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ON SKEW-SUPERCOMMUTING MAPS IN SUPERALGEBRAS

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Abstract

Let A be a semiprime superalgebra over a commutative ring F with $\frac{1}{2}$ and $f: A \to A$ a skew-supercommuting map on A. We show that $f = 0$. This gives a version of Brešar's theorem for superalgebras.

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1. Introduction

Let R be an associative ring. For any $x, y \in R$, we shall define $[x, y] = xy - yx$ and $x \circ y = xy + yx$. Let S be a subset of R. A mapping $f : S \to R$ is said to be *skew-commuting* on S if $f(x) \circ x = 0$ for all $x \in S$. A number of authors have discussed the skew-commuting maps and their generalizations [8, 12–14]. In these papers the authors have showed that nonzero derivations and ring endomorphisms cannot be skew-commuting on certain subsets of prime rings (for example, ideals). In [4] Brešar obtained theorems of this kind for general additive maps. He proved that there are no nonzero additive maps that are skew-commuting on either ideals of prime rings of characteristic not 2 [4, Theorem 1] or semiprime rings of 2-torsion free [4, Theorem 2]. In [5] Brešar obtained a different proof of [4, Theorem 1].

In recent years, some results on maps of associative rings have been extended to superalgebras by several authors (see, for example, [1, 2, 7, 9–11, 17]). In the present paper, we shall give a version of Brešar's theorem mentioned above for superalgebras.

2. Preliminaries

Throughout the article, algebras are over a unital commutative associative ring F. We shall assume without further mention that $\frac{1}{2} \in F$. Although this requirement is not always needed, it is assumed for the sake of simplicity.

We use standard terminology and refer the reader to [15] for background information on the constructions of graded algebras and superalgebras. A *superalgebra* A over F is a $\mathbb{Z}_2$-graded associative algebra over F; that is, A is the direct sum of two F-submodules A_0 and A_1 such that $A_0^2 \subseteq A_0$, $A_0 A_1 \subseteq A_1$, $A_1 A_0 \subseteq A_1$ and $A_1^2 \subseteq A_0$. We call A_0 the even part and A_1 the odd part of A. Elements in $H = A_0 \cup A_1$ are said to be *homogeneous*. We define $\sigma : A \to A$ by $(a_0 + a_1)^\sigma = a_0 - a_1$. Note that σ is an automorphism of A such that $\sigma^2 = 1$. Conversely, given an algebra A and an automorphism σ of A with $\sigma^2 = 1$, A then becomes a superalgebra by defining $A_0 = \{a \in A \mid a^\sigma = a\}$ and $A_1 = \{a \in A \mid a^\sigma = -a\}$. A σ-invariant F-submodule B of A is just a *graded* F-submodule, that is, $B^\sigma \subseteq B$ if and only if $B = B_0 \oplus B_1$ where $B_0 = B \cap A_0$ and $B_1 = B \cap A_1$. For instance, the center Z of A is clearly invariant under any automorphism of A and hence $Z = Z_0 \oplus Z_1$ where $Z_0 = Z \cap A_0$ and $Z_1 = Z \cap A_1$.

A superalgebra A is said to be *prime* if the product of any two nonzero graded ideals is nonzero. A is said to be *semiprime* if it has no nonzero nilpotent graded ideals. One can see readily that A is a prime superalgebra if and only if every nonzero graded ideal of A has zero annihilator in A. A superalgebra A is called a *trivial* superalgebra if $A_1 = 0$, or equivalently $\sigma = 1$.

For $a, b \in H$, the *skew-supercommutator* of a and b is defined to be $a \circ_s b = ab + (-1)^{|a||b|} ba$. Note that we have $a \circ_s b = ab - ba = [a, b]$, the ordinary commutator, if both a and b are odd, and $a \circ_s b = ab + ba = a \circ b$, if either a or b is even. The definition can be extended linearly to arbitrary $a, b \in A$, namely,

$$a \circ_s b = a_0 \circ_s b_0 + a_0 \circ_s b_1 + a_1 \circ_s b_0 + a_1 \circ_s b_1$$

for $a = a_0 + a_1$ and $b = b_0 + b_1$ with $a_i, b_i \in A_i$, $i = 0, 1$.

Let S be a subset of A. We say that a mapping $f : S \to A$ is *skew-supercommuting* on S if $f(x) \circ_s x = 0$ for all $x \in S$.

Let A be a semiprime superalgebra. It is well known that A and A_0 are semiprime as algebras [16, Lemma 1.2]. So we can construct the maximal right quotient ring Q and the extended centroid C of A. All these notions are explained in detail in the book [3, Ch. 2].

