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ON THE CENTRAL HAAGERUP TENSOR PRODUCT*

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For a large class of C^*-algebras including all von Neumann algebras, the central Haagerup tensor product of the multiplier algebra with itself has an isometric representation as completely bounded operators.

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1. Introduction

In an unpublished paper [25], Haagerup introduced a cross-norm on the algebraic tensor product of C^*-algebras which was named the *Haagerup norm* $\|\cdot\|_h$ by Effros and Kishimoto in [17]. During the elaboration of the theory of (multilinear) completely bounded operators on operator spaces, this norm emerged to be a fundamental device, in particular in cohomology theory, and it is distinguished by several extraordinary features amongst them the injectivity of the Haagerup norm and the commutant theorem for the Haagerup tensor product; see e.g. [**1**, **7**, **8**, **9**, **12**, **13**, **14**, **17**, **18**, **38**, **40**], as well as the recent work on Morita theory for operator algebras by Blecher, Muhly and Paulsen.

The Haagerup norm is designed to allow a canonical representation of the tensor product as completely bounded operators. It is shown in [25] that the mapping

$$\theta: B(H) \otimes_h B(H) \to CB(K(H)), \quad \theta(a \otimes b) = L_a R_b$$

is an isometry, where $K(H)$ and $B(H)$ denote the C^*-algebras of compact and all bounded linear operators on a Hilbert space H, respectively, and $CB(K(H))$ is the Banach algebra of all completely bounded linear operators on $K(H)$. When $K(H)$ and $B(H)$ are replaced by an arbitrary C^*-algebra A and its multiplier algebra $M(A)$, respectively, θ need no longer be injective, but it was observed in [32], see also [30], that $\ker\theta = \{0\}$ if and only if A is prime. In fact, in this case θ still is an isometry which was proved by the second-named author, but again not published, cf. [31]. Recently, Smith rediscovered Haagerup's result independently [39], and Chatterjee and Sinclair obtained the isometric property of θ for von Neumann factors on separable Hilbert spaces using several non-trivial results on injective subfactors [10]. In the sequel, Chatterjee and Smith introduced the notion of a *central Haagerup tensor product* and showed that θ induces an isometry θ_Z on this one, if A is a von Neumann algebra or a

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unital C^*-algebra with Hausdorff spectrum [11]. They also provided an example of a C^*-algebra such that θ_Z is injective, but not isometric.

If we aim for an isometric representation of the central Haagerup tensor product, we first of all, of course, have to guarantee that θ_Z is injective. This amounts to solving operator equations of the form $\sum_{j=1}^{n} L_{a_j}R_{b_j}=0$ with $a_1,\ldots,a_n$, $b_1,\ldots,b_n$ in $M(A)$. To do this, we appeal to the C^*-algebra $M_{\mathrm{loc}}(A)$ of *local multipliers* of A since this is the C^*-analogue of the symmetric ring of quotients $Q_s(R)$ of a semiprime ring R, studied and used in particular by Kharchenko [26, 27] and Passman [34], whose elements can be viewed as 'generalised fractions'. With their aid we can determine the kernel of θ for an arbitrary C^*-algebra.

Like in ring theory, it often suffices to add the central elements of $M_{\mathrm{loc}}(A)$ to the C^*-algebra A by passing to the *bounded central closure* ${}^cA$. This, for example, has fruitfully been exploited in obtaining a complete description of all centralising additive mappings on C^*-algebras [5]. If A is boundedly centrally closed, that is ${}^cA=A$, then everything takes place within the C^*-algebra itself (or rather the multiplier algebra, if A is non-unital). For these C^*-algebras, we therefore obtain the isometric representation of the central Haagerup tensor product (Theorem 3.7). Since von Neumann algebras are very easily seen to be in this class of C^*-algebras, we recover the result by Chatterjee and Smith without further effort.

2. Prerequisites

Throughout A will be a C^*-algebra and $A\otimes A$ denotes the algebraic tensor product of A with itself. We start by compiling a few facts on the Haagerup norm. If $u\in A\otimes A$, then

$$\|u\|_h=\inf\left\{\left\|\sum_{j=1}^{n} a_ja_j^*\right\|^{1/2}\left\|\sum_{j=1}^{n} b_j^*b_j\right\|^{1/2}\ \middle|\ u=\sum_{j=1}^{n} a_j\otimes b_j\right\}$$

where the infimum is taken over all representations of u. From now on, we shall consider each tensor product (over $\mathbf{C}$) of a C^*-algebra with itself as a normed space endowed with $\|\cdot\|_h$.

Let π be a (non-degenerate) representation of A in $B(H)$. By abuse of notation, we write π for the representation $\pi\otimes\pi$ of $A\otimes A$ in $B(H\bar{\otimes} H)$, too. Since $\|\cdot\|_h\geqq\|\cdot\|_{\max}$, the maximal C^*-tensor norm [25], we have that $\|u\|_h\geqq\|\pi(u)\|$. Another easy computation shows that $\|u\|_h\geqq\|\pi(u)\|_h$, whence π is a contraction from $A\otimes A$ into $B(H)\otimes B(H)$. (Note that, by the injectivity of the Haagerup norm [35, Theorem 4.4], see also [7, Theorem 3.6], $\|\pi(u)\|_h$ is the same in $\pi(A)\otimes\pi(A)$ and in $B(H)\otimes B(H)$.) This leads to the following observation.

