SEMIGROUP STRUCTURES FOR FAMILIES OF FUNCTIONS, II

CONTINUOUS FUNCTIONS

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1. Introduction

This is a continuation of [5] and we begin by recalling two definitions and a result of that paper which are needed here. Let $\mathscr{S}$ be a family of functions with domains contained in a set X and ranges contained in a set Y and let $\mathfrak{f}$ be a function with domain $\mathscr{D}(\mathfrak{f}) = Y$ and range $\mathscr{R}(\mathfrak{f}) \subseteqq X$ with the property $f \circ \mathfrak{f} \circ g \in \mathscr{S}$ for each pair of elements f and g of $\mathscr{S}$. Since the composition operation is associative, $\mathscr{S}$ is a semigroup if for f and g in $\mathscr{S}$, we define the product fg by $fg = f \circ \mathfrak{f} \circ g$.

Definition 1.1. $\mathscr{S}$ is referred to as an $\mathfrak{S}$-semigroup and is denoted by $\mathfrak{S}(X, Y, \mathfrak{f})$ if the following two conditions are satisfied.

1.1.1. $\mathscr{S}$ is point-separating, i.e., for each pair x_1 and x_2 of distinct points of X, there exists a function f in $\mathscr{S}$ whose domain contains both x_1 and x_2 with the property that $f(x_1) \neq f(x_2)$.

1.1.2. For each x in X and y in Y, there is a subset A of X containing x such that $A_y \in \mathscr{S}$ (A_y is the function whose domain is A and which is defined by $A_y(p) = y$ for all $p \in A$).

Definition 1.2. An $\mathfrak{S}$-semigroup $\mathfrak{S}(X, Y, \mathfrak{f})$ is referred to as an $\mathfrak{S}^*$-semigroup and is denoted by $\mathfrak{S}^*(X, Y, \mathfrak{f})$ if $\mathfrak{f}$ is a surjection onto X.

The main result we need from [5] is the following

Theorem 1.3. *A bijection φ from an $\mathfrak{S}^*$-semigroup $\mathfrak{S}^*(X, Y, \mathfrak{f})$ onto an $\mathfrak{S}^*$-semigroup $\mathfrak{S}^*(U, V, \mathfrak{g})$ is an isomorphism if and only if there exist bijections $\mathfrak{h}$ and $\mathfrak{t}$ from X onto U and Y onto V respectively such that for each f in $\mathfrak{S}^*(X, Y, \mathfrak{f})$, $\mathfrak{h}$ maps $\mathscr{D}(f)$ bijectively onto $\mathscr{D}(\varphi(f))$ and the following diagram commutes.*

$$\begin{tikzcd}
\mathscr{D}(f) \arrow[r, "f"] \arrow[d, "\mathfrak{h}"'] & Y \arrow[r, "\mathfrak{f}"] \arrow[d, "\mathfrak{t}"'] & X \arrow[d, "\mathfrak{h}"'] \\
\mathscr{D}(\varphi(f)) \arrow[r, "\varphi(f)"] & V \arrow[r, "\mathfrak{g}"] & U
\end{tikzcd}$$

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Moreover, the functions $\mathfrak{h}$ *and* $\mathfrak{t}$ *are unique in the sense that if* $\mathfrak{h}^*$ *and* $\mathfrak{t}^*$ *are two mappings from* X *into* U *and* Y *into* V *respectively with the property that the resulting diagram commutes when* $\mathfrak{h}$ *is replaced by* $\mathfrak{h}^*$ *and* $\mathfrak{t}$ *by* $\mathfrak{t}^*$, *then* $\mathfrak{h} = \mathfrak{h}^*$ *and* $\mathfrak{t} = \mathfrak{t}^*$.

Now suppose we let X and Y denote topological spaces and $\mathscr{S}$ the collection of all continuous functions whose domains equal X and whose ranges are subsets of Y. Suppose further that for any two distinct points p and q of X these exists a function f in $\mathscr{S}$ such that $f(p) \neq f(q)$. Then if X happens to be the image of Y under a continuous mapping $\mathfrak{f}$, $\mathscr{S}$ is an $\mathfrak{S}^*$-semigroup if we define $fg = f \circ \mathfrak{f} \circ g$ for all f, g in $\mathscr{S}$. This particular $\mathfrak{S}^*$-semigroup will be denoted by $\mathfrak{C}^*_T(X, Y, \mathfrak{f})$ and will be referred to as a $\mathfrak{C}^*_T$-semigroup. Now, Theorem (1.3) implies that any isomorphism from a $\mathfrak{C}^*_T$-semigroup $\mathfrak{C}^*_T(X, Y, \mathfrak{f})$ onto a $\mathfrak{C}^*_T$-semigroup $\mathfrak{C}^*_T(U, V, \mathfrak{g})$ uniquely determines two bijections $\mathfrak{h}$ and $\mathfrak{t}$ such that the diagram commutes. It is quite natural to ask if these bijections must be homeomorphisms. The answer is, in general, no. Section 2 is devoted to the task of finding conditions which will insure that the functions $\mathfrak{h}$ and $\mathfrak{t}$ will be homeomorphisms. These results are used in Section 3 to determine the automorphism groups of $\mathfrak{C}^*_T$-semigroups. It is shown for certain X, Y and $\mathfrak{f}$ that the automorphism group of $\mathfrak{C}^*_T(X, Y, \mathfrak{f})$ is isomorphic to a certain subgroup of the group of all homeomorphisms on Y. For example, if $X = [0, +\infty)$ and R denotes the space of real numbers and $\mathfrak{f}$ is defined by $\mathfrak{f}(x) = x^2$ for each in R, the automorphism group of $\mathfrak{C}^*_T(X, R, \mathfrak{f})$ is isomorphic to the group of all homeomorphisms on R which are symmetric about the origin. The results of Section 3 are then applied in Section 4 to the near-ring of all continuous functions from a topological space into a topological group. We recall that a near-ring is a system with two operations, addition and multiplication, which satisfies all the postulates for a ring with the possible exceptions of the commutative law of addition and one of the distributive laws. We construct near-rings of continuous functions as follows: let X denote a topological space, G an additive topological group, and $\mathfrak{f}$ a continuous function from G onto X. Let $\mathfrak{N}^*_T(X, G, \mathfrak{f})$ denote the family of all continuous mappings from X into G. For f and g in $\mathfrak{N}^*_T(X, G, \mathfrak{f})$, we define

$$fg = f \circ \mathfrak{f} \circ g \quad \text{and}$$
$$(f+g)(x) = f(x)+g(x) \text{ for all } x \text{ in } X.$$

