O}_Z(p_1^*A)$, *the canonical morphism* $\Pi_2 \to C$ *is a homeomorphism and the singularities of* Π_2 *are exactly the pre-image of points on* C *where the morphism* h *is ramified;*

(3) $\mathcal{O}_Z(\Pi_1) = \mathcal{O}(1)$, $\Pi_3 = p_1^*D$, *for a reduced divisor* D. *Let* $d := \deg D$; *then* $\alpha + d$ *is even, and* $\mathcal{O}_Z(\Pi) = \mathcal{O}(p+1) \otimes p_1^*\mathcal{O}_C(A+D)$;

(4) $\chi(\mathcal{O}_{X_0}) = (p-3)(q-1)/2 + (p-1)(\alpha+d)/4$.

Here we note that the last statement comes from Corollary 2.2.

We are going to run the canonical resolution of singularities (Definition 2.13) to $X_0 \to Z$ to obtain $\chi(\mathcal{O}_X)$. We first need to analyse the singularities of Π. From the above proposition, non-negligible singularities of Π are all lying on Π_2. Since Π_2 is homeomorphic to C via p_1, we shall use the following conventions: if $b_2 \in \Pi_2$ is a singularity of Π, we divide it into one of the four types below according to its image $b := p_1(b_2) \in C$, and use the notation ξ_b to denote ξ_{b_2} (see Definitions 2.13 and 2.14, here the flat double cover is taken to be $X_0 \to Z$). The four types of singularities are:

- Type I: $b \notin (A \cup D)$ and b is a ramification of h. The local function of Π near b_2 is $u^p - h$ in $\mathcal{O}_{C,b}[u]$;
- Type II: $b \in D\backslash A$. The local function of Π near b_2 is $t(u^p - h)$ in $\mathcal{O}_{C,b}[u]$, where t is a uniformiser of $\mathcal{O}_{C,b}$;
- Type III: $b \in A\backslash D$. The local function of Π near b_2 is $v(v^p - 1/h)$ in $\mathcal{O}_{C,b}[v]$;
- Type IV: $b \in (A \cap D)$. The local function of Π near b_2 is $tv(v^p - 1/h)$ in $\mathcal{O}_{C,b}[v]$, where t is a uniformiser of $\mathcal{O}_{C,b}$.

Denote
$\mathcal{S} := \{b \mid b \text{ is of type I, II, III or IV}\}$;
$\mathcal{T} := \{b \mid b \text{ is of type III or IV and } h \text{ is unramified or tamely ramified at } b\}$;
$\mathcal{W} := \{b \mid b \text{ is of type III or IV and } h \text{ is wildly ramified at } b\}$.

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By (2.9),

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_0}) - \sum_{b\in\mathcal{S}} \xi_b = \frac{(p-3)(q-1)}{2} + \frac{(p-1)(\alpha+d)}{4} - \sum_{b\in\mathcal{S}} \xi_b.$$

Set

$$d_b := \begin{cases} 1, & b \in D, \\ 0, & b \notin D. \end{cases}$$

Then

$$\chi(\mathcal{O}_X) = \frac{(p-3)(q-1)}{2} + \frac{(p-1)\alpha}{4} + \sum_{b\in\mathcal{S}} \left(\frac{(p-1)d_b}{4} - \xi_b \right). \tag{5.1}$$

Next we study in detail these four kinds of singularities. Before we continue, we shall introduce the notation of differential ramification index and ramification type.

Suppose that $\phi : D \to C$ is *a separable morphism* between two smooth curves over k. Let $d \in D$ be any closed point and $c := \phi(d)$. Choose an arbitrary uniformiser $s \in \mathcal{O}_{C,c}$ of c.

Definition 5.2. We define the *differential ramification index* of ϕ at d to be the number $R_d(\phi) := \dim_k(\Omega_{D/C})_d$.

And, we define the *type of ramification* at d to be a set $\Lambda_d(\phi)$ of numbers as below.