Since σ can be extended to Q such that $\sigma^2 = 1$ on Q [3, Proposition 2.5.3], thus Q is also a semiprime superalgebra. Since $C^\sigma = C$ we see that $C = C_0 \oplus C_1$ is a graded subalgebra of Q. It is well known that for any $a \in Q$ there exists an essential ideal I of A such that $aI \subseteq A$. We may assume that I is graded since otherwise we can replace it by $I \cap I^\sigma$. This fact will be used later.

We begin with some basic properties of prime superalgebras.

LEMMA 2.1. *Let A be a prime superalgebra and I a graded ideal of A. If $aI_0 = 0$ ($I_0 a = 0$), where $a \in A$, then $a = 0$ or $I = 0$.*

PROOF. Suppose that $aI_0 = 0$. Since $A_0 I_1 A_1 \subseteq I_0$ and $A_1 I_1 \in I_0$, we get $aAI_1A_1 = 0$ and so $a = 0$ or $I_1 A_1 = 0$ since A is a prime superalgebra. If $I_1 A_1 = 0$, then $I_1 = 0$

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by [10, Lemma 2.1(i)] and so $I = I_0$. Hence, we get by assumption that $aI = 0$, which results in $a = 0$ or $I = 0$. The case of $I_0 a = 0$ is proved in a similar way. □

LEMMA 2.2. *Let A be a prime superalgebra and I a nonzero graded ideal of A. If $aI_1 = 0$ ($I_1 a = 0$), where $a \in A$, then $a = 0$ or $A_1 = 0$.*

PROOF. Suppose that $aI_1 = 0$. Since $I_0 A_1 \subseteq I_1$ we get $aI_0 A_1 = 0$ and so $aI_0 = 0$ or $A_1 = 0$ by [10, Lemma 2.1(i)]. If $aI_0 = 0$, then $a = 0$ by Lemma 2.1. The case of $I_1 a = 0$ is proved in a similar way. □

LEMMA 2.3. *Let A be a prime superalgebra and I a graded ideal of A. Let $a \in A$ be such that $ax_0 + x_0 a = 0$ for all $x_0 \in I_0$. Then $a = 0$ or $I = 0$.*

PROOF. Suppose that $I \neq 0$. For any $x_0, y_0 \in I_0$,

$$-x_0 y_0 a = a x_0 y_0 = -x_0 a y_0 \quad \text{for all } x_0, y_0 \in I_0.$$

Hence $I_0[a, I_0] = 0$ and so $[a, I_0] = 0$ by Lemma 2.1. Thus $ax_0 = \frac{1}{2}(ax_0 + x_0 a) = 0$ for all $x_0 \in I_0$. By Lemma 2.1 again we obtain $a = 0$. □

LEMMA 2.4. *Let A be a superalgebra with 1 over F. Let f be an F-linear map of A into itself such that $f(x) \circ_s x = 0$ for all $x \in A$. Then $f = 0$.*

PROOF. Note that $1 \in A_0$. Then we have $2f(1) = f(1) \circ 1 = 0$, from our assumption, and so $f(1) = 0$. Since

$$f(1 + x) \circ_s (1 + x) = 0 \quad \text{for all } x \in A,$$

we have $f(x) \circ 1 + f(x) \circ_s x = 0$. Since $f(x) \circ_s x = 0$ for all $x \in A$, we have $2f(x) = f(x) \circ 1 = 0$ and so $f(x) = 0$ for all $x \in A$. □

LEMMA 2.5. *Let A be a noncommutative prime superalgebra and I a graded ideal of A. Let $g_1, g_2 : A \to A$ be F-linear maps. Suppose that one of the following conditions are satisfied:*

(i) $x_0 g_1(y) + y g_2(x_0) = 0$ *for all* $x_0 \in I_0$, $y \in I$;
(ii) $x_0 g_1(y) + y^\sigma g_2(x_0) = 0$ *for all* $x_0 \in I_0$, $y \in I$.

Then $g_1(I) = g_2(I_0) = 0$.

PROOF. Suppose first that (i) is fulfilled. If $A_1 = 0$, the result follows from [5, Lemma 4.4]. So we may assume that $A_1 \neq 0$. By assumption we have $r_0 x_0 g_1(y) = -r_0 y g_2(x_0)$ for all $r_0, x_0 \in I_0$, $y \in I$. But, on the other hand, since $r_0 x_0 \in I_0$, we have $r_0 x_0 g_1(y) = -y g_2(r_0 x_0)$. Thus, $r_0 y g_2(x_0) = y g_2(r_0 x_0)$ for all $r_0, x_0 \in I_0$, $y \in I$. In particular,

$$r_0 r y g_2(x_0) = r y g_2(r_0 x_0) \quad \text{for all } r_0, x_0 \in I_0,\ y \in I,\ r \in A.$$

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If $yg_2(x_0) \neq 0$ for some $x_0 \in I_0$, $y \in I$, then r_0 and 1 are C-dependent for any $r_0 \in I_0$ by [10, Theorem 3.3]. This implies that $[I_0, A_1] = 0$ and so $[A_0, A_1] = 0$ by [16, Lemma 1.8(i)]. It follows from [10, Lemma 2.1(vii)] that A is commutative or $A_1 = 0$, contradicting our assumption. Thus $Ig_2(I_0) = 0$ and so $g_2(I_0) = 0$ by Lemma 2.1. According to (i) we have $I_0 g_1(I) = 0$ and so $g_1(I) = 0$ as desired.

