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INDICIAL EQUIVALENTS OF MULTIPARAMETER DEFINITENESS CONDITIONS IN FINITE DIMENSIONS

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1. Introduction

Let T_m, V_{mn} be Hermitean linear operators on complex Hilbert spaces $H_m, m=1\dots k$. A nonzero column vector $\lambda=[\lambda_0\dots\lambda_k]^T\in\mathbb{R}^{k+1}$ satisfying

$$W_m(\lambda)x_m=0\neq x_m\in H_m, \tag{1.1}$$

where

$$W_m(\lambda)=\lambda_0T_m+\lambda_1V_{m1}+\dots+\lambda_kV_{mk}$$

will be called an *eigenvalue*. This type of problem has been studied extensively by Atkinson [2] from the viewpoint of determinantal operators on the tensor product $H^{\otimes}=\otimes_{m=1}^{k}H_m$. We shall connect his work with more recent investigations [5, 7] of eigenvalue indices based on minimax principles for $W(\lambda)=(W_1(\lambda),\dots,W_k(\lambda))$, which can be viewed as an operator on $H^{\times}=\times_{m=1}^{k}H_m$.

We should perhaps point out that the distinction between the T_m and the V_{mn} reflects the relative simplicity of the V_{mn} in certain differential equation problems, with corresponding implications for finite dimensional (e.g. difference equation) approximations. Indeed Atkinson's results in $H^{\otimes}$ have their roots in multiple Fourier expansion theorems for linked Sturm–Liouville problems, going back to Hilbert. Likewise the indicial results in $H^{\times}$ are developments of corresponding oscillation theorems dating back to Klein and Bôcher. Incidentally, an oscillation theory for linked difference equations can be found in [1; Chapter 6].

From now on, we shall assume that

$$d_m=\dim H_m \tag{1.2}$$

is finite. We shall discuss (1.1) under certain "definiteness conditions" which have received much recent attention. Our aim will be to relate the $H^{\otimes}$ and $H^{\times}$ theories, together with certain geometrical equivalents of these definiteness conditions [2; Chapter 9], [3]. In particular, we shall demonstrate *equivalence* of some of the definiteness conditions with certain indicial properties of the eigenvalues λ of (1.1).

The simplest case is that of right definiteness (RD) which requires

$$0\neq\delta_0(x):=\det[v_{mn}(x)]_{1\leq m,n\leq k}\quad\text{whenever each}\quad x_m\neq 0,$$

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where $x=(x_1,\dots,x_k)$ and $v_{mn}(x)=(x_m, V_{mn}x_m)$. δ_0 is clearly an $H^\times$ construction, and we label the corresponding definiteness condition $A_0^\pm$. The corresponding $H^\otimes$ construction involves the determinantal operator

$$\Delta_0 := \otimes \det [V_{mn}]_{1 \leqq m,n \leqq k}, \tag{1.3}$$

and $A_0^\pm$ is easily seen to be equivalent to definiteness of Δ_0 on decomposable tensors of $H^\otimes$. The following fundamental result is due to Atkinson.

Theorem 1.1. [**2**; Theorem 7.8.2 and Section 8.6] *$A_0^\pm$ is equivalent to definiteness of Δ_0 on $H^\otimes$.*

Remark. In [**3**] these conditions were labelled RD_δ and RD_Δ, respectively. The notation here is chosen for unification with other conditions. Also, since we wish to discuss the logical status of Theorem 1.1, we shall retain the distinction between the two forms of the condition.

Major results of the $H^\otimes$ and $H^\times$ theories under RD are as follows.

Theorem 1.2. [**2**; Theorem 7.6.2] *If Δ_0 is definite on $H^\otimes$ then for each $\boldsymbol{T}=(T_1,\dots,T_k)$ the following property of* (1.1) *obtains:*

($a_0^\pm$) *there exists a complete orthonormal basis of $H_0^\otimes$ consisting of decomposable tensors $x^\otimes = x_1 \otimes \dots \otimes x_k$ where the x_m are eigenvectors satisfying* (1.1) *with $\lambda_0 \neq 0$, and $H_0^\otimes$ is $H^\otimes$ endowed with the inner product $[x,y]_0=(x,|\Delta_0|y)$, where $(,)$ is the Hilbert space inner product and $|\Delta_0|$ is the positive square root of Δ_0^2.*

We remark that Atkinson's proofs (the only ones to date) of Theorem 1.1 depend on the above result.