(1) If ϕ is wildly ramified at d, $\Lambda_d(\phi) := \{v(s), R_d(\phi)\}$, where v is the normalised valuation at d. Note here that $v(s)$ is the usual ramification index and $p \mid v(s)$ by assumption; we also define $j_d(\phi) := v(s)/p$.

(2) If ϕ is tamely ramified at d, $\Lambda_d(\phi) := \{R_d(\phi)\}$. Note that in this case $p \nmid v(s) = R_d(\phi) + 1$. □

When no confusion can occur, we shall use R_d and Λ_d instead of $R_d(\phi)$ and $\Lambda_d(\phi)$.

Remark 5.3. By abuse of language, we can also talk about the differential ramification index and ramification type of a certain kind of function as below. Suppose that $s \in \mathcal{O}_{d,D} \backslash \mathcal{O}_{d,D}^p$ is an element in the maximal ideal of $\mathcal{O}_{d,D}$; then it defines a separable morphism (still denoted by s) $s : U \to \mathbb{P}^1$ canonically, where U is the open subset of D where s is regular. By mixing the function s and the associated morphism s, we are allowed to talk about its differential ramification index $R_d(s)$ and ramification type $\Lambda_d(s)$.

From our definition of differential ramification index, we have Hurwitz's formula (see [Liu02, Theorem 4.16 and Remark 4.17]).

Proposition 5.4. *Suppose that $\phi : D \to E$ is a separable morphism between smooth projective curves. Then*

$$2 \deg \phi (g(E) - 1) + \sum_d R_d(\phi) = 2g(D) - 2. \tag{5.2}$$

Now let us continue; we will find a relation between $((p-1)d_b)/4 - \xi_b$ and the differential ramification index $R_b(h)$ (defined above) for all b. In order to do this, we give a definition as follows.

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TABLE 1. Table of $\lambda_1 = 0$.

	m_0	l_0	Equation of B_1 on $\mathrm{Spec}(\mathcal{O}_{C,b}[x/t])$	Equation of singularities	$(l_0^2 - l_0)/2$
I	p	$\frac{p-1}{2}$	$t((x/t)^p - e_1)$	$t((x/t)^p - e_1)$	$\frac{(p-1)(p-3)}{8}$
II	$p+1$	$\frac{p+1}{2}$	$(x/t)^p - e_1$	$(x/t)^p - e_1$	$\frac{(p-1)(p+1)}{8}$
III	$p+1$	$\frac{p+1}{2}$	$(x/t)((x/t)^p - e_1)$	$(x/t)((x/t)^p - e_1)$	$\frac{(p-1)(p+1)}{8}$
IV	$p+2$	$\frac{p+1}{2}$	$t(x/t)((x/t)^p - e_1)$	$t(x/t)((x/t)^p - e_1)$	$\frac{(p-1)(p+1)}{8}$

DEFINITION 5.5. Suppose that $b \in C$ is a closed point, $t \in \mathcal{O}_{C,b}$ is a uniformiser, v is the canonical discrete valuation and $e \in t\mathcal{O}_{C,b} \backslash \mathcal{O}_{C,b}^p$; we consider an arbitrary flat double cover $S_0 \to Y_0 := \mathrm{Spec}(\mathcal{O}_{C,b}[x])$ with branch divisor $B_0 = \mathrm{div}(x^p - e)$ (respectively $\mathrm{div}(t(x^p - e))$, $\mathrm{div}(x(x^p - e))$, $\mathrm{div}(tx(x^p - e))$). Let Q denote the point associated to the maximal ideal (x, t) of $\mathrm{Spec}(\mathcal{O}_{C,b})[x]$; then we define the number $\xi_{\mathrm{I},e}$ (respectively $\xi_{\mathrm{II},e}$, $\xi_{\mathrm{III},e}$, $\xi_{\mathrm{IV},e}$) to be ξ_Q with respect to this flat double (see Definition 2.14).

Note that by definition we have that $\xi_b = \xi_{*,e}$ for some e such that $\Lambda_b(e) = \Lambda_b(h)$ (see Definition 5.2); here $*$ is the type of b (i.e., I, II, III or IV).