Then $\mathfrak{N}^*_T(X, G, \mathfrak{f})$, along with these two binary operations is a near-ring. Note that for any f, g, h in $\mathfrak{N}^*_T(X, G, \mathfrak{f})$, we always have

$$(f+g)h = fh+gh$$

while it need not be true that

$$h(f+g) = hf+hg.$$

If $X = G$ and $\mathfrak{f}$ is the identity mapping, $\mathfrak{N}_T^*(G, G, \mathfrak{f})$ is the near-ring of all continuous functions mapping G into G where multiplication is simply composition. We shall use the simpler notation $\mathfrak{N}_T^*(G)$ for such a near-ring. It is shown in Section 4 (for certain X, G and $\mathfrak{f}$) that the automorphism group of $\mathfrak{N}_T^*(X, G, \mathfrak{f})$ is isomorphic to a certain subgroup of the automorphism group of G. Applying this result, we obtain the fact that the automorphism group of $\mathfrak{N}_T^*(R_A)$ (R_A denotes the additive group of real numbers) is isomorphic to the multiplicative group of non-zero real numbers. Certain semigroups of continuous functions whose domains are subsets of a given space X are discussed in Section 5.

2. The semigroup $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$

Let φ be an isomorphism from the $\mathfrak{C}_T^*$-semigroup $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$ onto the $\mathfrak{C}_T^*$-semigroup $\mathfrak{C}_T^*(U, V, \mathfrak{g})$. According to Theorem (1.3), φ uniquely determines two bijections $\mathfrak{h}$ and $\mathfrak{t}$ from X onto U and Y onto V respectively such that for every f in $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$, the following diagram is commutative.

2.1
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathfrak{h}"'] & Y \arrow[r, "\mathfrak{f}"] \arrow[d, "\mathfrak{t}"'] & X \arrow[d, "\mathfrak{h}"'] \\ U \arrow[r, "\varphi(f)"] & V \arrow[r, "\mathfrak{g}"] & U \end{tikzcd}$$

As we mentioned in the introduction, it may happen that the mappings $\mathfrak{h}$ and $\mathfrak{t}$ are not homeomorphisms. There exist infinite spaces with the property that the only continuous functions mapping the space into itself are the constant functions and the identity function. De Groot has shown [2, p. 87, Theorem 3] that there are 2^c such subspaces of the Euclidean plane. Let X be such a space and let i be the identity mapping from X into X. Then $\mathfrak{C}_T^*(X, X, i)$ is a $\mathfrak{C}_T^*$-semigroup and the subsemigroup obtained by subtracting the identity is referred to as a left zero semigroup [1]. It has the property that $fg = f$ for any two elements f and g. Consequently, any bijection from $\mathfrak{C}_T^*(X, X, i)$ onto itself which leaves the identity fixed is an automorphism of $\mathfrak{C}_T^*(X, X, i)$. There are infinitely many of these automorphisms and each one uniquely determines a pair of bijections $\mathfrak{h}$ and $\mathfrak{t}$. However, only the identity automorphism determines bijections which are homeomorphisms.

From this point on, we shall always assume that the topological spaces discussed in this paper are T_1. Let X and Y be topological spaces and let $\mathfrak{f}$ be a continuous function from Y onto X. *The triple* $(X, Y, \mathfrak{f})$ *is said to be admissible if for each closed subset H of X and each point p in $X-H$, there exists a continuous function f mapping X into Y and a point q in X such* *that* $(\mathfrak{f} \circ f)(x) = q$ *for* x *in* H *and* $(\mathfrak{f} \circ f)(p) \neq q$. The following two theorems indicate that admissible triples are reasonably abundant.

THEOREM 2.2. *Suppose* X *is completely regular and* Y *contains two points* s *and* t *joined by an arc such that* $\mathfrak{f}(s) \neq \mathfrak{f}(t)$. *Then* $(X, Y, \mathfrak{f})$ *is an admissible triple.*

PROOF. Let H be a closed subset of X and suppose $p \in X - H$. Since X is completely regular, there exists a continuous function f from X into the closed unit interval I such that $f(x) = 0$ for x in H and $f(p) = 1$. Since s and t are joined by an arc, there exists a continuous function g from I into Y such that $g(0) = s$ and $g(1) = t$. Then $g \circ f$ is a continuous function from X into Y such that $(\mathfrak{f} \circ (g \circ f))(x) = \mathfrak{f}(s)$ for x in H and $(\mathfrak{f} \circ (g \circ f))(p) = \mathfrak{f}(t) \neq \mathfrak{f}(s)$.

THEOREM 2.3. *If* X *is* 0*-dimensional,* $(X, Y, \mathfrak{f})$ *is an admissible triple.*

PROOF. Suppose H is a closed subset of X and p belongs to $X - H$. Since X is 0-dimensional, there exists a subset G of X which is both open closed such that $p \in G$ and $H \subseteq X - G$. Choose any two points s and t of Y such that $\mathfrak{f}(s) \neq \mathfrak{f}(t)$ and define a function f from X into Y by $f(x) = s$ if $x \in G$ and $f(x) = t$ if $x \in X - G$. Then f is continuous and $(\mathfrak{f} \circ f)(x) = \mathfrak{f}(t)$ for x in H while $(\mathfrak{f} \circ f)(p) \neq \mathfrak{f}(t)$.