Theorem 1.3. [**6**; Theorem 2] *$A_0^\pm$ implies the following property of* (1.1), *for each* $\boldsymbol{T}$:

($\hat{b}_0$) *for each $\boldsymbol{i}=(i_1,\dots,i_k)$ where the i_m are nonnegative integers, there exists an eigenvalue $\boldsymbol{\lambda}=\boldsymbol{\lambda}^i$ satisfying* (1.1) *with $\lambda_0 \neq 0$ and*

$$\rho_m^{i_m}(\boldsymbol{\lambda})=0, \quad m=1\dots k, \tag{1.4}$$

where

$$\rho_m^1(\boldsymbol{\lambda}) \leqq \dots \leqq \rho_m^{d_m}(\boldsymbol{\lambda})$$

denote the eigenvalues of $W_m(\boldsymbol{\lambda})$, repeated according to multiplicity.

We shall say that any $\boldsymbol{\lambda}$ satisfying (1.1) and (1.4) has *index* $\boldsymbol{i}$.

In Section 2, we shall show, roughly, that $\hat{b}_0$ is central to the RD theory, $\hat{b}_0$ for all $\boldsymbol{T}$ being equivalent to $A_0^\pm$. We shall also show that $\hat{b}_0$ can be replaced by a "stable" version b_{0+}^s where for each $\boldsymbol{i}$, the $\boldsymbol{\lambda}^i$, normalised to $\lambda_0^i=1$, are unique and continuously dependent on the T_m and V_{mn}. As a consequence we shall deduce $a_0^\pm$ from $A_0^\pm$ using a minimum of tensor product machinery. In doing so, we shall reprove Theorem 1.1 via b_{0+}^s, thereby shortening the existing proofs.

As regards equivalence with $A_0^{\pm}$, we shall use the following result, which follows directly from [**4**; Lemma 2.3(i) and Corollary 4.4].

Theorem 1.4. *$A_0^{\pm}$ is equivalent to the following condition on the V_{mn}:*

(c_0) $\mathbf{0} \notin R_\sigma$ for each $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_k) \in \mathbb{R}^k$ such that $|\sigma_m| = 1$, $m = 1 \ldots k$.

Here R_σ, which was labelled V_σ in [**4**], is defined by

$$R_\sigma = \mathrm{co} \bigcup_{m=1}^{k} \sigma_m R_m, \quad R_m = \mathrm{co}\,\{\mathbf{v}_m(\dot{x}_m) : 0 \neq x_m \in H_m\} \tag{1.5}$$

where co denotes convex hull and

$$\mathbf{v}_m(x_m) = (v_{m1}(x_m), \ldots, v_{mk}(x_m)) \in \mathbb{R}^k.$$

We shall establish the converse of Theorem 1.3 by identifying $\boldsymbol{T}(\boldsymbol{\sigma})$ and $\boldsymbol{i}(\boldsymbol{\sigma})$ such that no eigenvalue of index $\boldsymbol{i}(\boldsymbol{\sigma})$ exists if $\mathbf{0} \in R_\sigma$.

The remaining sections are devoted to further definiteness conditions which, in general, involve both $\boldsymbol{T}$ and the V_{mn}. Atkinson [**2**; Chapters 7–10] has analysed (1.1) under three such conditions, the second and third being equivalent, essentially because of Theorem 1.1 [**2**; Theorem 7.8.2]. Consequences of the first condition [**2**; p. 117], which we label A, will be studied in Section 3. We shall use analogues of Theorems 1.1 and 1.2 to prove an analogue of Theorem 1.3. As far as the author is aware, there are no previous indicial results under condition A. The key changes from $\hat{b}_0$ are (i) λ_0 may vanish, although some eigenvalue-independent linear combination $\boldsymbol{\mu\lambda}$ will not do so (ii) $\delta_0(x)$ may vanish, and even if $\boldsymbol{\mu\lambda} = \lambda_0$ (in which case we denote A by A_0) then $\delta_0(x)$ may still be indefinite. We can circumnavigate (i) by a preliminary rotation of eigenvalue axes in $\mathbb{R}^{k+1}$ so as to ensure A_0, at the cost of destroying the special status of $\boldsymbol{T}$. As regards (ii), we may attach signs to the eigenvalues either by $\operatorname{sgn} \delta_0(x)$ as in [**5**] or by $\sigma = \operatorname{sgn}(\lambda_0 \delta_0(x))$, assuming A_0. These two methods differ in general, and it turns out that σ has certain advantages. In particular, A_0 implies

(b_0) for each signed index $(\boldsymbol{i}, \sigma)$, an eigenvalue $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(i,\sigma)}$ exists with $\lambda_0 \neq 0$.