Note also that if $R_b(e) \geqslant p$ (see Definition 5.2), then $e = t^p(\lambda_1 + e_1)$ for a unique $\lambda_1 \in k$ and $e_1 \in t\mathcal{O}_{C,b}$. In particular, $R_b(e_1) = R_b(e) - p$, and $\lambda_1 \neq 0$ if and only if $v(e) = p$. If we blow up $y_0 = Q$ to obtain the first step of the canonical resolution (see Definition 2.13), we can obtain a recursion relation as follows.

LEMMA 5.6. (1) *If $R_b(e) \geqslant p$, then*

$$\xi_{\mathrm{I},e} = \frac{(p-1)(p-3)}{8} + \xi_{\mathrm{II},e_1}, \quad \xi_{\mathrm{II},e} = \frac{(p-1)(p+1)}{8} + \xi_{\mathrm{I},e_1}. \tag{5.3}$$

(2) *If $R_b(e) \geqslant p$ and $v(e) > p$, then*

$$\xi_{\mathrm{III},e} = \frac{(p-1)(p+1)}{8} + \xi_{\mathrm{III},e_1}, \quad \xi_{\mathrm{IV},e} = \frac{(p-1)(p+1)}{8} + \xi_{\mathrm{IV},e_1}. \tag{5.4}$$

(3) *If $R_b(e) \geqslant p$ and $v(e) = p$, then*

$$\xi_{\mathrm{III},e} = \frac{(p-1)(p+1)}{8} + \xi_{\mathrm{I},e_1}, \quad \xi_{\mathrm{IV},e} = \frac{(p-1)(p+1)}{8} + \xi_{\mathrm{II},e_1}. \tag{5.5}$$

Proof. According to the process of canonical resolution, after blowing up we get two tables (Tables 1 and 2); everything then follows from the tables. We remark that it is clear outside the open subset $\mathrm{Spec}(\mathcal{O}_{C,b}[x/t])$ that B_1 could have at worst negligible singularities. □

LEMMA 5.7. (1) *The number $\xi_{*,e}$ depends on the ramification type $\Lambda_b(e)$ (see Definition 5.2) rather than e itself.*

(2) *If $*$ is I or II, then $\xi_{*,e}$ depends on $R_b(e)$ only.*

Since $\xi_{*,e}$ depends on the ramification type $\Lambda_b(e)$ rather than e, we shall also write $\xi_{*,\Lambda}$ to denote $\xi_{*,e}$ for those e with $\Lambda_b(e) = \Lambda$.

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TABLE 2. Table of $\lambda_1 \neq 0$.

	m_0	l_0	Equation of B_1 on $\mathrm{Spec}(\mathcal{O}_{C,b}[x/t])$	Equation of singularities	$(l_0^2 - l_0)/2$
I	p	$\frac{p-1}{2}$	$t((x/t-\lambda_1^{1/p})^p - e_1)$	$t((x/t-\lambda_1^{1/p})^p - e_1)$	$\frac{(p-1)(p-3)}{8}$
II	$p+1$	$\frac{p+1}{2}$	$(x/t-\lambda_1^{1/p})^p - e_1$	$(x/t-\lambda_1^{1/p})^p - e_1$	$\frac{(p-1)(p+1)}{8}$
III	$p+1$	$\frac{p+1}{2}$	$(x/t)((x/t-\lambda_1^{1/p})^p - e_1)$	$(x/t-\lambda_1^{1/p})^p - e_1$	$\frac{(p-1)(p+1)}{8}$
IV	$p+2$	$\frac{p+1}{2}$	$t(x/t)((x/t-\lambda_1^{1/p})^p - e_1)$	$t((x/t-\lambda_1^{1/p})^p - e_1)$	$\frac{(p-1)(p+1)}{8}$

Proof. By Lemma 5.6, if $R_b(e) \geqslant p$, then $\xi_{*,e}$ is determined by ξ_{*,e_1} and whether $\lambda_1 = 0$ or not. However, it is clear that the ramification type of $\Lambda_b(e)$ is also determined by $\Lambda_b(e_1)$ and whether $\lambda_1 = 0$ or not, so our lemma is true if it is true for cases $R_b(e) < p$; the latter is clear. □