Lemma 2.1. *If J is a closed ideal in $A\otimes A$ such that $J\subseteq\ker\pi$, then the induced homomorphism $\pi_J\colon A\otimes A/J\to B(H)\otimes B(H)$ is a contraction.*

In the sequel, J will *always* denote the ideal generated by $az\otimes b-a\otimes zb$, $a,b\in A$,

$z \in Z(A)$ (the centre of A). Then, the tensor product $A \otimes_{Z(A)} A$ as $Z(A)$-bimodules is nothing but $A \otimes A/J$. If J is closed, we define the *central Haagerup tensor norm* on $A \otimes_{Z(A)} A$ by

$$\|u\|_{Zh} = \inf\left\{\left\|\sum_{j=1}^{n} a_j a_j^*\right\|^{1/2} \left\|\sum_{j=1}^{n} b_j^* b_j\right\|^{1/2} \,\middle|\, u = \sum_{j=1}^{n} a_j \otimes_Z b_j\right\}$$

where the infimum is taken over all representations of $u \in A \otimes_{Z(A)} A$ in $A \otimes_{Z(A)} A$. A moment's reflection yields the following, which in particular implies that $\|\cdot\|_{Zh}$ is a norm.

Proposition 2.2. *The central Haagerup norm coincides with the quotient norm in* $A \otimes A/J$.

Let π be an irreducible representation of A. Since $\pi(Z(A)) \subseteq \mathbf{C}$, we clearly have $J \subseteq \ker \pi$. Consequently, from Lemma 2.1 and Proposition 2.2 we conclude:

Proposition 2.3. *For each* $u \in A \otimes_{Z(A)} A$ *we have* $\|u\|_{Zh} \geq \sup_\pi \|\pi(u)\|_h$, *where the supremum is taken over the set* $Irr(A)$ *of all irreducible representations of* A *and, in order to simplify the notation, we write* π *instead of* $\pi_J = (\pi \otimes \pi)_J$.

One of the main goals of the next section will be to establish the reverse inequality for a large class of C^*-algebras, including all von Neumann algebras.

Remark. In [11], Chatterjee and Smith define the *central Haagerup tensor product* $A \otimes_{Zh} A$ by first completing the tensor product with respect to the Haagerup norm $\|\cdot\|_h$ and then taking the quotient by the *closed* ideal generated by $az \otimes b - a \otimes zb$. If J is closed in $A \otimes A$, then the completion of $A \otimes_{Z(A)} A$ is clearly isomorphic to $A \otimes_{Zh} A$ wherefore we will identify these two spaces henceforth. Generally, it suffices and may be simpler to work in the uncompleted space.

We extract the following important observation from Lemma 2.3 and the proof of Theorem 2.4 of [11].

Lemma 2.4. *Let* A *be a* C^*-*algebra and* $u = \sum_{j=1}^{n} a_j \otimes b_j \in A \otimes A$. *For each* $\varepsilon > 0$, *there exist invertible* $n \times n$-*matrices* $S_1, \ldots, S_p$ *such that for every* $\pi \in Irr(A)$ *with* $\|\pi(u)\|_h \leq 1$, *there is* $i \in \{1, \ldots, p\}$ *satisfying*

$$\max\left\{\|(\pi(a_1), \ldots, \pi(a_n))\, S_i^{-1}\|, \left\| S_i \begin{pmatrix} \pi(b_1) \\ \vdots \\ \pi(b_n) \end{pmatrix} \right\| \right\} \leq 1 + \varepsilon.$$

Recall that a linear operator $T: A \to A$ is called *completely bounded*, if there is a real number majorising each of the norms $\|T_k\|$, $k \in \mathbf{N}$ where

$$T_k: A \otimes M_k \to A \otimes M_k,\ T_k = T \otimes 1;$$

in this case, $\|T\|_{cb} = \sup_k \|T_k\|$ is the *completely bounded norm* of T. (Here, of course,

$A \otimes M_k$ is endowed with its unique C^*-norm.) The Banach algebra of all completely bounded operators on A will be denoted by $CB(A)$.

Let $M(A)$ be the multiplier algebra of A. There is a canonical way to consider the elements of $M(A) \otimes M(A)$ as completely bounded operators on A via

$$\theta: M(A) \otimes M(A) \to CB(A), \quad \theta(a \otimes b) = L_a R_b$$

where L_a, R_b is the left resp. right multiplication by a resp. b. Let $u = \sum_{j=1}^n a_j \otimes b_j$. Then

$$\begin{aligned} \|\theta(u)_k x\| &= \left\| \sum_{j=1}^n (a_j \otimes 1) x (b_j \otimes 1) \right\| \\ &\leqq \left\| \sum_{j=1}^n (a_j \otimes 1)(a_j \otimes 1)^* \right\|^{1/2} \left\| \sum_{j=1}^n (b_j \otimes 1)^*(b_j \otimes 1) \right\|^{1/2} \|x\| \\ &= \left\| \sum_{j=1}^n a_j a_j^* \right\|^{1/2} \left\| \sum_{j=1}^n b_j^* b_j \right\|^{1/2} \|x\| \end{aligned}$$

for all $x \in A \otimes M_k$. Hence, $\|\theta(u)\|_{cb} \leqq \|u\|_h$ and since $J \subseteq \ker \theta$, we have an induced map θ_Z on $M(A) \otimes_{Z(M(A))} M(A)$, whenever the ideal J is closed. We summarise this in the following statement.