This will be shown in Corollary 3.7, where we shall also deduce uniqueness and continuous dependence on the data T_m and V_{mn} of each eigenvalue $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(i,\sigma)}$, normalised to $|\lambda_0| = 1$.

In Section 4, we prove equivalence of A_0, b_0 and a "stable" version $\hat{b}_0^s$ of $\hat{b}_0$. We also give indicial equivalents for A, and for Atkinson's second condition A^+ where change (i) above occurs, but (ii) does not. It turns out that one can retain the special status of $\boldsymbol{T}$ at the indicial level, and that the difference between A and A^+ reduces to the difference between the methods described above for attaching signs to the eigenvalues. One consequence of this analysis is a new set of equivalents for RD, in some ways more useful than those of Section 2. Finally we discuss left definiteness which is a combination of conditions on the cofactors of δ_0 and "definiteness" of $\boldsymbol{T}$[**3**; p. 321], [**8**; pp. 62–3]. We shall relate the cofactor condition to A^+ for all "definite" $\boldsymbol{T}$, and then deduce indicial equivalents from the theory for A^+.

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Remark. Equivalents of weaker conditions on the R_σ (1.5), involving the "fundamental" index $\boldsymbol{i}=(1,\dots,1)$ for all $\boldsymbol{T}$ satisfying various definiteness conditions, can be found for $k=2$ in [7], and they remain valid in infinite dimensions.

2. Right definiteness

In the first part of this section, we discuss some indicial equivalents for RD, beginning with a sufficiency property.

Lemma 2.1. *If $\mathbf{0}\in\mathbf{R}_\sigma$ then there are $\boldsymbol{T}=\boldsymbol{T}(\boldsymbol{\sigma})$ and $\boldsymbol{i}=\boldsymbol{i}(\boldsymbol{\sigma})$ so that no $\boldsymbol{\lambda}=\boldsymbol{\lambda}^i$ exists as in $\hat{b}_0$, i.e.* (1.1), $\lambda_0\neq 0$ *and* (1.4) *are incompatible.*

Proof. If $\mathbf{0}\in R_\sigma$ then there exist nonnegative a_{n_m} and nonzero $x_{n_m}\in H_m$, such that

$$\sum_{m,n_m=1}^{k}\alpha_{n_m}\sigma_m\mathbf{v}_m(x_{n_m})=\mathbf{0},\ \sum_{n_m=1}^{k}\alpha_{n_m}>0$$

in the notation of (1.5). Assuming $\lambda_0\neq 0$ and choosing $\boldsymbol{T}=\boldsymbol{T}(\boldsymbol{\sigma})$ so that each $\lambda_0\sigma_m T_m$ is negative definite, we deduce

$$\sum_{m,n_m}\alpha_{n_m}(x_{n_m},\sigma_m W_m(\boldsymbol{\lambda})x_{n_m})=\sum_{m,n_m}\alpha_{n_m}\sigma_m\lambda_0 t_m(x_{n_m})<0 \tag{2.1}$$

where we write

$$t_m(x_m)=(x_m,T_m x_m).$$

Now choose $\boldsymbol{i}=\boldsymbol{i}(\boldsymbol{\sigma})$ so that $i_m=1$ if $\sigma_m=1$ and $i_m=d_m$ (1.2) if $\sigma_m=-1$. If $\boldsymbol{\lambda}=\boldsymbol{\lambda}^i$ exists as in $\hat{b}_0$ then (1.4) forces each $\sigma_m W_m(\boldsymbol{\lambda})$ to be nonnegative definite, and this contradicts (2.1).

We shall improve Theorem 1.3 by means of the following general perturbation result, which requires no definiteness conditions.