LEMMA 5.8. (1) *For any e, we have*

$$\frac{(p-1)^2 R_b(e)}{8p} - \xi_{\mathrm{I},e} \geqslant 0; \tag{5.6}$$
$$\frac{(p-1)^2 R_b(e)}{8p} + \frac{p-1}{4} - \xi_{\mathrm{II},e} \geqslant 0. \tag{5.7}$$

(2) *For any e with tame ramification, we have*

$$\frac{(p-1)(p+1)R_b(e)}{8p} - \xi_{\mathrm{III},e} \geqslant -\frac{p-1}{4p}; \tag{5.8}$$
$$\frac{(p-1)(p+1)R_b(e)}{8p} + \frac{p-1}{4} - \xi_{\mathrm{IV},e} \geqslant -\frac{p-1}{4p}. \tag{5.9}$$

(3) *If e has wild ramification and $\Lambda_b(e) = \{pj, R_b(e)\}$, then*

$$\frac{(p-1)^2 R_b(e)}{8p} - \xi_{\mathrm{III},e} \geqslant -\frac{(p-1)j}{4}; \tag{5.10}$$
$$\frac{(p-1)^2 R_b(e)}{8p} + \frac{p-1}{4} - \xi_{\mathrm{IV},e} \geqslant -\frac{(p-1)j}{4}. \tag{5.11}$$

We shall put the proof of this lemma in the appendix as it is a bit long.

LEMMA 5.9. $\alpha = \sum_{b\in\mathcal{T}}(R_b(h)+1) + \sum_{b\in\mathcal{W}}(pj_b(h))$.

Proof. By definition,

$$A = \sum_{b\in\mathcal{T}}(R_b(h)+1)b + \sum_{b\in\mathcal{W}} pj_b(h)b.$$

Taking the degrees, we obtain our lemma. □

THEOREM 5.10. *Under the assumption that $g = (p-1)/2$ and X_η being pseudo-hyperelliptic, we have $\chi(\mathcal{O}_X) \geqslant (p^2-4p-1)(q-1)/4p$.*

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Proof. By (5.1),

$$\chi(\mathcal{O}_X) = \frac{(p-3)(q-1)}{2} + \frac{(p-1)(\alpha+d)}{4} - \sum_{b\in\mathcal{S}} \xi_b.$$

Lemmas 5.8 and 5.9 show that

$$\begin{aligned}\frac{(p-1)d}{4} - \sum_{b\in\mathcal{S}}\xi_b &\geqslant -\sum_{b\in\mathcal{S}}\frac{(p-1)^2R_b(h)}{8p} - \sum_{b\in\mathcal{T}}\frac{(p-1)(R_b(h)+1)}{4p} - \sum_{b\in\mathcal{W}}\frac{(p-1)j}{4}\\ &= -\sum_{b\in\mathcal{S}}\frac{(p-1)^2R_b(h)}{8p} - \frac{(p-1)\alpha}{4p}.\end{aligned}$$

Hence,

$$\begin{aligned}\chi(\mathcal{O}_X) &= \frac{(p-3)(q-1)}{2} + \frac{(p-1)(\alpha+d)}{4} - \sum_{b\in\mathcal{S}}\xi_b\\ &\geqslant \frac{(p-3)(q-1)}{2} + \frac{(p-1)\alpha}{4} - \sum_{b\in\mathcal{S}}\frac{(p-1)^2R_b(h)}{8p} - \frac{(p-1)\alpha}{4p}\\ &= \frac{(p^2-4p-1)(q-1)}{4p};\end{aligned}$$

by Hurwitz's formula,

$$2\alpha + 2(q-1) = \sum_{b\in\mathcal{S}} R_b(h). \tag{5.12}$$

□

COROLLARY 5.11. *Under the assumption that $g = (p-1)/2$ and X_η being pseudo-hyperelliptic, the optimal bound of χ/c_1^2 is $(p^2-4p-1)/4(3p^2-8p-3)$.*