Proposition 2.5. *For every C^*-algebra A such that $J \subseteq M(A) \otimes M(A)$ is closed, the $Z(M(A))$-bimodule homomorphism $\theta_Z: M(A) \otimes_{Z(M(A))} M(A) \to CB_Z(A)$ is a contraction, where $CB_Z(A)$ denotes the completely bounded $Z(M(A))$-bimodule maps on A.*

Remarks. 1. Replacing $M(A) \otimes M(A)$ by $M(A) \otimes M(A)^{op}$, where ^{op} denotes the opposite algebra, we have an *algebra* homomorphism θ_Z into $CB(A)$. However, this additional structure will not be needed in the following.
2. Dropping the assumtion that J is closed, we, of course, still have a contraction $\theta_Z: M(A) \otimes_{Zh} M(A) \to CB_Z(A)$.

The special case $A = B(H)$ was first treated in [**25**, Theorem 6], and independently rediscovered in [**39**, Theorem 4.3]; see also [**10**, Corollary 2] and [**13**, Corollary 6.2].

Proposition 2.6. *If $A = B(H)$, then $\theta = \theta_Z$ is an isometry.*

In the next section, we will reduce the general case to this particular situation, and to this end we need the notion of the local multiplier algebra and the bounded extended centroid. Recall that $M(A)$ is the largest C^*-algebra in which A is an essential closed ideal. Hence, if I_1, I_2 are essential closed ideals in A and $I_2 \subseteq I_1$, we get an embedding of $M(I_1)$ into $M(I_2)$ by "restricting the multipliers". The algebraic inductive limit of this directed system of C^*-algebras and $*$-isomorphisms is denoted by $Q_b(A)$ and called the *bounded symmetric algebra of quotients* of A. Its completion, $Q_b(A)^{\sim} = \varinjlim M(I)$ is the *local multiplier algebra* $M_{loc}(A)$ of A. This construction, under the name "essential multipliers", apparently was first pursued by Pedersen and Elliott who used it to obtain

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innerness of derivations and *-automorphisms [**36**, **19**]. Recently, a thorough investigation of its structure was started by the present authors [**2**, **3**, **4**, **33**] leading to the fundamental result that $Z:=Z(M_{\mathrm{loc}}(A))$ is the closure of $C_b:=Z(Q_b(A))$, the latter being called the *bounded extended centroid* of A [**4**, Theorem 1]. One of the useful properties of Z is that it is an AW^*-algebra [**4**, Corollary 1].

We define the *bounded central closure* ${}^cA$ of A by ${}^cA=\overline{AC_b}$, which is a C^*-subalgebra of $M_{\mathrm{loc}}(A)$. This is the C^*-analogue of the central closure of a semiprime ring (cf. e.g. [**34**]). The C^*-algebra A is *boundedly centrally closed* if ${}^cA=A$. These C^*-algebras will be of the main interest to us, and therefore we give a number of equivalent characterisations as follows.

Proposition 2.7. *The following conditions on a C^*-algebra A are equivalent.*

(a) ${}^cA=A$;

(b) $AC_b=A$;

(c) ${}^cM(A)=M(A)$;

(d) $M(A)C_b=M(A)$;

(e) $Z(M(A))=Z$;

(f) $Z(M(A))=C_b$.

Proof. Note at first that, since A is an essential ideal of $M(A)$, $Q_b(M(A))=Q_b(A)$ whence $M_{\mathrm{loc}}(M(A))=M_{\mathrm{loc}}(A)$ and $C_b(M(A))=C_b(A)=C_b$. Also, $C_b\subseteq Z(M(A)C_b)$ always holds, wherefore $Z\subseteq Z({}^cM(A))$ by the local Dauns–Hofmann theorem [**4**, Theorem 1].

Trivially, (b)$\Rightarrow$(a) and (d)$\Rightarrow$(c), and (f)$\Leftrightarrow$(e) follows from $Z=\bar{C}_b$. If ${}^cM(A)=M(A)$, then $Z({}^cM(A))=Z(M(A))\subseteq C_b$ which proves (c)$\Rightarrow$(f). It is clear that (f)$\Rightarrow$(d) and (f)$\Rightarrow$(b). In order to obtain the final implication (a)$\Rightarrow$(c), we use that ${}^cA$ is an essential ideal in ${}^cM(A)$ proved below (Lemma 2.8). Consequently, ${}^cM(A)\subseteq M({}^cA)$ but ${}^cA=A$ implies then that ${}^cM(A)\subseteq M(A)$ and the reverse inclusion is obvious. □

Lemma 2.8. *For every C^*-algebra A, ${}^cA$ is an essential closed ideal in ${}^cM(A)$.*