Lemma 2.2. *The set of eigenvalues $\boldsymbol{\lambda}^i$ of index $\boldsymbol{i}$ depends upper semi-continuously on the data T_m and V_{mn}.*

Remark. We use the Euclidean $\mathbb{C}^{k(k+1)}$ topology for the data.

Proof. The quadratic forms t_m and v_{mn} depend continuously on the data. By the minimax principle, the same goes for the eigenvalues $\rho_m^j(\boldsymbol{\lambda})$ (1.4) for each $\boldsymbol{\lambda}$. Thus the ρ_m^j functions have closed graphs, and the conclusion is now immediate.

We are now ready for the first indicial equivalences. We write b_{0+}^s to mean existence, uniqueness and continuous dependence on the data of each $\boldsymbol{\lambda}^i$, normalised to $\lambda_0^i=1$.

Theorem 2.3 *The following are equivalent*: (i) $A_0^\pm$, (ii) $\hat{b}_0$ *for all* $\boldsymbol{T}$, (iii) b_{0+}^s *for all* $\boldsymbol{T}$.

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Proof. Since (iii)$\Rightarrow$(ii) is trivial, and Lemma 2.1 proves (ii)$\Rightarrow$(i), it suffices to establish (i)$\Rightarrow$(iii). Evidently $A_0^{\pm}$ is equivalent to $|\delta_0(u)|$ being positively bounded below over $\|u_m\|=1$, $m=1\ldots k$, so continuity of δ_0 in the data implies that $A_0^{\pm}$ persists under small perturbations.

Let $T_m(\varepsilon)\rightarrow T_m$ as $\varepsilon\rightarrow 0$, $\varepsilon\in\mathbb{R}$, and similarly for $V_{mn}(\varepsilon)$. Now the proof of Theorem 1.3 actually gives uniqueness of eigenvalues of fixed index, normalised to $\lambda_0=1$. By persistence of $A_0^{\pm}$, there exist $\boldsymbol{\lambda}^i(\varepsilon)$ for all suitably small ε, with $\lambda_0^i(\varepsilon)=1$. Let $\boldsymbol{\lambda}^i$ be any limit point of $\boldsymbol{\lambda}^i(\varepsilon)$ as $\varepsilon\rightarrow 0$—necessarily $\lambda_0^i=1$. Lemma 2.2 then yields uniqueness of the limit point $\boldsymbol{\lambda}^i$, whence continuity of $\boldsymbol{\lambda}^i(\varepsilon)$ at $\varepsilon=0$.

The remainder of this section is devoted to consequences for $H^{\otimes}$ of the indicial theory. The first step is to show that b_{0+}^s implies an $H^{\times}$ version of $a_0^{\pm}$.

Theorem 2.4. *If b_{0+}^s holds then there exist $x_m=x_m^i$ satisfying* (1.1) *with $\boldsymbol{\lambda}=\boldsymbol{\lambda}^i$, $\lambda_0=1$, and such that $\det V^{ij}=0$ if $i\neq j$, where V^{ij} is the $k\times k$ matrix with (m,n)th entry $(x_m^i, V_{mn}x_m^j)$.*

Proof. If $\boldsymbol{\lambda}=\boldsymbol{\lambda}^i$ corresponds to x_m^i then (1.1) yields

$$(W_m(\boldsymbol{\lambda}^i)x_m^i, x_m^j)=0=(x_m^i, W_m(\boldsymbol{\lambda}^j)x_m^j)$$

whence

$$V^{ij}(\boldsymbol{\lambda}^i-\boldsymbol{\lambda}^j)=0 \tag{2.2}$$

so the conclusion is immediate if the $\boldsymbol{\lambda}^i$ are all distinct.

In the case of multiple eigenvalues, we follow Atkinson [**2**; Section 7.5] and perturb T_m to $T_m(\varepsilon)$ so that (i) $T_m(\varepsilon)\rightarrow T_m$ as $\varepsilon\rightarrow 0$ and (ii) all the eigenvalues of (1.1) are distinct if T_m is replaced by $T_m(\varepsilon)$. With $x_m^i(\varepsilon)$ as corresponding unit eigenvectors, we choose any limit points as $\varepsilon\rightarrow 0$ for the x_m^i. By b_{0+}^s, the x_m^i do satisfy (1.1) with $\boldsymbol{\lambda}=\boldsymbol{\lambda}^i$. From the perturbed version of (2.2) we have $\det V^{ij}(\varepsilon)=0$ in an obvious notation, and we now let $\varepsilon\rightarrow 0$.

