ON THE RELATION BETWEEN A CLUSTER SET INTRODUCED BY CONSTANTINESCU AND CORNEA, AND THE FINE CLUSTER SET OF CARTAN, BRELOT, AND NAÏM

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The field of boundary limit theorems in analytic function theory is usually considered to have begun about 1906, with the publication of Fatou's thesis [8]. In this remarkable memoir a theorem is proved, that now bears the author's name, which implies that any bounded holomorphic function defined on the unit disk possesses an angular limit almost everywhere (Lebesgue measure) on the frontier. Outstanding classical contributions to this field can be attributed to F. and M. Riesz, R. Nevanlinna, Lusin, Privaloff, Frostman, Plessner, and others. During the past decade, there have been serious attempts to generalise the classical theory in such a way as to include analytic functions from a hyperbolic Riemann surface R (that is, one which tolerates a Green's function) into an arbitrary Riemann surface R'. The most elaborate results in this direction can be found in two papers, one by Constantinescu and Cornea [6], henceforth to be designated by C-C, and the other by Doob [7]. Since the term "cluster set" is fundamental through this discussion, we briefly define this term. Let f be a function from R into R' and $\mathscr{F}$ a filter of subsets of R. The cluster set of f along $\mathscr{F}$, denoted by $C_{\mathscr{F}}(f)$, is defined in such a way that $p' \in C_{\mathscr{F}}(f)$ if and only if p' is a member of the closure of $f(F_\alpha)$ for every $F_\alpha \in \mathscr{F}$.

Both C-C and Doob employ the boundary of R. S. Martin [10], as a fundamental tool, but Doob makes use of the fine cluster set of Cartan-Brelot-Naïm [11], and its associated limit, whereas C-C apparently introduce their own cluster set and limit. In Math. Reviews (Vol. 23, A1025), Doob commented

that the cluster set of C-C was actually the fine cluster set of Cartan-Brelot-Naïm. In this article, we shall show that the fine cluster set is always contained in the C-C cluster set, and that the two sets are identical for a continuous function. Doob's remark is therefore true for analytic functions, but not in general. For example, the C-C cluster set of a super-harmonic function may strictly contain the fine cluster set. The following notation shall be employed throughout:

R, a hyperbolic Riemann surface.

$\triangle$, the Martin boundary of R, [10].

$\triangle_1$, the subset of $\triangle$ consisting of the minimal points.

$\hat{R}$, the compact metric space consisting of the set $R \cup \triangle$, endowed with the Martin metric, ([10], p. 147).

R', any Riemann surface.

$\triangle'$, the Martin boundary of R' as defined by C-C ([6], p. 44).

$\hat{R}'$, the compact Martin space $R' \cup \triangle'$.

f, an analytic function from R into R'.

1. The C-C Cluster Set and Associated Limit. Let G be a region in R, ∂G its frontier in R, and η the identity mapping from G into R. Following C-C, we define HP(G) to be the set of non-negative harmonic functions on G.

The Set HP(η), ([6], pp. 9, 10): In this section, F is defined to be an open subset of G such that F is relatively compact in R. We denote the frontier of F, (which is entirely contained in R), by ∂F. For any u $\in$ HP(G) we define u on ∂F as follows:

$$\left. \begin{array}{l} u_* \equiv u \text{ on } \partial F \cap G \\ \\ \equiv 0 \text{ on } \partial F \cap \partial G \end{array} \right\}.$$

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At the non-isolated points of ∂F, the function u_* is lower semi-continuous and therefore resolutive ([12], p. 111).

Let $H^F_{u_*}$ denote the generalised solution of the Dirichlet problem (Wiener function) on F which is defined by the boundary function u_*. We now define $HP(\eta) \subset HP(G)$ to be

$$HP(\eta) = \{\, u \in HP(G):\ u \equiv H^F_{u_*} \text{ on every } F \,\}.$$

The Operator I_G ([6], p. 18). For any $v \in HP(R)$, we define $I_G v = \sup \{\, u \in HP(\eta): u \leq v \text{ on } G \,\}$. C-C have shown that $I_G v$ is itself a member of the set $HP(\eta)$. The term "inverse extremalisation" employed to describe this operator comes from Kuramochi ([9], p. 577).