Note that if $(X, Y, \mathfrak{f})$ is an admissible triple, the family of all continuous functions mapping X into Y is point-separating and hence the semigroup of all continuous functions mapping X into Y is a $\mathfrak{C}_T^*$-semigroup. *In the sequel, only admissible triples will be considered.* For any two such triples, we have the following

THEOREM 2.4. *Let* $(X, Y, \mathfrak{f})$ *and* $(U, V, \mathfrak{g})$ *be admissible triples and let* φ *be an isomorphism from* $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$ *onto* $\mathfrak{C}_T^*(U, V, \mathfrak{g})$. *Then the bijection* $\mathfrak{h}$ *from* X *onto* U *determined by* φ *is a homeomorphism.*

PROOF. For each point p in X and f in $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$, let

$$H(p, f) = \{x \in X : (\mathfrak{f} \circ f)(x) = p\}.$$

Similarly, for q in U and h in $\mathfrak{C}_T^*(U, V, \mathfrak{g})$, we let

$$H(q, h) = \{x \in U : (\mathfrak{g} \circ h)(x) = q\}.$$

According to diagram 2.1, the following statements are successively equivalent:

$$u \in \mathfrak{h}[H(p, f)], \; u = \mathfrak{h}(x) \quad \text{and} \quad (\mathfrak{f} \circ f)(x) = p,$$
$$\mathfrak{h}(p) = (\mathfrak{h} \circ \mathfrak{f} \circ f)(x) = (\mathfrak{g} \circ \varphi(f) \circ \mathfrak{h})(x) = (\mathfrak{g} \circ \varphi(f))(u),$$
$$u \in H(\mathfrak{h}(p), \varphi(f)).$$

Therefore, $\mathfrak{h}[H(p, f)] = H(\mathfrak{h}(p), \varphi(f))$. In a similar manner, $\mathfrak{h}^{\leftarrow}[H(q, h)] = H(\mathfrak{h}^{\leftarrow}(q), \varphi^{\leftarrow}(h))$ for all q in U and h in $\mathfrak{S}_T^*(U, V, \mathfrak{g})$.

Since all spaces concerned are T_1 spaces, sets of the form $H(p, f)$ are closed. Our proof will be complete when we show that for an admissible triple, $(X, Y, \mathfrak{f})$, the family of all such sets forms a basis for the closed subsets of X. With this in mind, let F be a nonempty closed subset of X and x a point in $X-F$. Then there exists a point p_x in X and a function f_x in $\mathfrak{S}_T^*(X, Y, \mathfrak{f})$ such that $(\mathfrak{f} \circ f_x)(y) = p_x$ for y in F and $(\mathfrak{f} \circ f_x)(x) \neq p_x$. Hence,

$$F = \cap \{H(p_x, f_x) : x \in X - F\}.$$

This proves the theorem.

The following example shows that the bijection $\mathfrak{t}$ need not be a homeomorphism.

EXAMPLE 2.5. Let both X and U consist of the single point x. Let Y and V be any two spaces with the same cardinality and define functions $\mathfrak{f}$ and $\mathfrak{g}$ from Y onto X and V onto U respectively by $\mathfrak{f}(y) = x$ for all y in Y and $\mathfrak{g}(v) = x$ for all v in V. Then $(X, Y, \mathfrak{f})$ and $(U, V, \mathfrak{g})$ are both admissible triples and $\mathfrak{S}_T^*(X, Y, \mathfrak{f})$ and $\mathfrak{S}_T^*(U, V, \mathfrak{g})$ are left zero semigroups, i.e., the product of two elements is the element on the left. Since the two semigroups have the same cardinality (which is the cardinality of Y and V), there exist bijections from one onto the other and any such bijection is an isomorphism. It follows that the bijection $\mathfrak{t}$ from Y onto V determined by any such isomorphism need not be a homeomorphism since we can choose Y and V to be non-homeomorphic spaces.

We say two admissible triples $(X, Y, \mathfrak{f})$ *and* $(U, V, \mathfrak{g})$ *are isomorphic if the semigroups* $\mathfrak{S}_T^*(X, Y, \mathfrak{f})$ *and* $\mathfrak{S}_T^*(U, V, \mathfrak{g})$ *are isomorphic. Two isomorphic triples are said to be compatible if for any isomorphism* φ *from* $\mathfrak{S}_T^*(X, Y, \mathfrak{f})$ *onto* $\mathfrak{S}_T^*(U, V, \mathfrak{g})$, *the bijection* $\mathfrak{t}$ *from* Y *onto* V *determined by* φ *is a homeomorphism.* With this convention, the following result is an immediate consequence of Theorems 1.3 and 2.4.

THEOREM 2.6. *Suppose* $(X, Y, \mathfrak{f})$ *and* $(U, V, \mathfrak{g})$ *are compatible triples. Then a bijection* φ *from* $\mathfrak{S}_T^*(X, Y, \mathfrak{f})$ *onto* $\mathfrak{S}_T^*(U, V, \mathfrak{g})$ *is an isomorphism if and only if there exists a unique homeomorphism* $\mathfrak{h}$ *from* X *onto* U *and a unique homeomorphism* $\mathfrak{t}$ *from* Y *onto* V *such that for each* f *in* $\mathfrak{S}_T^*(X, Y, \mathfrak{f})$, *the following diagram is commutative.*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{\mathfrak{f}} & X \\ \mathfrak{h}\downarrow & & \mathfrak{t}\downarrow & & \mathfrak{h}\downarrow \\ U & \xrightarrow{\varphi(f)} & V & \xrightarrow{\mathfrak{g}} & U \end{array}$$

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The remainder of this section will be devoted to the task of finding conditions under which two triples will be compatible.