Suppose next that (ii) is fulfilled. Thus

$$x_0 g_1(y^\sigma) + yg_2(x_0) = 0 \quad \text{for all } x_0 \in I_0,\ y \in I$$

since $\sigma^2 = 1$ and $I^\sigma = I$. Then the assertion of (ii) follows from the assertion of (i). □

LEMMA 2.6. *Let A be a prime superalgebra and I a graded ideal of A. Then I_0 is a semiprime subalgebra of A.*

PROOF. Suppose that $a_0 I_0 a_0 = 0$ for some $a_0 \in I_0$. We want to prove $a_0 = 0$. Since $a_0 I_0 A_0 a_0 I_0 = 0$, we get $a_0 I_0 = 0$ by the semiprimeness of A_0. So $a_0 = 0$ by Lemma 2.1. □

LEMMA 2.7. *Let A be a prime superalgebra with extended centroid C. If $C_1 \neq 0$, then A_0 is a prime subalgebra of A.*

PROOF. Suppose that $a_0 A_0 b_0 = 0$, where $a_0, b_0 \in A_0$. We want to show that either $a_0 = 0$ or $b_0 = 0$. Pick a nonzero λ_1 in C_1 and a nonzero graded ideal J of A such that $\lambda_1 J \subseteq A$. Write $J = J_0 \oplus J_1$ where $J_0 = J \cap A_0$ and $J_1 = J \cap A_1$. Thus, $a_0 \lambda_1 J_1 b_0 = 0$ and so $a_0 J_1 b_0 = 0$. On the other hand, we have by assumption that $a_0 J_0 b_0 = 0$. This implies that $a_0 J b_0 = 0$ and so $a_0 A J b_0 = 0$. Hence, either $a_0 = 0$ or $J b_0 = 0$; but if $J b_0 = 0$, then $b_0 = 0$. This proves our lemma. □

LEMMA 2.8. *Let A be a prime superalgebra and I a nonzero graded ideal of A. If $[a_1, I_0] = 0$, where $a_1 \in A_1$, then $a_1 \in C_1$.*

PROOF. For any $x_0 \in I_0$, $y_0 \in A_0$, since $x_0 y_0 \in I_0$, by assumption we have $[a_1, x_0 y_0] = 0$ and so $x_0 [a_1, y_0] = 0$. It follows from Lemma 2.1 that $[a_1, A_0] = 0$. Thus $[a_1, a_1 A_0 A_1] = 0$ and so $a_1 A_0 [a_1, A_1] = 0$. Similarly, we can get $[a_1, A_1] A_0 a_1 = 0$. By [10, Lemma 2.1(iii)] we get $[a_1, A_1] = 0$. Hence $[a_1, A] = 0$ as desired. □

LEMMA 2.9. *Let A be a prime superalgebra and I a nonzero graded ideal of A. If $[I_0, I_0] = 0$, then $[A_0, A_0] = 0$.*

PROOF. Since $I_0 [A_0, I_0] = [I_0 A_0, I_0] = 0$, we get $[A_0, I_0] = 0$ by Lemma 2.1. Furthermore, $[A_0, A_0] I_0 = [A_0, A_0 I_0] = 0$, which implies that $[A_0, A_0] = 0$ as desired. □

The following result is of crucial important for the proof of our main result.

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LEMMA 2.10. *Let A be a prime superalgebra with extended centroid C and I a graded ideal of A. Suppose that $C_1 = 0$. If $F : I_0 \to A_1$ is an F-linear map such that*

$$[F(r_0), r_0] = 0 \quad \textit{for all } r_0 \in I_0. \tag{2.1}$$

then $F(I_0) = 0$ or $[A_0, A_0] = 0$.