Proof. Clearly, ${}^cA$ is a closed ideal in ${}^cM(A)$. If L is a non-zero ideal in ${}^cM(A)$ and $z\in L$, $\|z\|=1$, then there is $y\in M(A)C_b$ such that $\|z-y\|<\frac{1}{4}$. Write $y=\sum_i x_ic_i$ with $x_i\in M(A)$ and $c_i\in Z(M(K))$ for some essential closed ideal K of A (using that $C_b=\xrightarrow{\text{alg lim}}Z(M(I))$ [**2**, Proposition 2.2]). Then $y\in M(K)$ and $\|y\|\geq\frac{3}{4}$ yield an element $x\in K$, $\|x\|=1$ such that $\|xy\|\geq\frac{1}{2}$. From $\|xz-xy\|<\frac{1}{4}$ it follows that $\|xz\|\geq\frac{1}{4}$ and, moreover,

$$xz\in L\cap K\overline{M(A)C_b}\subseteq L\cap\overline{KM(A)C_b}\subseteq L\cap\overline{AC_b}$$

finally shows that L intersects ${}^cA$ non-trivially. □

We write $\check{A}$ for the primitive spectrum of A endowed with its natural topology. In addition to the algebraic descriptions above, we have the following topological characterisation of boundedly centrally closed C^*-algebras.

Proposition 2.9. *A C^*-algebra A is boundedly centrally closed if and only if $\check{A}$ is extremally disconnected.*

Proof. It is well-known that a topological space is extremally disconnected if and only if every bounded continuous complex-valued function on a dense open subset can be uniquely extended to a bounded continuous function on the whole space (see, e.g. [**23**, 1.H.6]). The dense open subsets of $\check{A}$ are of the form $\check{I}$, where I is an essential closed ideal of A. Thus, using the Dauns–Hofmann theorem, $\check{A}$ is extremally disconnected if and only if $Z(M(I))=\mathscr{C}_b(\check{I})=\mathscr{C}_b(\check{A})=Z(M(A))$ for all essential closed ideals, which in turn is equivalent to $C_b=Z(M(A))$ by [**2**, Proposition 2.2]. From Proposition 2.7 we therefore conclude that $\check{A}$ is extremally disconnected if and only if ${}^cA=A$. □

Remark. If $Z(M(A))=Z$, then the Stone–Čech compactification $\beta\check{A}$ of $\check{A}$ is extremally disconnected since Z is an AW^*-algebra. However, this is not sufficient for A being boundedly centrally closed, cf. [**4**].

The following examples illustrate that a boundedly centrally closed C^*-algebra is either rich in central projections or is very non-commutative.

Examples 2.10. (a) Every AW^*-algebra A is boundedly centrally closed, in fact $A=M_{\mathrm{loc}}(A)$ [**5**, Proposition 3.3].

(b) $M_{\mathrm{loc}}(A)$ is boundedly centrally closed [**4**, Theorem 2].

(c) ${}^cA$ is boundedly centrally closed (Proposition 3.10 below).

(d) Every prime C^*-algebra is boundedly centrally closed; in this case, $C_b\cong\mathbf{C}$ [**32**, Proposition 2.5].

3. Main results

Our first aim in this section is to determine the kernel of θ for an arbitrary C^*-algebra A. As pointed out in the Introduction, $\ker\theta=\{0\}$ if and only if A is prime [**32**, Corollary 4.4]. The special cases $A=B(H)$ and $A=C(H)$, the Calkin algebra on a separable Hilbert space H, had been treated before in [**22**]. Our main tool here is the *bounded extended centroid* C_b of A (this is already reminiscent in [**32**]), which is the C^*-analogue of the *extended centroid* C of a semiprime ring (see [**27**]).

The following lemma, known in the case of von Neumann algebras, see, e.g. [**28**, Lemma 2.1], requires some technical, but purely algebraic effort (mainly a thorough study of the A-subbimodules of A^n), and we therefore merely cite it from [**6**]. For $a_1,\dots,a_n,b_1,\dots,b_n\in M(A)$ we will use the following notation

$$a=\begin{pmatrix} a_1\dots a_n \\ 0 \end{pmatrix},\quad {}^tb=\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} 0 \quad \in M_n(M(A)),$$

and $u=\sum_{j=1}^{n} a_j \otimes b_j$ will be abbreviated as $u=a\otimes {}^t b$.

Lemma 3.1. *An element $u=a\otimes {}^t b\in M(A)\otimes M(A)$ belongs to $\ker\theta$ if and only if there exists a projection $p\in M_n(C_b)$ such that $ap=a$ and $p^t b=0$.*

We may consider ${}^c M(A)$ as a Z-bimodule since $Z\,{}^c M(A)=\overline{C_b\overline{M(A)C_b}}\subseteq {}^c M(A)$. If $u\in M(A)\otimes M(A)$, then u_Z denotes its image in ${}^c M(A)\otimes_Z {}^c M(A)$.

Proposition 3.2. *For every C^*-algebra A we have $\ker\theta=\{u\,|\,u_Z=0\}$.*

Proof. If $u=a\otimes {}^t b$, then, by Lemma 3.1, $\theta(u)=0$ if and only if $ap=a$, $p^t b=0$ for some projection $p\in M_n(C_b)$. As a result, $u_Z=a\otimes_Z {}^t b=ap\otimes_Z {}^t b=a\otimes_Z p^t b=0$.