Proof. Since $\chi(\mathcal{O}_X) \geqslant (p^2-4p-1)(q-1)/4p$, we see that

$$\frac{\chi(\mathcal{O}_X)}{K_X^2} = \frac{\chi(\mathcal{O}_X)}{12\chi(\mathcal{O}_X) - c_2(X)} \geqslant \frac{\chi(\mathcal{O}_X)}{12\chi(\mathcal{O}_X) + 4(q-1)} \geqslant \frac{p^2-4p-1}{4(3p^2-8p-3)}.$$

On the other hand, Raynaud's example in § 4.1 gives examples whose χ/c_1^2 is equal to $(p^2-4p-1)/4(3p^2-8p-3)$. □

Another consequence of this corollary is the value of κ_5.

COROLLARY 5.12. *We have $\kappa_5 = 1/32$.*

Proof. When $g = (p-1)/2$, X_η is automatically hyperelliptic and hence the best bound of χ/c_1^2 is $1/32$ for these surfaces by the former corollary. Combining this with Proposition 3.19, we obtain $\kappa_5 = 1/32$. □

Finally, as a consequence of Corollary 5.11, we make the following conjecture.

CONJECTURE 5.13. For any $p \geqslant 5$, we have $\kappa_p = (p^2-4p-1)/4(3p^2-8p-3)$.

Remark 5.14. The part $\kappa_p \leqslant (p^2-4p-1)/4(3p^2-8p-3)$ follows from Raynaud's example; see § 4, Remark 4.2(3).

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6. Proof of Lemma 5.8

Assume that $\mathrm{char}(k) \neq 2$, $a, b \in \{0, 1\}$ and $m, n \in \mathbb{N}_+$ are two numbers co-prime to each other. Let $S := \mathrm{Spec}(k[[x, y, t]]_{(x,y,t)}/(y^2 - x^a t^b(x^m - t^n)))$ and $f : \widetilde{S} \to S$ be an arbitrary desingularisation; we define $\xi(a, b, m, n) := \dim_k R^1 f_* \mathcal{O}_{\widetilde{S}}$.

PROPOSITION 6.1. *If $2 \nmid m$, then*

$$\xi(a, b, m, n) \leqslant \frac{(m-1)^2(n-1)}{8m} + \frac{(m-1)n}{4m}a + \frac{m-1}{4}b.$$

First we point out an algorithm for calculating $\xi(a, b, m, n)$.

LEMMA 6.2. (1) *If $m = 1$ or $n = 1$, then $\xi(a, b, m, n) = 0$.*

(2) *If $m > n > 1$, then*

$$\xi(a, b, m, n) = \begin{cases} \xi(0, b, m-n, n) + (a+b+n)(a+b+n-2)/8, & \text{if } 2 \mid a+b+n, \\ \xi(1, b, m-n, n) + (a+b+n-1)(a+b+n-3)/8, & \text{if } 2 \nmid a+b+n. \end{cases}$$

(3) *If $n > m > 1$, then*

$$\xi(a, b, m, n) = \begin{cases} \xi(a, 0, m, n-m) + (a+b+m)(a+b+m-2)/8, & \text{if } 2 \mid a+b+m, \\ \xi(a, 1, m, n-m) + (a+b+m-1)(a+b+m-3)/8, & \text{if } 2 \nmid a+b+m. \end{cases}$$

Proof. In fact, S is obtained as a flat double cover of $Y := \mathrm{Spec}(k[x, t])$ with branch divisor $B = \mathrm{div}(x^a t^b(x^m - t^n))$. Our lemma follows from the process of the canonical resolution (see Definition 2.13). □

LEMMA 6.3. *Proposition 6.1 holds if it holds for all $n < m$.*

Proof. Let $n = m + n'$. If $2 \mid a + b + m$, then, by Lemma 6.2, we have

$$\begin{aligned} &\frac{(m-1)^2(n-1)}{8m} + \frac{(m-1)n}{4m}a + \frac{m-1}{4}b - \xi(a, b, m, n) \\ &\quad \geqslant \frac{(m-1)^2(n'-1)}{8m} + \frac{(m-1)n'}{4m}a - \xi(a, 0, m, n'). \end{aligned}$$