THEOREM 2.7. *Let* $(X, Y, \mathfrak{f})$ *and* $(U, V, \mathfrak{g})$ *be two isomorphic triples such that* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are homeomorphisms onto* X *and* U *respectively. Then* $(X, Y, \mathfrak{f})$ *and* $(U, V, \mathfrak{g})$ *are compatible.*

PROOF. Let φ be an isomorphism from $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$ onto $\mathfrak{C}_T^*(U, V, \mathfrak{g})$. Then by diagram (2.1), $\mathfrak{t} = \mathfrak{g}^{\leftarrow} \circ \mathfrak{h} \circ \mathfrak{f}$. Now $\mathfrak{g}^{\leftarrow}$ and $\mathfrak{f}$ are homeomorphisms by hypothesis and $\mathfrak{h}$ is a homeomorphism by Theorem 2.4. Thus $\mathfrak{t}$ is a homeomorphism.

Let us recall [3, page 230] that a space X is called a k-space if it satisfies the condition: if a subset H of X intersects each closed, compact set in a closed set, then H is closed. The class of k-spaces includes all locally compact, Hausdorff spaces and all Hausdorff spaces which satisfy the first axiom of countability. The important fact about k-spaces is that for any bijection h from one such space into another, it is sufficient to show that both h and $h^{\leftarrow}$ take closed, compact sets into closed, compact sets in order to conclude h is a homeomorphism. Of course, for a Hausdorff space, the family of all closed, compact subsets coincides with the family of all compact subsets.

THEOREM 2.8. *Let* $(X, Y, \mathfrak{f})$ *and* $(U, V, \mathfrak{g})$ *be isomorphic triples and suppose* X *and* U *are Hausdorff spaces and* Y *and* V *are Hausdorff, k-spaces. Suppose further that for every compact subset* K *of* Y, *there exists a continuous function* f *from* X *into* Y *such that* $K \subseteqq f[X]$ *and* $f^{\leftarrow}[K]$ *is a compact subset of* X. *Finally, suppose a similar condition holds for compact subsets of* V. *Then the triples* $(X, Y, \mathfrak{f})$ *and* $(U, V, \mathfrak{g})$ *are compatible.*

PROOF. Let K be a compact subset of Y. Then there exists a continuous function f mapping X into Y such that $K \subseteqq f[X]$ and $f^{\leftarrow}[K]$ is compact. By Theorem 2.4, $\mathfrak{h}$ is a homeomorphism from X onto U and it follows that $\varphi(f)[\mathfrak{h}[f^{\leftarrow}[K]]]$ is a compact subset of V. But it follows from diagram (2.1) that this set is actually $\mathfrak{t}[K]$. Therefore $\mathfrak{t}$ takes compact subsets into compact subsets and, in a similar manner, $\mathfrak{t}^{\leftarrow}$ also takes compact subsets into compact subsets. Since both Y and V are Hausdorff k-spaces, this implies $\mathfrak{t}$ is a homeomorphism.

Let us recall that a compact, connected, locally connected metric space is referred to as a Peano space.

THEOREM 2.9. *Let* $(X, Y, \mathfrak{f})$ *and* $(U, V, \mathfrak{g})$ *be isomorphic triples and suppose* X *and* U *are completely regular, Hausdorff spaces each containing a compact, connected subset with nonempty interior and more than one point. Suppose also that* Y *and* V *are Hausdorff k-spaces with the property that each compact subspace is contained in a Peano subspace. Then* $(X, Y, \mathfrak{f})$ *and* $(U, V, \mathfrak{g})$ *are compatible triples.*

PROOF. Let us first consider the case where Y is compact. It follows from the hypothesis that Y is a Peano space and is therefore a connected space. X must also be compact and connected since it is the image of Y under the continuous mapping $\mathfrak{f}$. Choose any two distinct points p and q in X. Since X is completely regular, there exists a continuous function f mapping X into the closed unit interval I such that $f(p) = 0$ and $f(q) = 1$. Since X is connected, it follows that $f[X] = I$. Since Y is a Peano space, it follows from the Hahn-Mazurkiewicz Theorem that Y is the image of I under some continuous function g. Then $g \circ f$ is a continuous function from X onto Y with the property that $(g \circ f)^{\leftarrow}[K]$ is compact for each compact subset of Y. Thus we have shown, in the case Y is compact, the existence of a continuous function from X into Y satisfying the conditions of Theorem 2.8.

Now suppose Y is not compact and let K be a nonempty compact subset of Y. Choose q in $Y - K$. Then according to hypothesis, there exists a Peano subspace K^* which contains $K \cup \{q\}$. Also according to hypothesis, there exists a point p of X, an open subset G of X and a compact, connected subset W of X containing more than one point such that $p \in G \subseteqq W$. Choose $r \in W - \{p\}$ and let $G^* = G - \{r\}$. Since X is completely regular, there exists a continuous function f mapping X into the closed unit interval I such that $f(p) = 1$ and $f(x) = 0$ for x in $X - G^*$. Since W is connected and contains both p and r, it follows that $f[W] = I$. Again we appeal to the Hahn-Mazurkiewicz Theorem to conclude the existence of a continuous function g mapping I onto K^*. A check of the proof of that theorem will convince one that g can be chosen such that $g(0) = q$. Therefore, $g \circ f$ is a continuous mapping from X into Y such that $K \subseteqq (g \circ f)[X]$. Now we want to show that $(g \circ f)^{\leftarrow}[K]$ is a compact subset of X. Since $q \notin K$, there exists an open subset H of Y containing q such that $H \cap K = \emptyset$. Since $0 \in g^{\leftarrow}[H]$, it follows that there exists a number a between 0 and 1 such that $[0, a) \subseteqq g^{\leftarrow}[H]$ which implies $g^{\leftarrow}[K] \subseteqq [a, 1]$. But then, $(g \circ f)^{\leftarrow}[K] = f^{\leftarrow}[g^{\leftarrow}[K]] \subseteqq f^{\leftarrow}[a, 1] \subseteqq W$. That is, $(g \circ f)^{\leftarrow}[K]$ is a closed subset of a compact set W and is therefore also compact. Thus, in this case also, there exists a continuous function from X into Y satisfying the conditions of Theorem 2.8. Since the arguments given here can be repeated for the spaces U and V, it follows from Theorem 2.8 that $(X, Y, \mathfrak{f})$ and $(U, V, \mathfrak{g})$ are compatible.