The Set $\Delta_1(G) \subset \Delta_1$ ([6], p. 40).

Let $\hat{p} \in \Delta_1$, and $K_{\hat{p}}$ the associated minimal harmonic function. Following C-C, we define $\Delta_1(G) = \{\hat{p} \in \Delta_1: I_G K_{\hat{p}} \not\equiv 0\}$. For a given $\hat{p} \in \Delta_1$, we define $\{G_\alpha\}$ to be the family of open subsets of R possessing the property that $\hat{p} \in \Delta_1(G_\alpha)$ for every G_α in this family. C-C have shown that $\{G_\alpha\}$ satisfies the axioms of a filter base on R ([6], p. 25). The filter on R, whose base is $\{G_\alpha\}$, shall be denoted by the symbol Φ_{C-C}.

The C-C Cluster Set $\hat{M}_f(\hat{p})$ ([6], p. 46).

We recall that f is an analytic function from a hyperbolic Riemann surface R into an arbitrary Riemann surface R'. Following C-C, we define $\hat{M}_f(\hat{p})$ to be the cluster set (contained in $\hat{R}'$) of f at $\hat{p}$ along the filter Φ_{C-C}. Actually C-C defined $\hat{M}_f(\hat{p}) = \bigcap_\alpha \overline{f(G_\alpha)}$ where closure is relative to $\hat{R}'$. Henceforth, we shall refer to $\hat{M}_f(\hat{p})$ as the C-C cluster set of f at $\hat{p}$, and in the special circumstance where $\hat{M}_f(\hat{p})$ reduces to a singleton subset of $\hat{R}'$, we say that f possesses a C-C limit at $\hat{p}$ and denote this limit by $\hat{f}(\hat{p})$.

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2. The Fine Cluster Set of Cartan, Brelot, and Naïm.

While it is true that some of the following definitions were originally given in relation to classical Euclidean space, such definitions extend naturally to hyperbolic Riemann surfaces.

Thin Sets ([1], p. 327).

Let R be a hyperbolic Riemann surface, $p_0 \in R$ and $E \subset R$. Following Brelot, we define E to be thin (effilé) at p_0 if one of the following conditions is satisfied:

(i) p_0 is isolated (relative to the usual topology) with respect to the set $E \cup \{p_0\}$.

(ii) There exists a non-negative, superharmonic function $V(p)$ on R possessing the property that $V(p_0) < \varliminf_{\substack{p \to p_0 \\ p \in E}} V(p)$;

($\varliminf$ means limit inferior).

Pseudo-Limit ([2], p. 29).

Brelot ([1], p. 327) observed the following properties of thin sets:

(i) If E is thin at p_0 and $E' \subset E$ then E' is thin at p_0.

(ii) If E_1 is thin at p_0, and E_2 is thin at p_0, then $E_1 \cup E_2$ is also thin at p_0.

Let $\{E\}$ be the family of subsets of R such that every member of $\{E\}$ is thin at $p_0 \in R$, and $\{R - E\}$ the family of the complements of $\{E\}$. Then $\{R - (E \cup \{p_0\})\}$ satisfies the axioms of a filter on R. Following Brelot, we define the pseudo-limit of f at p_0 to be the limit of f along the filter $\{R - (E \cup \{p_0\})\}$ provided that such a limit exists.

The Fine Topology of H. Cartan ([5], pp. 236-237).