Remark. Although Atkinson's arguments depend explicitly on definiteness of Δ_0 on $H^{\otimes}$, an inspection, together with the above use of indexed eigenvalues, shows that b_{0+}^s in fact suffices.

We are now ready to draw the principal conclusions for $H^{\times}$. The following simple observation will be used at various points in the sequel.

Remark 2.5. The number of indices $\boldsymbol{i}$ equals the dimension of $H^{\otimes}$.

Corollary 2.6. *Theorem* 1.1 *holds.*

Proof. Setting $x^{i\otimes}=x_1^i\otimes\ldots\otimes x_k^i$, we deduce

$$(x^{i\otimes}, \Delta_0 x^{j\otimes})=0 \quad \text{if} \quad \boldsymbol{i}\neq\boldsymbol{j} \tag{2.3}$$

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from Theorems 2.3 and 2.4. The result now follows immediately from Remark 2.5 and the expansion

$$\left(\sum_i \alpha^i x^{i\otimes}, \Delta_0 \sum_j \alpha^j x^{j\otimes}\right) = \sum_i \delta_0(x^i)|\alpha^i|^2$$

—see [**2**; Lemma 7.8.1].

Corollary 2.7. *$A_0^{\pm}$ implies $a_0^{\pm}$.*

Proof. By the previous result, we may assume definiteness of Δ_0, so $H_0^{\otimes}$ is constructible. Since the $x^{i\otimes}$ may be scaled to unit norm in $H_0^{\otimes}$ by $|\delta_0(x)| = 1$, the result follows from (2.3) and Remark 2.5.

Remark. By utilising only [**2**; Lemma 7.8.1] for Corollary 2.6, we have shortened the proof of [**2**; Theorem 7.8.2]. Similarly the proof of Corollary 2.7 is more direct than that of [**2**; Theorem 7.6.2], and can be viewed as a generalisation of the procedure outlined on [**2**; p. 135] for a special case.

3. Some consequences of Condition A

Atkinson's "Definiteness Condition I" [**2**; p. 117], which we label A here, is defined as rank $W(x) = k$ whenever each $x_m \neq 0$, where $W(x)$ is the $k \times (k+1)$ matrix with mth row

$$\boldsymbol{w}_m(x_m) = [t_m(x_m) v_{m1}(x_m) \dots v_{mk}(x_m)]. \tag{3.1}$$

Closely associated with this condition is the determinant

$$\delta(\boldsymbol{\mu}, x) = \det \begin{bmatrix} \boldsymbol{\mu} \\ W(x) \end{bmatrix}$$

where $\boldsymbol{\mu} = [\mu_0 \dots \mu_k] \in \mathbb{R}^{k+1}$. The analogous $H^{\otimes}$ construction is

$$\Delta(\boldsymbol{\mu}) = \otimes \det \begin{bmatrix} \boldsymbol{\mu} \\ W \end{bmatrix}$$

where W has mth row $[T_m V_{m1} \dots V_{mk}]$. Evidently

$$\delta(\boldsymbol{\mu}, x) = (x^{\otimes}, \Delta(\boldsymbol{\mu}) x^{\otimes})$$

and it is easily seen that A forces positivity of this expression for each nonzero x_m, for some $\boldsymbol{\mu}$ which in general depends on the x_m.

We start with a similar result from the $H^{\otimes}$ theory, but where $\boldsymbol{\mu}$ is fixed.

Theorem 3.1. [**2**; Theorem 10.4.1] *A implies nonsingularity of $\Delta(\boldsymbol{\mu})$ for some $\boldsymbol{\mu}$.*

To make a closer parallel with Theorem 1.1, we use $\boldsymbol{e}_0, \ldots, \boldsymbol{e}_k$ to denote the coordinate basis of $\mathbb{R}^{k+1}$, and we write $\Delta_n = \Delta(\boldsymbol{e}_n)$, in agreement with (1.3). Condition A_0 will mean A and nonsingularity of Δ_0.