By a *manifold*, we mean a connected metric space with the property that for some positive integer N, each point of the space is contained in an open subset which is homeomorphic to the Euclidean N-space E^N. Suppose K is a compact subset of a manifold X. Then there exists a finite collection $\{B_n\}_{n=1}^{M}$ of subsets of X, each homeomorphic to the closed unit ball in E^N, with the property that $K \subset \cup \{B_n\}_{n=1}^{M}$. Choose a point p_n in B_n for each n. Since a manifold is arcwise connected (see [6], page 55, Theorem 2—17 for a proof of the 2-dimensional case which generalizes easily), we may join p_n to p_m by means of an arc $A_{n,m}$. Since there are only a finite number of such arcs, the subspace K^* consisting of $\cup \{B_n\}_{n=1}^{M}$ along with the arcs $A_{n,m}$ is compact. In addition, K^* is connected (in fact, arcwise connected), locally connected and metric, i.e., K^* is a Peano space. Thus, every compact subspace of X is contained in a Peano subspace and we may apply Theorem 2.9 to obtain the following

COROLLARY 2.10. *Let $(X, Y, \mathfrak{f})$ and $(U, V, \mathfrak{g})$ be isomorphic triples and suppose X and U are completely regular, Hausdorff spaces, each containing a compact, connected subset with nonempty interior and more than one point. Then if Y and V are either manifolds or Peano spaces, $(X, Y, \mathfrak{f})$ and $(U, V, \mathfrak{g})$ are compatible triples.*

3. The automorphism groups of $\mathfrak{C}_T^*$-semigroups

In this section, we determine the automorphism group of $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$ whenever $(X, Y, \mathfrak{f})$ is a strongly admissible triple. *A triple $(X, Y, \mathfrak{f})$ is said to be strongly admissible if it is admissible, compatible with itself, and $\mathfrak{f}$ is either a closed or an open mapping*. Concerning strongly admissible triples, we have the following three results.

THEOREM 3.1. *Suppose X is either 0-dimensional or completely regular with two distinct points joined by an arc. Then for any homeomorphism $\mathfrak{f}$ from Y onto X, $(X, Y, \mathfrak{f})$ is a strongly admissible triple.*

THEOREM 3.2. *Suppose X and Y are Peano spaces and X has more than one point. Then, for any continuous mapping $\mathfrak{f}$ from Y onto X, $(X, Y, \mathfrak{f})$ is a strongly admissible triple.*

THEOREM 3.3. *Suppose X is a completely regular, Hausdorff space which contains a compact, connected subset with nonempty interior and more than one point. Suppose also that Y is a manifold and that $\mathfrak{f}$ is a continuous, closed or open function from Y onto X with the property $\mathfrak{f}(p) \neq \mathfrak{f}(q)$ for two points p and q of Y which are joined by an arc. Then $(X, Y, \mathfrak{f})$ is a strongly admissible triple.*

Theorem 3.1 is an immediate consequence of Theorems 2.2, 2.3, and 2.7. To prove Theorem 3.2, we first recall that any Peano space is arcwise connected. It follows from this fact and Theorem 2.2 that $(X, Y, \mathfrak{f})$ is admissible. Corollary 2.10 implies that $(X, Y, \mathfrak{f})$ is compatible with itself. Finally, since any continuous mapping from a compact space into a Hausdorff space is a closed mapping, it follows that $(X, Y, \mathfrak{f})$ is strongly admissible. Theorem 3.2 is an immediate consequence of Theorem 2.2 and Corollary 2.10.

Now let $(X, Y, \mathfrak{f})$ be any strongly admissible triple and let $\mathfrak{D}_{\mathfrak{f}}$ denote the family of all point inverses of $\mathfrak{f}$, i.e.,

$$\mathfrak{D}_{\mathfrak{f}} = \{\mathfrak{f}^{\leftarrow}\{x\} : x \in X\}.$$

$\mathfrak{D}_{\mathfrak{f}}$ is a family of mutually disjoint nonempty subsets of Y whose union is all of Y. We let $G(\mathfrak{D}_{\mathfrak{f}})$ denote the group of all homeomorphisms h on Y (where the binary operation is composition) with the property $h[A] \in \mathfrak{D}_{\mathfrak{f}}$ for each $A \in \mathfrak{D}_{\mathfrak{f}}$. The main result of this section concerns this group and is the following

THEOREM 3.4. *If $(X, Y, \mathfrak{f})$ is strongly admissible, the automorphism group of $\mathfrak{C}^*_T(X, Y, \mathfrak{f})$ is isomorphic to $G(\mathfrak{D}_{\mathfrak{f}})$.*

Before proving this result, it will be convenient to have a lemma.