Conversely, we can extend θ to ${}^c\theta\colon {}^c M(A)\otimes {}^c M(A)\to CB({}^c A)$, since ${}^c M(A)\subseteq M({}^c A)$ by Lemma 2.8, such that $\theta(u)=0$ if and only if ${}^c\theta(u)=0$. Thus, $u_Z=0$ implies that $0={}^c\theta_Z(u_Z)={}^c\theta(u)$, i.e. $u\in\ker\theta$. □

Since Z is an AW^*-algebra, every normal element in $M_n(Z)$ can be diagonalised by [**16**, Corollary 3.3], see also [**24**, Theorem 3.2]. We therefore can describe elements in $\ker\theta$ in more detail. If $p\in M_n(C_b)$ is a projection with $ap=a$ and $p^t b=0$, there exists a unitary $v\in M_n(Z)$ such that $p'=v^*pv=\mathrm{diag}(p'_1,\dots,p'_n)$ where $p'_j\in Z$ are projections. Putting $a'=av$, ${}^t b'=v^{*t}b\in M_n({}^c M(A))$ we have $a'p'=a'$ and $p'^t b'=0$, that is, $p'_j a'_j=a'_j$ and $p'_j b'_j=0$ for all $1\leqq j\leqq n$. Moreover,

$$a'\otimes_Z {}^t b'=av\otimes_Z v^{*\,t}b=avv^*\otimes_Z {}^t b=a\otimes_Z {}^t b.$$

Hence, if $\sum_{j=1}^{n} a_j x b_j=0$ for all x (i.e. $u=a\otimes {}^t b\in\ker\theta$), then a and ${}^t b$, respectively, can be written as $a=a'v^*$ and ${}^t b=v^t b'$, respectively, for some unitary $v\in M_n(Z)$ such that $a'_j x b'_j=0$ for all x and all $1\leqq j\leqq n$.

Corollary 3.3. *Let $u=a\otimes {}^t b\in M(A)\otimes M(A)$ be such that $\{b_1,\dots,b_n\}$ is Z-independent. Then $\theta(u)=0$ if and only if $a_j=0$ for all $1\leqq j\leqq n$.*

Proof. If $p=(p_{ij})\in M_n(Z)$ is a projection such that $ap=a$ and $p^t b=0$, then $\sum_{j=1}^{n} p_{ij}b_j=0$ for all $1\leqq i\leqq n$ together with the Z-independence yields that $p_{ij}=0$ for all i,j. Hence, $a_j=0$. □

Corollary 3.4. *Let A be boundedly centrally closed. Then $\ker\theta=J$ and therefore θ_Z is injective.*

Proof. The assertion is immediate from Proposition 3.2 since $Z(M(A))=Z$ by Proposition 2.7. □

As a consequence, the ideal J is closed for boundedly centrally closed C^*-algebras so that Proposition 2.5 applies.

Our next aim is to show that θ_Z is in fact an isometry, if A is boundedly centrally closed. This will be done in two steps.

Theorem 3.5. *Let A be a boundedly centrally closed C^*-algebra. For every $u \in M(A) \otimes_Z M(A)$ we have $\|u\|_{Zh} = \sup_\pi \|\pi(u)\|_h$, where the supremum is taken over all irreducible representations of A.*

Proof. Every $\pi \in Irr(A)$ can be extended to an irreducible representation of $M(A)$, again denoted by π (since all extensions are equivalent [37, 4.1.11], it is irrelevant in the following which extension we take). By Proposition 2.3, $\|u\|_{Zh} \geqq \sup_\pi \|\pi(u)\|_h$ whence it suffices to conclude from $\sup_\pi \|\pi(u)\|_h \leqq 1$ that, for every $\varepsilon > 0$, there is a representation of u in $M(A) \otimes_Z M(A)$ whose norm is at most $(1+\varepsilon)^2$.

Let $\varepsilon > 0$ and $u = a \otimes_Z {}^t b$. By Lemma 2.4, there exist invertible matrices $S_1, \ldots, S_p \in M_n$ such that for each π there is $i \in \{1, \ldots, p\}$ satisfying $\max\{\|\pi_n(aS_i^{-1})\|, \|\pi_n(S_i {}^t b)\|\} \leqq 1 + \varepsilon$. Define $g_i : \check{A} \to \mathbf{R}_+$ by $g_i(t) = \max\{\|\pi_n(aS_i^{-1})\|, \|\pi_n(S_i {}^t b)\|\}$ if $t = \ker \pi \cap A$. Then, g_i is lower semi-continuous [37, 4.4.6 and 4.4.7], hence $U_i = g_i^{-1}(1+\varepsilon, \infty)$ is open. Since $\bigcap_{i=1}^p U_i = \emptyset$ and $\check{A}$ is extremally disconnected by Proposition 2.9, $\bigcap_{i=1}^p \bar{U}_i = \emptyset$ wherefore $\{V_i \mid 1 \leqq i \leqq p\}$ with $V_i = \check{A} \setminus \bar{U}_i$ forms a covering of $\check{A}$ by closed and open subsets. Put $W_1 = V_1$ and $W_i = V_i \cap \bigcap_{l=1}^{i-1} \bar{U}_l$ for $2 \leqq i \leqq p$. Then $\{W_i \mid 1 \leqq i \leqq p\}$ is a family of pairwise disjoint closed and open subsets such that $W_i \subseteq V_i$ and $\check{A} = \bigcup_{i=1}^p W_i$ as