PROOF. We may assume without loss of generality that both $I \neq 0$ and $A_1 \neq 0$. A linearization of (2.1) yields

$$[F(x_0), y_0] = [x_0, F(y_0)] \quad \text{for all } x_0, y_0 \in I_0.$$

Therefore the map $D : I_0 \times I_0 \to A_1$ defined by $D(x_0, y_0) = [F(x_0), y_0]$ is a biderivation, so it follows from [6, Lemma 3.1] that

$$[F(x_0), y_0]w_0[x_0', z_0] = [x_0, y_0]w_0[F(x_0'), z_0] \quad \text{for all } x_0, x_0', y_0, z_0, w_0 \in I_0.$$

Right-multiplying by $t_1 \in A_1$ and substituting w_0 with w_0u_0 where $u_0 \in A_0$, we get

$$([F(x_0), y_0]w_0)u_0([x_0', z_0]t_1) = ([x_0, y_0]w_0)u_0([F(x_0'), z_0]t_1).$$

Since $C_1 = 0$, it follows from [10, Lemma 3.4] that

$$[F(x_0), y_0]w_0u_0[x_0', z_0]t_1 = 0 = [x_0, y_0]w_0u_0[F(x_0'), z_0]t_1$$

and so

$$[F(x_0), y_0]w_0u_0[x_0', z_0] = 0 = [x_0, y_0]w_0u_0[F(x_0'), z_0]$$

in view of [10, Lemma 2.1(i)]. This implies that

$$[F_1(x_0), y_0]w_0A_0[x_0', z_0]w_0' = 0 = [x_0', z_0]w_0'A_0[F_1(x_0), y_0]w_0 = 0$$

for all $x_0, x_0', y_0, z_0, w_0, w_0' \in I_0$. It follows from [10, Lemma 2.1(iii)] that either $[F_1(x_0), y_0]w_0 = 0$ or $[x_0', z_0]w_0' = 0$ for all $x_0, y_0, w_0, x_0', z_0, w_0' \in I_0$. That is, either $[F_1(I_0), I_0]I_0 = 0$ or $[I_0, I_0]I_0 = 0$. Hence, either $[F_1(I_0), I_0] = 0$ or $[I_0, I_0] = 0$ by Lemma 2.1. If $[F_1(I_0), I_0] = 0$, we get $F_1(I_0) \in C_1 = 0$ by Lemma 2.8. Otherwise, if $[I_0, I_0] = 0$, then $[A_0, A_0] = 0$ by Lemma 2.9. This completes the proof. □

The following lemma is a slight generalization of [4, Theorem 1] and can be verified by the same proof.

LEMMA 2.11. *Let R be a prime ring of characteristic not 2 and I an ideal of R. If an additive mapping $f : I \to R$ is skew-commuting on I, then $f(I) = 0$.*

3. A result for prime superalgebras

Throughout this section $A = A_0 \oplus A_1$ will be a prime superalgebra over F, $I = I_0 \oplus I_1$ a graded ideal of A and $f : A \to A$ will denote a skew-supercommuting F-linear map on I, that is

$$f(x) \circ_s x = 0 \quad \text{for all } x \in I. \tag{3.1}$$

A linearization of (3.1) yields

$$f(x) \circ_s y + f(y) \circ_s x = 0 \quad \text{for all } x, y \in I. \tag{3.2}$$

Our goal is to prove $f(I) = 0$. Without loss of generality we may assume that both $I \neq 0$ and $[A_0, A_1] \neq 0$. Indeed, if $I = 0$, there is nothing to prove; if $[A_0, A_1] = 0$, then either A is commutative or $A_1 = 0$ [10, Lemma 2.1(vii)]. In the former case, any nonzero element of H has zero annihilator in A. It follows from (3.1) that

$$2f(x_0)x_0 = f(x_0)x_0 + x_0 f(x_0) = 0 \quad \text{for all } x_0 \in I_0$$

and so $f(x_0) = 0$ for all $x_0 \in I_0$. According to (3.2) we have $2f(y)x_0 = f(y)x_0 + x_0 f(y) = 0$. Hence, $f(y) = 0$ for all $y \in I$ as desired. In the latter case that $A_1 = 0$, A is a prime algebra and f is skew-commuting on I, so the assertion follows from Brešar's theorem [4, Theorem 1].

We begin with two useful lemmas.

LEMMA 3.1. *If $f(I_0) = 0$, then $f(I) = 0$.*

PROOF. We only need to prove $f(I_1) = 0$. By assumption we get from (3.2) that

$$f(y_1)x_0 + x_0 f(y_1) = 0 \quad \text{for all } x_0 \in I_0,\ y_1 \in I_1.$$

It follows from Lemma 2.3 that $f(y_1) = 0$ for all $y_1 \in I_1$ as desired. □

LEMMA 3.2. *Suppose that $[A_0, A_0] = 0$. Then $f(I) = 0$.*

PROOF. According to [16, Lemma 1.9], we get that $Z_0 \neq 0$ and the central closure $\overline{A} = Z_0^{-1}A$ is a field, or the direct sum of two fields, or a quaternion algebra. By assumption we easily see that $\overline{A} = Z_0^{-1}I$ is a four-dimensional central-simple algebra.