LEMMA 3.5. *Suppose $\mathfrak{f}$ is a continuous mapping which is either open or closed from Y onto X and $\mathfrak{t}$ is a homeomorphism from Y onto Y. Then there exists a homeomorphism $\mathfrak{h}$ from X onto X such that $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{f} \circ \mathfrak{t}$ if and only if $\mathfrak{t}[A] \in \mathfrak{D}_{\mathfrak{f}}$ for each $A \in \mathfrak{D}_{\mathfrak{f}}$.*

PROOF. We make use of Lemma 3.2 of [5] which is stated there as follows:

3.5.1. Suppose $\mathfrak{f}$ maps Y onto X, $\mathfrak{g}$ maps Y onto Z and $\mathfrak{t}$ is a bijection from Y onto Y. Then there exsists a bijection $\mathfrak{h}$ from X onto Z such that $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{g} \circ \mathfrak{t}$ if and only if $\mathfrak{t}[A] \in \mathfrak{D}_{\mathfrak{g}}$ for each $A \in \mathfrak{D}_{\mathfrak{f}}$.

First suppose $\mathfrak{t}$ is a homeomorphism from Y onto Y such that $\mathfrak{t}[A] \in \mathfrak{D}_{\mathfrak{f}}$ for each $A \in \mathfrak{D}_{\mathfrak{f}}$. Taking $X = Z$ and $\mathfrak{f} = \mathfrak{g}$ in 3.5.1, it follows that there exists a bijection $\mathfrak{h}$ from X onto X such that $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{f} \circ \mathfrak{t}$. Now suppose $\mathfrak{f}$ is a closed mapping and let H be any closed subset of X. Then, since $\mathfrak{f}$ is continuous and closed, $\mathfrak{f}[\mathfrak{t}[\mathfrak{f}^{\leftarrow}[H]]]$ is a closed subset of X. But this latter set is equal to $\mathfrak{h}[H]$. In a similar manner, $\mathfrak{h}^{\leftarrow}$ takes closed sets into closed sets and $\mathfrak{h}$ is a homeomorphism. One uses open sets for the proof if $\mathfrak{f}$ is an open mapping. The remaining portion of the proof is an immediate consequence of 3.5.1.

Now let us proceed with the proof of Theorem 3.4. Let $\mathfrak{A}$ denote the automorphism group of $\mathfrak{C}^*_T(X, Y, \mathfrak{f})$ and let φ denote an element of $\mathfrak{A}$. Then, according to Theorem 2.6, there exists a unique homeomorphism $\mathfrak{h}$ from X onto X and a unique homeomorphism $\mathfrak{t}$ from Y onto Y such that for each f in $\mathfrak{C}^*_T(X, Y, \mathfrak{f})$, the following diagram is commutative.

3.6
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathfrak{h}"'] & Y \arrow[r, "\mathfrak{f}"] \arrow[d, "\mathfrak{t}"'] & X \arrow[d, "\mathfrak{h}"'] \\
X \arrow[r, "\varphi(f)"] & Y \arrow[r, "\mathfrak{f}"] & X
\end{tikzcd}$$

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Then $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{f} \circ \mathfrak{t}$ and $\mathfrak{t} \in G(\mathfrak{D}_\mathfrak{f})$ by the previous lemma. Since $\mathfrak{t}$ is uniquely determined by φ, we can define a mapping Φ from $\mathfrak{A}$ into $G(\mathfrak{D}_\mathfrak{f})$ by $\Phi(\varphi) = \mathfrak{t}$. It can be verified in a straightforward manner that Φ is a homomorphism. Now if $\mathfrak{t}$ is any element in $G(\mathfrak{D}_\mathfrak{f})$, it follows from Lemma 3.5 that there exists a homeomorphism $\mathfrak{h}$ from X onto X such that $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{f} \circ \mathfrak{t}$. Then Theorem 2.6 implies that the bijection φ from $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$ onto itself defined by $\varphi(f) = \mathfrak{t} \circ f \circ \mathfrak{h}^{\leftarrow}$ is an automorphism. This implies Φ is an epimorphism onto $G(\mathfrak{D}_\mathfrak{f})$. Finally, suppose $\Phi(\varphi) = i$, the identity mapping on Y. Then there is a homeomorphism $\mathfrak{h}$ from X onto X such that the resulting diagram commutes when $\mathfrak{t}$ is replaced by i in diagram 3.6. For every x in X, there exists a y in Y such that $\mathfrak{f}(y) = x$. Then $\mathfrak{h}(x) = \mathfrak{h}(\mathfrak{f}(y)) = \mathfrak{f}(i(y)) = \mathfrak{f}(y) = x$, i.e. $\mathfrak{h}$ is the identity mapping on X. This implies that φ is the identity automorphism. Hence, the kernel of Φ consists of the identity and we conclude Φ is an isomorphism.

Example 3.7. Let $X = [0, +\infty)$, let R denote the space of real numbers and define a mapping $\mathfrak{f}$ from R onto X by $\mathfrak{f}(x) = x^2$ for each x in R. Since $\mathfrak{f}$ is a closed mapping from R onto X, it follows from Theorem 3.3 that $(X, R, \mathfrak{f})$ is a strongly admissible triple. Now $\mathfrak{D}_\mathfrak{f} = \{\{x, -x\} : 0 \leqq x\}$ and it follows that a homeomorphism $\mathfrak{t}$ from Y onto Y belongs to $G(\mathfrak{D}_\mathfrak{f})$ if and only if $\mathfrak{t}(-x) = -\mathfrak{t}(x)$ for each x in R. $G(\mathfrak{D}_\mathfrak{f})$ is the group of all homeomorphisms mapping R onto R which are symmetric about the origin. By Theorem (3.4), this group is isomorphic to the automorphism group of $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$.