$$
\begin{aligned}
\check{A} &= V_1 \cup \bar{U}_1 = V_1 \cup (\bar{U}_1 \cap (V_2 \cup \bar{U}_2)) \\
&= V_1 \cup (V_2 \cap \bar{U}_1) \cup (\bar{U}_1 \cap \bar{U}_2) \\
&\vdots \\
&= V_1 \cup (V_2 \cap \bar{U}_1) \cup \cdots \cup (V_p \cap \bar{U}_1 \cap \cdots \cap \bar{U}_{p-1}) \cup (\bar{U}_1 \cap \cdots \cap \bar{U}_p).
\end{aligned}
$$

Let I_i be the closed ideal of A corresponding to W_i; these are pairwise orthogonal, and $I_1 + \cdots + I_p = A$. Denoting by $e_i = c(I_i)$ the central supports of I_i in C_b, we thus have $\sum_{i=1}^p e_i = 1$, and $e_i \in Z(M(A))$ since A is boundedly centrally closed. Hence

$$
u = u \sum_{i=1}^p e_i = \sum_{i=1}^p aS_i^{-1} e_i \otimes_Z e_i S_i {}^t b = \sum_{i,k=1}^p aS_i^{-1} e_i \otimes_Z e_k S_k {}^t b
$$

yields a representation $u = a' \otimes_Z {}^t b'$ with $a' = \sum_{i=1}^p aS_i^{-1} e_i$, ${}^t b' = \sum_{k=1}^p e_k S_k {}^t b$. Denoting the entries in the first row of a' by $a'_1, \ldots, a'_n$ we have

$$
\begin{aligned}
\|a'\|^2 &= \left\| \sum_{j=1}^n a'_j a'^*_j \right\| \\
&= \sup_\pi \left\| \sum_{j=1}^n \pi(a'_j) \pi(a'_j)^* \right\|
\end{aligned}
$$

$$=\sup_{\pi}\left\|\sum_{j=1}^{n}\pi(a_j')\pi(a_j')^*\pi(e_{i(\pi)})\right\| \quad \text{where } \ker\pi\cap A\in W_{i(\pi)}$$

$$=\sup_{\pi}\|\pi_n(a)S_{i(\pi)}^{-1}\|^2\leqq(1+\varepsilon)^2$$

since $W_{i(\pi)}\subseteq V_{i(\pi)}\subseteq g_{i(\pi)}^{-1}[0,1+\varepsilon]$ by construction.

Similarly, $\|{}^t b'\|\leqq 1+\varepsilon$ and thus $\|a'\|\,\|{}^t b'\|\leqq(1+\varepsilon)^2$, i.e. $u=a'\otimes_Z {}^t b'$ is the desired representation of u. □

In the second step we will now describe the cb-norm of the operators in $\theta(M(A)\otimes M(A))$ via irreducible representations. We denote this subalgebra of $CB(A)$ by $\mathscr{E}\ell(A)$; its elements are called *elementary operators* on A. Both spectral and structural properties of elementary operators have been thoroughly studied during the past decades, cf. [**15**] and [**21**], but little seems to be known on the norm of such operators. As a consequence of our results, we will obtain a description of the cb-norm of an elementary operator on an arbitrary C^*-algebra. The ideas exploited follow the same lines as in calculating the norm of a two-sided multiplication $M_{a,b}:=L_aR_b$ in [**29**, Corollary 4.9].

Let $S\in\mathscr{E}\ell(A)$. Since $S=\sum_{j=1}^n M_{a_j,b_j}$ for some $a_1,\dots,a_n,b_1,\dots,b_n\in M(A)$, for each $k\in\mathbf{N}$ we have that $S_k=\sum_{j=1}^n M_{(a_j),(b_j)}\in\mathscr{E}\ell(M_k(A))$ where (a_j) respectively (b_j) denote the $k\times k$ diagonal matrices with a_j respectively b_j along the diagonal. If (π,H) is a (non-degenerate) representation of A, then $(\tilde\pi,H)$ denotes its ultraweakly continuous extension to A^{**} [**41**, III.2.2]. Since $\ker\pi$ is S-invariant, we obtain an elementary operator S_π on $\pi(A)$ via $S_\pi\circ\pi=\pi\circ S$. This one can be extended to the ultraweak closure $\pi(A)''$ to obtain $\tilde S_\pi$ satisfying $\tilde S_\pi\circ\tilde\pi=\tilde\pi\circ S^{**}$; simplifying the notation we write $S_{\tilde\pi}$ instead. As a consequence of the Kaplansky density theorem, one has equality of the unit balls $\pi(A)''_1=\tilde\pi(A_1^{**})$. Consequently, $\|S_\pi\|\leqq\|S_{\tilde\pi}\|\leqq\|S^{**}\|=\|S\|$.