Cartan introduced a topology on R defined to be the least topology that makes the superharmonic functions on R continuous. He called this topology the fine topology, and noticed that a set

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$N \subset R$ is a fine neighbourhood of $p_o \in R$ if and only if $(R - N)$ is thin at p_o in the sense of Brelot. The cluster set of f at p_o along the filter of deleted fine neighbourhoods is defined to be the _fine cluster set_ of f at p_o, and if the fine cluster set at p_o is a singleton subset, say $\{\hat{p}'_o\} \subset \hat{R}'$, then we say that $\hat{p}'_o$ is the _fine limit_ of f at p_o. A function f has a fine limit at p_o in the sense of Cartan if and only if it has a pseudo-limit at p_o in the sense of Brelot and the two limits are equal.

The Fine Topology of L. Naïm ([11], Chapt. 2).
Let $G(p, q)$ be the Green's function of R and $K(p, q) \equiv \dfrac{G(p, q)}{G(p, q_o)}$ be the _normalized Green's function_ defined on $(R - \{q_o\}) \times R$. The _Martin kernel function_, denoted by $K(\hat{p}, q)$ is defined on $(\hat{R} - \{q_o\}) \times R$ where $K(\hat{p}, q)$ is the normalized Green's function on $(R - \{q_o\}) \times R$. For any _mass distribution_ μ on R, (a non-negative Borel measure), we define the function

$$\bigcup(\hat{p}) \equiv \int_R K(\hat{p}, q) d\mu_q$$

to be the _Martin potential_ of the mass distribution μ. We must keep in mind that $\bigcup$ is defined on $\hat{R} - \{q_o\}$. On the subset $R - \{q_o\}$ of its domain of definition, however,

$$\bigcup(p) \equiv \int_R \frac{G(p, q)}{G(p, q_o)} d\mu_q \equiv \frac{V(p)}{G(p, q_o)}$$

where $V(p)$ is the _Green potential_ of μ, and in particular a positive superharmonic function.

In her thesis, L. Naïm defined a set $E \subset R$ to be _thin_ at a point $\hat{p}_o \in \hat{R}$ if one of the following conditions is satisfied:

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(i) $\hat{p}_0$ is an isolated point (in the Martin topology) of the set $E \cup \{\hat{p}_0\}$.

(ii) There exists a Martin potential $\bigcup(\hat{p})$ of some mass distribution μ on R such that $\bigcup(\hat{p}_0) < \underset{\substack{p \to \hat{p}_0 \\ p \in E}}{\lim} \bigcup(p)$.

A set $E \subset R$ is thin at $p_0 \in R$ in the sense of Naïm if and only if it is thin there in the sense of Brelot. Naïm proved that R itself is thin at $\hat{p} \in \Delta$ if and only if $\hat{p}$ is not a minimal point of Δ, that is $\hat{p} \in (\Delta - \Delta_1)$ ([11], p. 25).

In order to be able to define potentials of a mass distribution on $\hat{R} - \{q_0\}$, Naïm ([11], pp. 28-32) introduced a new kernel function, denoted by $\Theta(\hat{p}, \hat{q})$, defined on $(\hat{R} - \{q_0\}) \times (\hat{R} - \{q_0\})$. This function has the pleasant property of being symmetric in both arguments. On the subset $(R - \{q_0\}) \times (R - \{q_0\})$ of its domain of definition, Θ is defined to be

$$\Theta(p, q) \equiv \frac{G(p, q)}{G(p, q_0) \, G(q, q_0)} \, ,$$

and on $(\hat{R} - \{q_0\}) \times (R - \{q_0\})$, Θ is defined to be

$$\Theta(\hat{p}, q) \equiv \frac{K(\hat{p}, q)}{G(q, q_0)} \, .$$

Naïm ([11], p. 30-32) shows how Θ can be extended in the second argument to $(\hat{R} - \{q_0\})$ so that $\Theta(\hat{p}, \hat{q})$ is lower semi-continuous in each argument.