4. Applications to near-rings

Let us recall once again that a near-ring $\mathfrak{N}$ is a system with two binary operations, addition and multiplication, such that $\mathfrak{N}$ is a group under addition, a semigroup under multiplication and $(a+b)c = ac+bc$ for all a, b, c in $\mathfrak{N}$. Now let G be a topological group and let $\mathfrak{f}$ be a continuous mapping from G onto a topological space X. We recall from the introduction that $\mathfrak{N}_T^*(X, G, \mathfrak{f})$, the family of all continuous functions mapping X into G is a near-ring when addition is defined pointwise and multiplication is defined by $fg = f \circ \mathfrak{f} \circ g$ for all f and g in $\mathfrak{N}_T^*(X, G, \mathfrak{f})$. By an isomorphism from one topological group onto another, we mean a mapping that is both an algebraic isomorphism and a homeomorphism. Our first result is an easy consequence of Theorem 2.6.

Theorem 4.1. *Let X and Y be topological spaces, G and H topological groups and suppose $(X, G, \mathfrak{f})$ and $(Y, H, \mathfrak{g})$ are compatible. Then a bijection φ from the near-ring $\mathfrak{N}_T^*(X, G, \mathfrak{f})$ onto the near-ring $\mathfrak{N}_T^*(Y, H, \mathfrak{g})$ is an isomorphism if and only if there exists a homeomorphism $\mathfrak{h}$ from X onto Y and an*

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isomorphism $\mathfrak{t}$ *from* G *onto* H *such that for every* f *in* $\mathfrak{N}_T^*(X, G, \mathfrak{f})$, *the following diagram commutes.*

$$\begin{array}{ccccc} X & \xrightarrow{f} & G & \xrightarrow{\mathfrak{f}} & X \\ \downarrow{\scriptstyle \mathfrak{h}} & & \downarrow{\scriptstyle \mathfrak{t}} & & \downarrow{\scriptstyle \mathfrak{h}} \\ Y & \xrightarrow{\varphi(f)} & H & \xrightarrow{\mathfrak{g}} & Y \end{array}$$

Moreover, the homeomorphism $\mathfrak{h}$ *and the isomorphism* $\mathfrak{t}$ *are unique.*

PROOF. Suppose φ is an isomorphism from $\mathfrak{N}_T^*(X, G, \mathfrak{f})$ onto $\mathfrak{N}_T^*(Y, H, \mathfrak{g})$. Then according to Theorem 2.6, there exist homeomorphisms $\mathfrak{h}$ and $\mathfrak{t}$ such that the diagram above commutes. Now for an arbitrary element a of G, let $\langle a \rangle$ denote the constant function of $\mathfrak{N}_T^*(X, G, \mathfrak{f})$ which is defined by $\langle a \rangle(x) = a$ for each x in X. It follows that for any pair of elements a and b of G, we have $\langle a+b \rangle = \langle a \rangle + \langle b \rangle$. Thus, using the diagram and the fact that φ is additive, we get

$$\begin{aligned} \langle \mathfrak{t}(a+b) \rangle &= \varphi(\langle a+b \rangle) = \varphi(\langle a \rangle + \langle b \rangle) \\ &= \varphi(\langle a \rangle) + \varphi(\langle b \rangle) = \langle \mathfrak{t}(a) \rangle + \langle \mathfrak{t}(b) \rangle. \end{aligned}$$

From this it follows that $\mathfrak{t}(a+b) = \mathfrak{t}(a)+\mathfrak{t}(b)$ and hence that $\mathfrak{t}$ is an isomorphism from G onto H.

Now suppose φ is a bijection from $\mathfrak{N}_T^*(X, G, \mathfrak{f})$ onto $\mathfrak{N}_T^*(Y, H, \mathfrak{g})$ and that there exists a homeomorphism $\mathfrak{h}$ from X onto Y and an isomorphism $\mathfrak{t}$ from G onto H such that the diagram above commutes. By previous considerations, $\varphi(fg) = \varphi(f)\varphi(g)$ for all f and g in $\mathfrak{N}_T^*(X, G, \mathfrak{f})$. Using the fact that $\mathfrak{t}$ is an isomorphism, we also have

$$\varphi(f+g) = \mathfrak{t} \circ (f+g) \circ \mathfrak{h}^{\leftarrow} = (\mathfrak{t} \circ f \circ \mathfrak{h}^{\leftarrow}) + (\mathfrak{t} \circ g \circ \mathfrak{h}^{\leftarrow}) = \varphi(f)+\varphi(g).$$

Hence, φ is a near-ring isomorphism.

Now let $G_A(\mathfrak{D}_\mathfrak{f})$ denote the group of all automorphisms of G with the property $\mathfrak{t}[A] \in \mathfrak{D}_\mathfrak{f}$ for each A in $\mathfrak{D}_\mathfrak{f}$. Then using Theorem 4.1, the proof of Theorem 3.4 can be modified to yield a proof of the following

THEOREM 4.2. *If* $(X, G, \mathfrak{f})$ *is strongly admissible, the automorphism group of* $\mathfrak{N}_T^*(X, G, \mathfrak{f})$ *is isomorphic to* $G_A(\mathfrak{D}_\mathfrak{f})$.

By taking $X = G$ and $\mathfrak{f}$ to be a homeomorphism, we immediately get

COROLLARY 4.3. *If* $(G, G, \mathfrak{f})$ *is strongly admissible, the automorphism group of the near-ring* $\mathfrak{N}_T^*(G, G, \mathfrak{f})$ *is isomorphic to the automorphism group of the group* G.

EXAMPLE 4.4. Let us take G to be R_A, the additive group of real numbers. Then $\mathfrak{N}_T^*(R_A)$ is the near-ring of all continuous functions mapping R into R where addition is defined pointwise and multiplication is composition. According to Theorem 3.1, (R, R, i) is a strongly admissible triple (where i denotes the identity mapping). Hence, it follows from Corollary 4.3 that the automorphism group of $\mathfrak{N}_T^*(R_A)$ is isomorphic to the group of all (homeomorphic) additive automorphisms on R. But it is well known that any such automorphism $\mathfrak{t}$ is given by

$$\mathfrak{t}(x) = ax \text{ for all } x \text{ in } R \quad (a \neq 0).$$

From this it follows that the automorphism group of $\mathfrak{N}_T^*(R_A)$ is isomorphic to the multiplicative group of all non-zero real numbers.