In extending this to $k\times k$-matrices we use the identifications $M_k(M(A))=M(M_k(A))$, $M_k(A^{**})=M_k(A)^{**}$, $(\tilde\pi)_k=\tilde{\pi_k}=:\tilde\pi_k$, and

$$M_k(\pi(A)'')=M_k(\tilde\pi(A^{**}))=\tilde\pi_k(M_k(A^{**}))=\pi_k(M_k(A))''$$

from which $(S_\pi)_k=(S_k)_{\pi_k}=:S_{\pi_k}$ and $(\tilde S_\pi)_k=(\tilde S_k)_{\pi_k}=:S_{\tilde\pi_k}$ follow. Consequently, $\|S_{\pi_k}\|\leqq\|S_{\tilde\pi_k}\|\leqq\|S_k^{**}\|=\|S_k\|$ for all $k\in\mathbf{N}$.

Finally, we appeal to the fact that the reduced atomic representation (π^a,H^a) of A is faithful [**37**, 4.3.11], whence $\|S_{\pi_k^a}\|=\|S_{\tilde\pi_k^a}\|=\|S_k\|$ for all k. Taking all this together yields the following result.

Theorem 3.6. *For each elementary operator S on a C^*-algebra A and all $k\in\mathbf{N}$ we have that $\|S_k\|=\sup_\pi\|S_{\tilde\pi_k}\|$ and hence $\|S\|_{cb}=\sup_\pi\|S_{\tilde\pi}\|_{cb}$, where the supremum is taken over all irreducible representations of A.*

Proof. The second statement is immediate from the first. By the above inequalities, we obtain $\|S_k\|\geqq\sup_\pi\|S_{\tilde\pi_k}\|$. For the reverse inequality note that in order to calculate the

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norm of a $k\times k$-matrix via irreducible representations, it suffices to use the irreducible representations of A, since π_k^a is faithful. Therefore, for each $(x_{ij})\in M_k(A)$,

$$\begin{aligned}\|S_k((x_{ij}))\| &= \sup_\pi \|\pi_k\circ S_k((x_{ij}))\| \\ &= \sup_\pi \|S_{\pi_k}\circ \pi_k((x_{ij}))\| \\ &\leqq \sup_\pi \|S_{\tilde\pi_k}\|\,\|(x_{ij})\|\end{aligned}$$

which gives $\|S_k\|\leqq \sup_\pi\|S_{\tilde\pi_k}\|$. □

Remark. Note that, by Theorem 3.6, it in particular suffices to calculate the norm of an elementary operator on $B(H)$ in order to determine it in general.

Putting together Theorems 3.5 and 3.6 with Proposition 2.6 now yields the main result of this paper.

Theorem 3.7. *For every boundedly centrally closed C^*-algebra A the canonical homomorphism θ_Z is an isometry from $M(A)\otimes_{Zh}M(A)$ into $CB_Z(A)$.*

Proof. If $u\in M(A)\otimes_Z M(A)$ has a pre-image $\sum_{j=1}^n a_j\otimes b_j\in M(A)\otimes M(A)$, then, for each irreducible representation π of A, by Proposition 2.6 we have that

$$\|S_{\tilde\pi}\|_{cb}=\left\|\sum_{j=1}^n M_{\pi(a_j),\pi(b_j)}\right\|_{cb}=\left\|\sum_{j=1}^n \pi(a_j)\otimes\pi(b_j)\right\|_h=\|\pi(u)\|_h,$$

where $S=\theta(\sum_{j=1}^n a_j\otimes b_j)\in\mathscr{E}\ell(A)$. Consequently,

$$\|S\|_{cb}=\sup_\pi\|S_{\tilde\pi}\|_{cb}=\sup_\pi\|\pi(u)\|_h=\|u\|_{Zh}$$

whence θ_Z extends to an isometry on $M(A)\otimes_{Zh}M(A)$. □

Corollary 3.8. *For every AW^*-algebra, θ_Z is an isometry.*

This was obtained for von Neumann algebras by Chatterjee and Smith in [**11**, Theorem 2.4].

Corollary 3.9. *For every prime C^*-algebra, θ is an isometry.*

This, in particular, covers the case of von Neumann factors treated by Chatterjee and Sinclair in [**10**, Theorem 3]. Note that Corollary 3.9 is also directly deduced from Haagerup's result (Proposition 2.6) using arguments as in Theorem 3.6 and *one* faithful irreducible representation of an S-invariant primitive C^*-subalgebra, which exists by [**20**, Proposition 3.1] and [**37**, 4.3.6.].

Remark. In [11], Chatterjee and Smith provide an example of a unital C^*-algebra B such that θ_Z is *not* isometric. It is easily seen that $Z(B) \cong \mathbf{C}$ while $C_b(B) \cong \mathbf{C}^2$, whence B is not boundedly centrally closed.

In order to treat the case of a general C^*-algebra we need the following important stability property of the bounded central closure.