In the particular case where $R = \{z: |z| < 1\}$, $q_0 = 0$; then

$$G(z, o) \equiv \log \frac{1}{|z|}, \quad K(e^{i\theta}, z) \equiv \frac{1 - |z|^2}{|e^{i\theta} - z|^2}, \quad G(z, w) \equiv \log \left| \frac{1 - \bar{w}z}{z - w} \right| \, .$$

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Then

$$\Theta(z, w) \equiv \frac{G(z, w)}{(\log \frac{1}{|z|})(\log \frac{1}{|w|})} \quad \text{on} \quad (R - \{o\}) \times (R - \{o\}),$$

$$\Theta(e^{i\theta}, w) \equiv \frac{K(e^{i\theta}, w)}{\log \frac{1}{|w|}} \quad \text{on} \quad \Delta_1 \times (R - \{o\}),$$

$$\Theta(e^{i\theta}, e^{i\emptyset}) \equiv \lim_{w \to e^{i\emptyset}} \frac{K(e^{i\theta}, w)}{\log \frac{1}{|w|}} \equiv \frac{2}{|e^{i\theta} - e^{i\emptyset}|^2} \quad \text{on} \quad \Delta_1 \times \Delta_1.$$

For any mass distribution μ on $\hat{R} - \{q_o\}$, Naïm defined the function

$$V(\hat{p}) \equiv \int_{(\hat{R} - \{q_o\})} \Theta(\hat{p}, \hat{q}) d\mu_{\hat{q}}$$

to be the $\underline{\Theta \text{ potential}}$ of μ on $\hat{R} - \{q_o\}$. In case μ is only defined on $R - \{q_o\}$, then $V(\hat{p})$ is a Martin potential of some mass distribution on $R - \{q_o\}$.

Following Naïm ([11], p. 40) we now define the fine topology on $\hat{R}$ to be the least topology that makes the Θ potentials continuous. The fine topology of Naïm induces on R the fine topology of H. Cartan, and the frontier of R relative to this topology is the minimal Martin boundary Δ_1. Any set $N \subset R$ is a fine neighbourhood of $\hat{p} \in \Delta_1$ if and only if $R - N$ is thin at $\hat{p}$. For an analytic function f from R into R' we denote the cluster set of f at $\hat{p} \in \Delta_1$ along the filter of fine neighbourhoods as the fine cluster set of f at $\hat{p}$. In case this fine cluster set reduces to a singleton subset of $\hat{R}'$ we say that f possesses a fine limit at $\hat{p}$.

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3. **The $\mathcal{E}$ Operator of Brelot and its Relation to the I Operator of C-C.** A set $E \subset R$ is defined to be <u>polar</u> if there exists a non-negative superharmonic function V on R such that $\lim_{q \to p} V(q) = +\infty$ for every $p \in E$, and $V(q) \not\equiv +\infty$. A function f on R that possesses a given property everywhere on R except on (at most) a polar subset of R is said to possess this property <u>quasi-everywhere</u> on R. For instance, we define V to be quasi-superharmonic on R if V is superharmonic on R quasi-everywhere.

Let $u \in HP(R)$, and G a region in R. We now consider the set $\{V\}$ of positive superharmonic functions on R possessing the property that every member of $\{V\}$ dominates u on $(R - G)$. The lower envelope of $\{V\}$ is a quasi-superharmonic function on R denoted by Σ_u^G ([3], pp. 4, 6).

We now define $\overline{\overline{\Sigma}}_u^G \equiv \Sigma_u^G$ on $S \subset R$ where S is the set on which Σ_u^G is superharmonic, $\overline{\overline{\Sigma}}_u^G(p) = \underline{\lim}_{q \to p} \Sigma_u^G(q)$ for every $p \in R - S$.

The superharmonic function $\overline{\overline{\Sigma}}_u^G$ is said to <u>regularize</u> Σ_u^G, and coincides with the superharmonic function $\mathcal{E}_u^G$ called the <u>extremal</u> of u on G by Brelot ([3], pp. 6, 10). The function

$$\left. \begin{array}{l} \mathcal{E}_u^G \equiv H_{u_*}^G \text{ on } G , \\ \\ \equiv u \text{ quasi-everywhere on } R - G \end{array} \right\}$$

where $H_{u_*}^G$ is the generalized solution of the Dirichlet problem on G defined by the boundary function

$$\begin{cases} u_* \equiv u \text{ on } \partial G \text{ (boundary of } G \text{ in } R) , \\ \\ = 0 \text{ on } \cdots \text{ (one point compactification of } R). \end{cases}$$

It is common nowadays to refer to $\mathcal{E}$ as the extremalisation operator of Brelot.

