NOTE ON A THEOREM ON SINGULAR MATRICES

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J. A. Erdös proved recently [1] that every singular matrix over a field F is a product of idempotent matrices. He gave two proofs, one valid for matrices which are similar to triangular matrices and the other valid in general. We shall give a simple geometric proof of the above result. Instead of matrices we use linear operators. Moreover we get an explicit factorization in terms of projectors (idempotent operators).

Let A be a singular linear operator in n-dimensional vector space V over F. By a well known decomposition theorem ([2], p. 189) V decomposes into a direct sum

$$V = V_1 \oplus \dots \oplus V_k$$

such that each subspace V_i is A-cyclic and the minimal polynomial of the restriction of A to V_i is a power of a prime polynomial over F. Since V_i is A-cyclic it has a basis e^i_j $(j = 1, \dots, m_i)$ such that

$$e^i_1 A \in V_i \quad \text{and} \quad e^i_j A = e^i_{j-1} \quad \text{for } j = 2, \dots, m_i .$$

A being singular, we can assume that $e^1_1 A = 0$. The vectors e^i_j $(i = 1, \dots, k ; j = 1, \dots, m_i)$ form a basic set of V. If e is any of these vectors let $V(e)$ be the $(n-1)$-dimensional subspace of V spanned by all basic vectors e^i_j except e. If $x \in V(e)$, we define $P(e, x)$ to be the operator which maps e onto x and leaves $V(e)$ pointwise fixed. It is obvious that $P(e, x)$ is a projector of nullity 1. If

$$(1) \quad \begin{aligned} P_0 &= P(e^1_2, e^1_1)P(e^1_3, e^1_2)\dots P(e^1_{m_1}, e^1_{m_1-1}) , \\ P_i &= P(e^i_1, e^1_1)P(e^i_2, e^i_1)P(e^i_3, e^i_2)\dots P(e^i_{m_i}, e^i_{m_i-1})P(e^1_1, e^i_1 A) \end{aligned}$$

for $i=2, \dots, k$, then we claim that

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(2) $$A = P(e_1^1, 0)P_2P_3 \ldots P_kP_0.$$

This is easy to verify since both sides in (2) have the same effect when applied to basic vectors e_j^i.

REFERENCES

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2. F.R. Gantmacher, The theory of matrices. Vol. 1. (Chelsea Publishing Company, New York 1960).

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