Proposition 3.10. *For every C^*-algebra A, the bounded central closure ${}^cA$ of A is boundedly centrally closed, and the centre of ${}^cM(A)$ coincides with Z, the centre of $M_{\mathrm{loc}}(A)$.*

Proof. We prove the second assertion first. We already observed in the proof of Proposition 2.7 that $Z \subseteq Z({}^cM(A))$. Hence, it suffices to show that $xy = yx$ for all $y \in Q_b(A)$ whenever $x \in Z({}^cM(A))$, as this implies $Z \supseteq Z({}^cM(A))$.

Let $y \in M(I)$ for some essential closed ideal I of A, and for $x \in Z({}^cM(A))$ choose a sequence $(x_n)_{n \in \mathbf{N}}$ in $Q_b(A)$ with $x = \lim_{n \to \infty} x_n$. Take essential closed ideals I_n of A such that $x_n \in M(I_n)$ and suppose, without loss of generality, that $I_n \subseteq I$ whence $y \in M(I_n)$ for all $n \in \mathbf{N}$. Then, for each $z \in I_n$ with $\|z\| \leqq 1$, we have

$$\begin{aligned} \|(x_n y - y x_n)z\| &\leqq \|x_n - x\| \, \|yz\| + \|xyz - y x_n z\| \\ &\leqq \|x_n - x\| \, \|y\| \, \|z\| + \|y\| \, \|x - x_n\| \, \|z\| \\ &\leqq 2\|y\| \, \|x_n - x\| \end{aligned}$$

where we used that $xyz = yzx = yxz$ since yz, $z \in I_n \subseteq A \subseteq {}^cM(A)$. As I_n is essential in $M(I_n)$, it follows that

$$\|x_n y - y x_n\| = \sup_{\substack{z \in I_n \\ \|z\| = 1}} \|(x_n y - y x_n)z\|$$

whence by the above $\lim_{n \to \infty} \|x_n y - y x_n\| = 0$, which yields finally that $xy = yx$ as desired.

Towards the first assertion, ${}^{cc}A = {}^cA$, note that it suffices to prove ${}^{cc}M(A) = {}^cM(A)$. In fact, by Lemma 2.8, ${}^cA$ is essential in ${}^cM(A)$ wherefore $Z({}^cA) = Z({}^cM(A)) \cap {}^cA$, ${}^cM(A) \subseteq M({}^cA)$ as well as $Z({}^cM(A)) \subseteq Z(M({}^cA))$. As a result,

$$C_b({}^cA) = C_b({}^cM(A)) = Z({}^cM(A)) \subseteq Z(M({}^cA)) \subseteq C_b({}^cA) \tag{1}$$

by Proposition 2.7(f). Therefore. $C_b({}^cA) = Z(M({}^cA))$ which, again by Proposition 2.7, implies that ${}^{cc}A = {}^cA$.

However, ${}^cM(A) = \overset{\lim}{\to} B_I$ where for each essential closed ideal I of A, B_I denotes the C^*-algebra generated by $M(A)$ and $Z(M(I))$. Consequently, the proof of the fact that $M_{\mathrm{loc}}(A)$ is boundedly centrally closed [4, Theorem 2] immediately adopts to the present situation and may hence be omitted. □

Corollary 3.11. *For every C^*-algebra A we have $C_b({}^cA) = Z$.*

Proof. By (1), $C_b({}^cA)=Z({}^cM(A))=Z$. □

We can now combine Theorem 3.7 with Proposition 3.10 to obtain our final result.

Theorem 3.12. *For every C^*-algebra A, the Z-bimodule homomorphism*

$$ {}^c\theta_Z: {}^cM(A)\otimes_{Zh} {}^cM(A)\to CB_Z({}^cA)$$

is an isometry.

Remark. The above isomorphism may be exploited to further investigate the central Haagerup tensor product, for example its centre. If $u\in Z(M(A)\otimes_Z M(A))$, then $\theta_Z(u)$ is the multiplication by some $y\in Z(M(A))$ since $\theta_Z(u)$ commutes with all left and all right multiplications by elements in A. As $\theta_Z(y\otimes_Z 1)=\theta_Z(u)$, $y\otimes_Z 1=u$ if A is boundedly centrally closed by Corollary 3.4, whence $Z(M(A)\otimes_Z M(A))\cong Z(M(A))=Z$ by Proposition 2.7. Observing that θ_Z is multiplicative (respectively anti-multiplicative) if one of the factors is in $M(A)\otimes_Z 1$ (respectively $1\otimes_Z M(A)$), a similar reasoning applies to $M(A)\otimes_{Zh} M(A)$, from which we obtain that

$$Z(M(A)\otimes_{Zh} M(A))\cong Z(M(A)) \tag{2}$$

for boundedly centrally closed C^*-algebras. In the general case, Theorem 3.12 together with Proposition 3.10 yields

$$Z({}^cM(A)\otimes_{Zh} {}^cM(A))\cong Z \tag{3}$$

isometrically, since the cb-norm of a (one-sided) multiplication coincides with the norm of the multiplying element. As $Z\otimes_{Zh} Z\cong Z$, we thus obtain an analogue of a property noted in [**1**, Theorem 2.13], see also [**39**, Corollary 4.7], namely that $Z(A\otimes_h B)=Z(A)\otimes_h Z(B)$ for arbitrary C^*-algebras A and B.

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