We define

$$g(z_0^{-1}x) = z_0^{-1}f(x), \qquad z_0 \in Z_0,\ x \in I.$$

We claim that g is well-defined map of $\overline{A}$. Indeed, if $z_0^{-1}x = 0$ for some $x \in I$, then $x = 0$ and so $z_0^{-1}f(x) = 0$ as desired. Next, since

$$g(z_0^{-1}x) \circ_s z_0^{-1}x = (z_0^{-1})^2(f(x) \circ_s x) = 0 \quad \text{for all } x \in I,$$

we see that g is skew-supercommuting on $\overline{A}$. Then Lemma 2.4 tells us that $g = 0$ on $\overline{A}$ and so, in particular, $f(I) = 0$ as desired. □

We now consider two special cases of the map f.

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LEMMA 3.3. *If $f(A_0) \subseteq A_0$ and $f(A_1) \subseteq A_1$, then*

$$[r_0, f(yr_0) - f(y)r_0] = 0 \quad \text{for all } r_0 \in I_0,\ y \in I.$$

PROOF. Set

$$\pi(x_0, y) = f(x_0)y + yf(x_0) + f(y)x_0 + x_0 f(y) \quad \text{for all } x_0 \in I_0,\ y \in I.$$

Following the same argument as that of [5, Corollary 2.5], we can check that

$$\begin{aligned} &x_0[r_0, f(yr_0) - f(y)r_0] + y[r_0, f(x_0r_0) - f(x_0)r_0] \\ &\quad = \pi(x_0, y)r_0^2 - (\pi(x_0r_0, y) + \pi(x_0, yr_0))r_0 + \pi(x_0r_0, yr_0) \end{aligned}$$

for all $x_0, r_0 \in I_0$, $y \in I$. On the other hand, we see from (3.2) that $\pi(x_0, y) = 0$ for all $x_0 \in I_0$, $y \in I$. Hence,

$$x_0[r_0, f(yr_0) - f(y)r_0] + y[r_0, f(x_0r_0) - f(x_0)r_0] = 0$$

for all $x_0, r_0 \in I_0$, $y \in I$. So the result follows from Lemma 2.5(i). □

LEMMA 3.4. *If $f(A_0) \subseteq A_0$ and $f(A_1) \subseteq A_1$ and $[f(x_0), x_0] = 0$ for all $x_0 \in I_0$, then $f(I) = 0$.*

PROOF. Since $f(x_0) \circ x_0 = 0$ for all $x_0 \in I_0$, we get from our assumption that

$$f(x_0)x_0 = x_0 f(x_0) = \tfrac{1}{2}(f(x_0)x_0 + x_0 f(x_0)) = 0 \quad \text{for all } x_0 \in I_0. \tag{3.3}$$

A linearization of (3.3) yields

$$f(x_0)y_0 + f(y_0)x_0 = 0 \quad \text{for all } x_0, y_0 \in I_0. \tag{3.4}$$

Right-multiplying (3.4) by $f(y_0)$, we get $f(y_0)I_0 f(y_0) = 0$ for all $y_0 \in I_0$. Since I_0 is a semiprime subalgebra of A by Lemma 2.6, we get $f(y_0) = 0$ for all $y_0 \in I_0$. But then $f(I) = 0$ by Lemma 3.1. □

LEMMA 3.5. *If $f(A_0) \subseteq A_0$ and $f(A_1) \subseteq A_1$, then $f(I) = 0$.*

PROOF. We first assume that $C_1 \neq 0$. Then A_0 is a prime subalgebra of A by Lemma 2.7. It follows from (3.1) that

$$f(x_0) \circ x_0 = 0 \quad \text{for all } x_0 \in I_0.$$

Since I_0 is an ideal of A_0, we get from [4, Theorem 1] that $f(I_0) = 0$ and so $f(I) = 0$ by Lemma 3.1.

We now consider the case when $C_1 = 0$. For any $y_1 \in I_1$, we set

$$F_1(r_0) = f(y_1r_0) - f(y_1)r_0 \quad \text{for all } r_0 \in I_0.$$

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It follows from Lemma 3.3 that

$$[F_1(r_0), r_0] = 0 \quad \text{for all } r_0 \in I_0. \tag{3.5}$$

By Lemma 2.10 we get that $F_1(I_0) = 0$ or $[A_0, A_0] = 0$.