We now prove a theorem that relates the I Operator of C-C with the $\mathcal{E}$ operator of Brelot.

It is to be noted that C-C ([6], p. 21, Hilfsatz 4) have proved a similar result.

THEOREM 1. For any $u \in HP(R)$, it follows that

$$u \equiv I_G u + \mathcal{E}_u^G \quad \text{on } G, \quad \text{where } G \text{ is any region in } R.$$

Proof. Let us define $V \equiv u - \mathcal{E}_u^G$ on G, remembering that $\mathcal{E}_u^G \equiv H_u^G$ on G, and let F be an open subset of G which is relatively compact in R. We shall first establish that $V \in HP(\eta)$ and hence $V \le I_G u$.

We now define

$$\left. \begin{array}{l} V_* \equiv V \quad \text{on } \partial F \cap G \\[2mm] \equiv 0 \quad \text{on } \partial F \cap \partial G \end{array} \right\},$$

and compare V with $H_{V_*}^F$ on F. For any regular point $p \in \partial G$ ([4], p. 4) it follows that $\lim\limits_{q \to p} H_{u_*}^G = u(p)$, and therefore $\lim\limits_{q \to p} V = 0$. Therefore $V \equiv V_*$ on the regular points of ∂F, and $V \equiv H_{V_*}^F$ on F ([4], p. 1). It follows that $V \in HP(\eta)$, and that $u - \mathcal{E}_u^G \le I_G u$ on G by definition of $I_G u$.

Proceeding in the other direction, we shall establish that $\mathcal{E}_u^G \le u - I_G u$ on G. Since $I_G u \in HP(\eta)$, and since $\overline{\lim}\limits_{q \to p} I_G u < + \infty$ for every $p \in \partial G$, (because $I_G u \le u$ on G),

therefore $\lim_{q \to p} I_G u = 0$ for every regular $p \in \partial G$ ([3], p. 10).

Hence the boundary behaviour of $(u - I_G u)$ is the same as that of u at the regular points of ∂G. By the (lower) extremal property of $\mathcal{E}_u^G$ (and hence H_u^G) on G, it follows that $\mathcal{E}_u^G \leq u - I_G u$ on G, or that $u - \mathcal{E}_u^{G*} \geq I_G u$.

Combining this last result with an earlier one, it follows that $u \equiv I_G u - \mathcal{E}_u^G$ on G.

4. <u>The Relation Between the C-C Cluster Set and the Fine Cluster Set.</u> In order to prove our main result we require the following theorem of Naïm ([11], p. 27, Theorem 5).

THEOREM 2. A set $E \subset R$ is thin at $\hat{p} \in \wedge_1$ if and only if $K_{\hat{p}} \not\equiv \mathcal{E}_{K_{\hat{p}}}^{(R-E)}$ on $(R-E)$, where $K_{\hat{p}}$ is the minimal harmonic function associated with $\hat{p}$.

We are now in a position to prove that the C-C cluster set always contains the fine cluster set.

THEOREM 3. The C-C cluster set contains the fine cluster set of Naïm at $\hat{p} \in \wedge_1$ for any function g from R into R'.

<u>Proof.</u> Let G be any region in R which is a member of the filter $\bar{\Phi}_{C-C}$ at $\hat{p}$. Then $I_G K_{\hat{p}} \not\equiv 0$ on G, and by Theorem 1, $K_{\hat{p}} \not\equiv \mathcal{E}_{K_{\hat{p}}}^G$ on G. <u>It is a consequence of Theorem 2 that $(R-G)$ is thin at $\hat{p}$, and hence G is a deleted fine neighbourhood of $\hat{p}$. It follows that the filter $\bar{\Phi}_{C-C}$ at $\hat{p}$ is contained in the filter of fine neighbourhoods of $\hat{p}$ relativized to R.</u> Therefore the fine cluster set of g at $\hat{p}$ is contained in the C-C cluster set of g at $\hat{p}$.