EXAMPLE 4.5. Again take G to be R_A. Let $X = [0, +\infty)$ and define a mapping $\mathfrak{f}$ from R_A onto X by $\mathfrak{f}(x) = x^2$ for all x in R_A. Then, as we noted in Example 3.7, $(X, R_A, \mathfrak{f})$ is a strongly admissible triple and $\mathfrak{D}_\mathfrak{f} = \{\{x, -x\} : 0 \leqq x\}$. It follows from Theorem 4.2 that the automorphism group of $\mathfrak{N}^*(X, R_A, \mathfrak{f})$ is isomorphic to $G_A(\mathfrak{D}_\mathfrak{f})$. But $G_A(\mathfrak{D}_\mathfrak{f})$ coincides with the automorphism group of R_A which, as we noted previously, is isomorphic to the multiplicative group of non-zero real numbers. Then the near-rings $\mathfrak{N}_T^*(X, R_A, \mathfrak{f})$ and $\mathfrak{N}_T^*(R_A)$ have isomorphic automorphism groups but are not isomorphic themselves since X and R are not homeomorphic.

5. Remarks on some other semigroups of continuous functions

Results analogous to those we have obtained for $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$ can be obtained for other semigroups of continuous functions. Since one may use techniques similar to those used previously in this paper to obtain these other results, we will not prove them but merely discuss them in a somewhat informal manner. Let $\mathfrak{f}$ be a continuous mapping from a topological space Y onto a topological space X and let $\mathfrak{C}_F^*(X, Y, \mathfrak{f})$ denote the collection of all functions f such that $\mathfrak{D}(f)$, the domain of f, is a closed subset of X, the range of f is a subset of Y, and f is continuous on $\mathfrak{D}(f)$. Note that for any two functions f and g of $\mathfrak{C}_F^*(X, Y, \mathfrak{f})$, $\mathfrak{D}(f \circ \mathfrak{f} \circ g) = (\mathfrak{f} \circ g)^{\leftarrow}[\mathfrak{D}(f)]$. Now $\mathfrak{f} \circ g$ is a continuous mapping from $\mathfrak{D}(g)$ into X and since $\mathfrak{D}(f)$ is a closed subset of X, $(\mathfrak{f} \circ g)^{\leftarrow}[\mathfrak{D}(f)]$ is a closed subset of $\mathfrak{D}(g)$. But this implies $(\mathfrak{f} \circ g)^{\leftarrow}[\mathfrak{D}(f)]$ is a closed subset of X since $\mathfrak{D}(g)$ is a closed subset of X. Thus, $f \circ \mathfrak{f} \circ g$ is also an element of $\mathfrak{C}_F^*(X, Y, \mathfrak{f})$. Suppose p and q are distinct points of X. Since X is the image of Y under the mapping $\mathfrak{f}$, Y must also contain two distinct points r and s. Define a function f with $\mathfrak{D}(f) = \{p, q\}$ by $f(p) = r$ and $f(q) = s$. Then $f \in \mathfrak{C}_F^*(X, Y, \mathfrak{f})$ and we see that $\mathfrak{C}_F^*(X, Y, \mathfrak{f})$ is point-separating. Thus $\mathfrak{C}_F^*(X, Y, \mathfrak{f})$ is an $\mathfrak{C}^*$-semigroup.

The notion of an admissible triple was introduced in Section 2 to insure that the mapping $\mathfrak{h}$ from X onto Y determined by an isomorphism φ from $\mathfrak{C}_T^*(X, Y, \mathfrak{f})$ onto $\mathfrak{C}_T^*(U, V, \mathfrak{g})$ be a homeomorphism. For semigroups of the form $\mathfrak{C}_F^*(X, Y, \mathfrak{f})$, no such restriction need be placed on the triples to insure that $\mathfrak{h}$ be a homeomorphism. This is due to the fact that $\mathfrak{h}[\mathfrak{D}(f)] = \mathfrak{D}(\varphi(f))$ for every f in $\mathfrak{C}^*_F(X, Y, \mathfrak{f})$ and every closed subset of X is the domain of a function in $\mathfrak{C}^*_F(X, Y, \mathfrak{f})$. Just as in the case of $\mathfrak{C}^*_T$-semigroups, however, the mapping $\mathfrak{t}$ from Y onto V determined by φ need not be a homeomorphism. Example 2.5 serves to establish this fact. Thus, further restrictions must be placed on the triples in order that the mapping $\mathfrak{t}$ be a homeomorphism.

We conclude by mentioning that results analogous to those we have obtained for $\mathfrak{C}^*_T(X, Y, \mathfrak{f})$ can also be obtained for $\mathfrak{C}^*_G(X, Y, \mathfrak{f})$, the semigroup of all continuous functions whose domains are open subsets of X and whose ranges are contained in Y.

References

[1] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Mathematical Surveys, Number 7, Amer. Math. Soc. (1961).

[2] J. de Groot, 'Groups represented by homeomorphism groups I', *Math. Annelen* 138 (1959), 80–102.

[3] J. L. Kelley, *General topology*, Van Nostrand, Princeton, 1955.

[4] E. S. Ljapin, *Semigroups*, Translations of Mathematical Monographs, Vol. 3, Amer. Math. Soc., 1963.

[5] K. D. Magill, Jr., 'Semigroup structures for families of functions, I. Some homomorphism theorems', *Journ. Australian Math. Soc.* (to appear).

[6] G. Springer, *Introduction to Riemann Surfaces*, Addison-Wesley, Reading, Mass., 1957.

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