If $2 \nmid a + b + m$, then we also have

$$\begin{aligned} &\frac{(m-1)^2(n-1)}{8m} + \frac{(m-1)n}{4m}a + \frac{m-1}{4}b - \xi(a, b, m, n) \\ &\quad \geqslant \frac{(m-1)^2(n'-1)}{8m} + \frac{(m-1)n'}{4m}a + \frac{m-1}{4} - \xi(a, 1, m, n'). \end{aligned}$$

So, it is sufficient to prove the inequality for the pair (m, n'). □

Proof of Proposition 6.1. We shall proceed by induction on m. When $m = 1$, the statement holds trivially. Assume that our proposition holds for odd numbers smaller than m; we need to show that it also holds for m. By Lemma 6.3, we can assume that $n < m$.

If $2 \nmid n$, then

$$\begin{aligned}\xi(a,b,m,n) = \xi(b,a,n,m) &\leqslant \frac{(n-1)^2(m-1)}{8n} + \frac{(n-1)m}{4n}b + \frac{n-1}{4}a \\ &\leqslant \frac{(m-1)^2(n-1)}{8m} + \frac{m-1}{4}b + \frac{(m-1)n}{4m}a.\end{aligned}$$

If $2 \mid n$, let $m = n + m'$; then, by Lemma 6.2, we have

$$\begin{aligned}\xi(a,0,m,n) &= \xi(a,0,m',n) + \frac{n(n-2)}{8} \\ &\leqslant \frac{(m'-1)^2(n-1)}{8m'} + \frac{(m'-1)n}{m'}a + \frac{n(n-2)}{8} \\ &< \frac{(m-1)^2(n-1)}{8m} + \frac{(m-1)n}{m}a,\end{aligned}$$

$$\begin{aligned}\xi(0,1,m,n) &= \xi(1,1,m',n) + \frac{n(n-2)}{8} \\ &\leqslant \frac{(m'-1)^2(n-1)}{8m'} + \frac{(m'-1)n}{4m'} + \frac{m'-1}{4} + \frac{n(n-2)}{8} \\ &< \frac{(m-1)^2(n-1)}{8m} + \frac{m-1}{4},\end{aligned}$$

$$\begin{aligned}\xi(1,1,m,n) &= \xi(0,1,m',n) + \frac{n(n+2)}{8} \\ &\leqslant \frac{(m'-1)^2(n-1)}{8m'} + \frac{m'-1}{4} + \frac{n(n+2)}{8} \\ &\leqslant \frac{(m-1)^2(n-1)}{8m} + \frac{(m-1)n}{m} + \frac{m-1}{4}.\end{aligned}$$

Here we note that

$$\frac{(m-1)^2(n-1)}{8m} - \frac{(m'-1)^2(n-1)}{8m'} = \frac{n(n-1)}{8} - \frac{n(n-1)}{8mm'},$$

and the last equality holds only if $n = m - 1$. □

Proof of Lemma 5.8. The previous proposition asserts our statements (a) and (b) immediately by Lemma 5.6.

For (c), we assume that $\Lambda(e) = \{pj, R_b(e)\}$; then $e = t^{pj}(\lambda + e')$, with $\lambda \neq 0$. Hence, by Lemma 5.6, we have

$$\begin{aligned}\frac{(p-1)^2R_b(e)}{8p} - \xi_{\mathrm{III},e} &= \frac{(p-1)^2R_b(e')}{8p} - \xi_{\mathrm{I},e'} + \frac{(p-1)^2j}{8} - \frac{(p+1)(p-1)j}{8} \\ &= \left(\frac{(p-1)^2R_b(e')}{8p} - \xi_{\mathrm{I},e'}\right) - \frac{(p-1)j}{4} \\ &\geqslant -\frac{(p-1)j}{4}\end{aligned}$$

and similarly

$$\frac{(p-1)^2R_b(e)}{8p} + \frac{p-1}{4} - \xi_{\mathrm{IV},e} \geqslant -\frac{(p-1)j}{4}.$$ □

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