Suppose that $F_1(I_0) = 0$, that is

$$f(y_1 r_0) = f(y_1) r_0 \quad \text{for all } y_1 \in I_1, r_0 \in I_0. \tag{3.6}$$

It follows from (3.2) that

$$f(x_0) y_1 + y_1 f(x_0) + f(y_1) x_0 + x_0 f(y_1) = 0 \quad \text{for all } x_0 \in I_0, y_1 \in I_1. \tag{3.7}$$

Substituting y_1 with $y_1 x_0$ in (3.7) and making use of (3.6) we obtain

$$f(x_0) y_1 x_0 + y_1 x_0 f(x_0) + f(y_1) x_0 x_0 + x_0 f(y_1) x_0 = 0 \quad \text{for all } x_0 \in I_0, y_1 \in I_1. \tag{3.8}$$

On the other hand, right-multiplying (3.7) by x_0 yields

$$f(x_0) y_1 x_0 + y_1 f(x_0) x_0 + f(y_1) x_0 x_0 + x_0 f(y_1) x_0 = 0 \quad \text{for all } x_0 \in I_0, y_1 \in I_1. \tag{3.9}$$

Combining (3.8) with (3.9) yields $y_1[x_0, f(x_0)] = 0$ for all $x_0 \in I_0$, $y_1 \in I_1$ and so $[x_0, f(x_0)] = 0$ for all $x_0 \in I_0$ by Lemma 2.2. But then $f(I) = 0$ by Lemma 3.4.

Finally, if $[A_0, A_0] = 0$, then the result follows from Lemma 3.2. The proof of the lemma is now complete. □

LEMMA 3.6. *If $f(A_0) \subseteq A_1$ and $f(A_1) \subseteq A_0$, then*

$$[r_0, f(y r_0) - f(y) r_0] = 0 \quad \textit{for all } r_0 \in I_0, y \in I.$$

PROOF. According to (3.2),

$$\begin{aligned} f(x_0) y_0 + y_0 f(x_0) + f(y_0) x_0 + x_0 f(y_0) &= 0, \\ f(x_0) y_1 - y_1 f(x_0) + f(y_1) x_0 + x_0 f(y_1) &= 0, \end{aligned}$$

for all $x_0, y_0 \in I_0$, $y_1 \in I_1$. Adding the above two equations yields

$$f(x_0) y + y^\sigma f(x_0) + f(y) x_0 + x_0 f(y) = 0 \quad \text{for all } x_0 \in I_0, y \in I.$$

Set

$$\pi_2(x_0, y) = f(x_0) y + y^\sigma f(x_0) + f(y) x_0 + x_0 f(y) \quad \text{for all } x_0 \in I_0, y \in I.$$

Following the same argument as that of [5, Corollary 2.5], we easily check that

$$\begin{aligned} &x_0[r_0, f(y r_0) - f(y) r_0] + y^\sigma [r_0, f(x_0 r_0) - f(x_0) r_0] \\ &\quad = \pi(x_0, y) r_0^2 - (\pi(x_0 r_0, y) + \pi(x_0, y r_0)) r_0 + \pi(x_0 r_0, y r_0) \end{aligned}$$

for all $x_0, r_0 \in I_0$, $y \in I$. Note that $\pi(x_0, y) = 0$ for all $x_0 \in I_0$, $y \in I$. Hence,

$$x_0[r_0, f(y r_0) - f(y) r_0] + y^\sigma [r_0, f(x_0 r_0) - f(x_0) r_0] = 0 \quad \text{for all } x_0, r_0 \in I_0, y \in I.$$

So the result follows from Lemma 2.5(ii). □

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Lemma 3.7. *If $f(A_0) \subseteq A_1$ and $f(A_1) \subseteq A_0$, then $f(I) = 0$.*

Proof. We first assume that $C_1 \neq 0$. In this case, A_0 is prime as an algebra by Lemma 2.7. For $0 \neq \lambda_1 \in C_1$, there exists a nonzero graded ideal J of A such that $\lambda_1 J \subseteq A$ and $J \subseteq I$. By assumption, we have

$$f(\lambda_1 x_0) \circ \lambda_1 x_0 = 0 \quad \text{for all } x_0 \in J_0.$$

Note that all nonzero homogeneous elements in C are invertible [10, Lemma 3.1]. So

$$f(\lambda_1 x_0) \circ x_0 = 0 \quad \text{for all } x_0 \in J_0.$$

Thus, Lemma 2.11 tells us that $f(\lambda_1 J_0) = 0$. We shall claim that $f(I_0) = 0$. Indeed, according to (3.2) we see that

$$f(\lambda_1 x_0) y_0 + y_0 f(\lambda_1 x_0) + [f(y_0), \lambda_1 x_0] = 0 \quad \text{for all } x_0 \in J_0,\ y_0 \in I_0.$$

Since $f(\lambda_1 J_0) = 0$ we get

$$[f(y_0), \lambda_1 x_0] = 0 \quad \text{for all } x_0 \in J_0,\ y_0 \in I_0$$

and so $[f(y_0), J_0] = 0$ for all $y_0 \in I_0$. It follows from Lemma 2.8 that $f(I_0) \subseteq C_1$. But then

$$f(x_0)x_0 = \tfrac{1}{2}(f(x_0)x_0 + x_0 f(x_0)) = 0 \quad \text{for all } x_0 \in I_0.$$

If $f(x_0) \neq 0$ for some $x_0 \in I_0$, then $x_0 = 0$, which is a contradiction. Thus $f(I_0) = 0$ and so $f(I) = 0$ by Lemma 3.1.