We now observe that the two cluster sets are not in general equivalent. Let $R = \{ z : |z| < 1 \}$, and $g(z)$ be defined as follows:

$$\begin{cases} g(z) \equiv 0 & \text{on the rational points of } R, \\ \equiv 1 & \text{on the irrational points of } R. \end{cases}$$

Since the set of rational points is thin at every boundary point $e^{i\theta}$ of R, g has the fine limit 1 at $e^{i\theta}$. But the rationals are dense in R, so the C-C cluster set of g at $e^{i\theta}$ is $\{0\} \cup \{1\}$.

We now give an example of a superharmonic function $\bigcup(z)$ on R, where $\bigcup(z)$ has a fine limit at $z = 1$, but the C-C cluster set of $\bigcup$ at $z = 1$ strictly contains the singleton fine cluster set. Let μ be a mass distribution on R such that $\mu(\{q_i\}) = c_i > 0$ where q_i is an arbitrary rational point of R, and $\mu(\mathscr{I}) = 0$ where $\mathscr{I}$ is the set of irrational points of R. Then the Martin potential of this mass distribution is

$$\bigcup(z) = \sum_{i=1}^{\infty} c_i K(z, q_i)$$

on R where $K(z, q_i)$ is the normalized Green's function on R. If $\bigcup(z) \not\equiv +\infty$ then $\bigcup(z)$ is a superharmonic function on R. On the Martin boundary of R, $\bigcup(e^{i\theta}) \equiv \sum_{i=1}^{\infty} c_i K(e^{i\theta}, q_i)$ where $K(e^{i\theta}, q_i)$ is the Poisson Kernel function. We now choose the set $\{c_i\}$ such that

$$\bigcup(1) = \sum_{i=1}^{\infty} c_i \frac{1 - |q_i|^2}{|1 - q_i|^2}$$

is a finite number, say A.

Because $\bigcup(z)$ is continuous in the fine topology, therefore the fine $\lim_{z \to 1} \bigcup(z) = A$, or the fine cluster set of $\bigcup(z)$ at $z = 1$ is $\{A\}$. Since $\bigcup(q_i) = +\infty$ for all rational points, therefore

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the C-C cluster set of $\bigcup(z)$ at $z = 1$ is $[A, +\infty]$.

We shall now prove that if $g(p)$ is a continuous function from R into R' then the fine cluster set of g at $\hat{p} \in \Delta_1$ is identical to the C-C cluster set $\hat{M}_g(\hat{p})$.

THEOREM 4. Let $g(p)$ be a continuous function from R into R'. Then the fine cluster set of g at $\hat{p} \in \Delta_1$ is the C-C cluster set $\hat{M}_g(\hat{p}) \subset \hat{R}'$.

Proof. Let $\hat{q}' \in \hat{M}_g(\hat{p})$, and $\overline{N}$ a closed neighbourhood of $\hat{q}'$. Since g is continuous, $g^{-1}(\overline{N})$ is closed in R. If the set $g^{-1}(\overline{N})$ were thin at $\hat{p}$, then $R - g^{-1}(\overline{N})$ would be a fine, open (usual topology) neighbourhood of $\hat{p}$, and this would contradict the definition of $\hat{q}'$. It follows that $g^{-1}(\overline{N})$ cannot be thin at $\hat{p} \in \Delta_1$.

We now let F be any fine neighbourhood of $\hat{p}$. Since $g^{-1}(\overline{N})$ is not thin at $\hat{p}$ it follows that $g^{-1}(\overline{N}) \cap F \neq \emptyset$. The equivalence of the two cluster sets follows immediately.

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