We next consider the case when $C_1 = 0$. For any $x_0 \in I_0$, we set

$$F_2(r_0) = f(x_0 r_0) - f(x_0) r_0 \quad \text{for all } r_0 \in I_0.$$

According to Lemma 3.6 we see that $[F_2(r_0), r_0] = 0$ for all $r_0 \in I_0$. Thus, by Lemma 2.10 we infer that either $F_2(I_0) = 0$ or $[A_0, A_0] = 0$.

Suppose that $F_2(I_0) = 0$, that is,

$$f(x_0 r_0) = f(x_0) r_0 \quad \text{for all } x_0,\ r_0 \in I_0. \tag{3.10}$$

It follows from (3.2) that

$$f(x_0) y_0 + y_0 f(x_0) + f(y_0) x_0 + x_0 f(y_0) = 0 \quad \text{for all } x_0,\ y_0 \in I_0. \tag{3.11}$$

Substituting y_0 with $y_0 r_0$ in (3.11) and making use of (3.10) we obtain

$$f(x_0) y_0 r_0 + y_0 r_0 f(x_0) + f(y_0) r_0 x_0 + x_0 f(y_0) r_0 = 0 \quad \text{for all } x_0,\ y_0,\ r_0 \in I_0. \tag{3.12}$$

Right-multiplying (3.11) by r_0 yields

$$f(x_0) y_0 r_0 + y_0 f(x_0) r_0 + f(y_0) x_0 r_0 + x_0 f(y_0) r_0 = 0 \quad \text{for all } x_0,\ y_0,\ r_0 \in I_0. \tag{3.13}$$

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Combining (3.12) with (3.13) we obtain

$$y_0[r_0, f(x_0)] + f(y_0)[r_0, x_0] = 0 \quad \text{for all } x_0, y_0, r_0 \in I_0.$$

Substituting r_0 by $w_0 r_0$ with $w_0 \in I_0$,

$$y_0 w_0[r_0, f(x_0)] + f(y_0) w_0[r_0, x_0] = 0 \quad \text{for all } x_0, y_0, r_0, w_0 \in I_0.$$

Substituting w_0 by $w_0 u_0$ with $u_0 \in A_0$ and left-multiplying by $t_1 \in A_1$, we get

$$(t_1 y_0 w_0) u_0[r_0, f(x_0)] + (t_1 f(y_0) w_0) u_0[r_0, x_0] = 0.$$

Since $C_1 = 0$, we get from [10, Lemma 3.4] that

$$t_1 y_0 w_0 u_0[r_0, f(x_0)] = 0 = t_1 f(y_0) w_0 u_0[r_0, x_0] \quad \text{for all } x_0, y_0, r_0, w_0 \in I_0, u_0 \in A_0$$

and so $[r_0, f(x_0)] = 0$ for all $x_0, r_0 \in I_0$ by Lemmas 2.1 and 2.2. Thus, Lemma 2.8 tells us that $f(I_0) \subseteq C_1$, resulting in $f(I_0) = 0$ since $C_1 = 0$. Therefore, $f(I) = 0$ in view of Lemma 3.1.

The case that $[A_0, A_0] = 0$ follows from Lemma 3.2. The proof of the lemma is now complete. $\square$

Now we are ready to prove the following important result.

THEOREM 3.8. *Let A be a prime superalgebra over a commutative ring F with $\frac{1}{2}$ and I a graded ideal of A. Let $f : A \to A$ be an F-linear map such that $f(x) \circ_s x = 0$ for all $x \in I$. Then $f(I) = 0$.*

PROOF. For $i = 0$ or 1, let π_i be the canonical projection of A onto A_i and let $f_0 = \pi_0 f \pi_0 + \pi_1 f \pi_1$ and $f_1 = \pi_0 f \pi_1 + \pi_1 f \pi_0$. Then each f_i is an F-linear map of A into itself and $f = f_0 + f_1$. Moreover, $f_0(A_0) \subseteq A_0$, $f_0(A_1) \subseteq A_1$, $f_1(A_0) \subseteq A_1$ and $f_1(A_1) \subseteq A_0$.

We claim that each f_i satisfies the condition that $f_i(x) \circ_s x = 0$ for all $x \in I$. For $i = 0$ or 1,

$$f(x_i) \circ_s x_i = f_0(x_i) \circ_s x_i + f_1(x_i) \circ_s x_i = 0 \quad \text{for all } x_i \in I_i.$$

Since $f_0(x_i) \circ_s x_i$ is even and $f_1(x_i) \circ_s x_i$ is odd, we obtain that

$$f_0(x_i) \circ_s x_i = 0 \quad \text{for all } x_i \in I_i \tag{3.14}$$

and

$$f_1(x_i) \circ_s x_i = 0 \quad \text{for all } x_i \in I_i. \tag{3.15}$$

For $x_0 \in A_0$, $x_1 \in A_1$ we have $f(x_0) \circ_s x_1 + f(x_1) \circ_s x_0 = 0$ and so

$$f_0(x_0) \circ_s x_1 + f_0(x_0) \circ_s x_1 + f_1(x_1) \circ_s x_0 + f_1(x_1) \circ_s x_0 = 0$$

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for all $x_0 \in I_0$, $x_1 \in I_1$. Thus the odd part is

$$f_0(x_0) \circ_s x_1 + f_0(x_0) \circ_s x_1 = 0 \quad \text{for all } x_0 \in I_0,\, x_1 \in I_1 \tag{3.16}$$

and the even part is

$$f_1(x_1) \circ_s x_0 + f_1(x_1) \circ_s x_0 = 0 \quad \text{for all } x_0 \in I_0,\, x_1 \in I_1. \tag{3.17}$$

Hence we have $f_0(x) \circ_s x = 0$ for all $x \in I$ by (3.14) and (3.16), and $f_1(x) \circ_s x = 0$ for all $x \in I$ by (3.15) and (3.17).

Therefore, by Lemma 3.5, $f_0(I) = 0$ and by Lemma 3.7, $f_1(I) = 0$. This proves the theorem. □

4. Main result

In this section, we always assume that A is a semiprime superalgebra over F and $f : A \to A$ is a skew-supercommuting F-linear map on A, that is

$$f(x) \circ_s x = 0 \quad \text{for all } x \in A. \tag{4.1}$$

A graded ideal P of A is said to be *graded-prime ideal* of A if A/P is a prime superalgebra. One can see readily that P is a graded-prime ideal of A if and only if for $aAb \subseteq P$, where $a, b \in H$, then $a \in P$ or $b \in P$.

We begin with the following useful result.

LEMMA 4.1. *Let A be a semiprime superalgebra. The intersection of all graded-prime ideals in A is zero.*

PROOF. Since A is semiprime as an algebra, it is well known that the intersection of all prime ideals in A is zero. For any prime ideal P of A, we claim that $P \cap P^\sigma$ is a graded-prime ideal of A. Indeed, it is obvious that $P \cap P^\sigma$ is a graded ideal of A. If $aAb \subseteq P \cap P^\sigma$, where $a, b \in H$, then $aAb \subseteq P$, which implies that $a \in P$ or $b \in P$ since P is a prime ideal of A. If $a \in P$, then $a = \pm a^\sigma \in P^\sigma$, that is, $a \in P \cap P^\sigma$. Similarly, if $b \in P$, then $b \in P \cap P^\sigma$. This implies that $P \cap P^\sigma$ is a graded-prime ideal of A. It is obvious that the intersection of all $P \cap P^\sigma$, where P is a prime ideal of A, is zero. This proves the lemma. □

We now give our main result as follows.

THEOREM 4.2. *Let A be a semiprime superalgebra over a commutative ring F with $\frac{1}{2}$. Let $f : A \to A$ be a skew-supercommuting F-linear map on A. Then $f = 0$.*

PROOF. Pick any graded-prime ideal P. We want to show that P is invariant under f. A linearization of (4.1) gives

$$f(x_0) \circ_s y + f(y) \circ x_0 = 0 \quad \text{for all } x_0 \in A_0,\, y \in A.$$

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Hence

$$f(p) \circ x_0 \in P \quad \text{for all } p \in P,\, x_0 \in A_0. \tag{4.2}$$

In particular, $f(p) \circ x_0 y_0 \in P$ for all $p \in P$, x_0, $y_0 \in A_0$. That is,

$$(f(p) \circ x_0)y_0 - x_0[f(p), y_0] \in P.$$

According to (4.2) we get that $x_0[f(p), y_0] \in P$ for all x_0, $y_0 \in A_0$, $p \in P$. Since A/P is a prime superalgebra, we can get from Lemma 2.1 that $[f(p), y_0] \in P$ for all $p \in P$, $y_0 \in A_0$. Combining this relation with (4.2) we obtain $f(P)A_0 \subseteq P$ and so $f(P) \subseteq P$ by Lemma 2.1.

Since $f(P) \subseteq P$ and A/P is a prime superalgebra, we easily see that f induces a skew-supercommuting F-linear map on R/P. Hence, we can get from Theorem 3.8 that $f(A) \subseteq P$ and so $f(A) \subseteq \cap P = 0$ by Lemma 4.1. This proves our theorem. □

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