Corollary 3.2. *A implies A_0 after a nonsingular linear eigenvalue transformation.*

Indeed it suffices to rotate axes in $\mathbb{R}^{k+1}$ so that $\boldsymbol{\mu}$ in Theorem 3.1 becomes $\boldsymbol{e}_0$.

Two remarks are pertinent here. First, definiteness of Δ_0 (or $\Delta(\boldsymbol{\mu})$ in Theorem 3.1) cannot be guaranteed, at least when $k > 2$ [**2**; Section 9.9]. Second, the transformation of Corollary 3.2 may destroy the special status of $\boldsymbol{T}$. We shall return to these points in Section 4.

Our present purposes will be served by the following analogue of Theorem 1.2, where a_0 means $a_0^{\pm}$ except that the inner product $(x, y)_0 = (x, \Delta_0 y)$ may be indefinite, so the eigenvector basis is only $H_0^{\otimes}$ – orthogonal and not normalised to $|(x^{\otimes}, x^{\otimes})_0| = 1$. (Actually such normalisation *is* possible under A_0, as will be seen later.) We write a if the eigenvalues satisfy $\boldsymbol{\mu\lambda} \neq 0$ where $\boldsymbol{\mu}$ is eigenvalue-independent and the (perhaps indefinite) inner product in $H^{\otimes}$ is induced by $\Delta(\boldsymbol{\mu})$ instead of Δ_0.

Theorem 3.3. *A_0 implies a_0 and A implies a.*

Proof. By [**2**; Theorem 10.6.1], an orthogonal basis exists as required, assuming A, and it remains to prove that the eigenvalues $\boldsymbol{\lambda}$ satisfy $\boldsymbol{\mu\lambda} \neq 0$. Now [**2**; Theorem 6.4.2] gives

$$\lambda_0 \Delta_n x^{\otimes} = \lambda_n \Delta_0 x^{\otimes}, \quad n = 1 \ldots k \tag{3.2}$$

whenever (1.1) is satisfied. Then A_0 and $\boldsymbol{\lambda} \neq \boldsymbol{0}$ force $\lambda_0 \neq 0$. This establishes the first contention, and the second follows from

$$(\boldsymbol{\mu\lambda}) \Delta_n x^{\otimes} = \lambda_n \Delta(\boldsymbol{\mu}) x^{\otimes}, \quad n = 0 \ldots k.$$

For the indicial theory in $H^{\otimes}$, it turns out to be convenient to normalise λ_0 (or $\boldsymbol{\mu\lambda}$) to ± 1, and to specify the sign via the following construction. We define

$$\delta_n(x) = \delta(\boldsymbol{e}_n, x) = (x^{\otimes}, \Delta_n x^{\otimes})$$

and we set

$$\boldsymbol{\delta}(x) = [\delta_0(x) \ldots \delta_k(x)]^T.$$

In particular, $\delta(\boldsymbol{\mu}, x) = \boldsymbol{\mu\delta}(x)$.

Lemma 3.4. *A is equivalent to $\boldsymbol{\delta}(x) \neq \boldsymbol{0}$ whenever each $x_m \neq 0$. If in addition $\boldsymbol{\lambda} \in \mathbb{R}^{k+1}$ satisfies $W(x)\boldsymbol{\lambda} = \boldsymbol{0} \neq \boldsymbol{\lambda}$ then $\boldsymbol{\lambda} = \alpha\boldsymbol{\delta}(x)$ for some $\alpha \in \mathbb{R}$, and the sign of α depends only on $\boldsymbol{\lambda}$, i.e. not on x.*

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Proof. The first contention follows from the definition of determinant rank. For the second, A implies that the $\boldsymbol{w}_m(x_m)$ (3.1) span a k-dimensional subspace $S \subset \mathbb{R}^{k+1}$. Thus $\boldsymbol{\lambda} \in S^{\perp}$, and it suffices to prove $\boldsymbol{\delta}(x) \in S^{\perp}$. Now

$$\boldsymbol{w}_m(x)\boldsymbol{\delta}(x) = \delta(\boldsymbol{w}_m(x), x) = 0$$

so indeed $W(x)\boldsymbol{\delta}(x) = \mathbf{0}$.

For the final contention, we note that $W(x)\boldsymbol{\lambda} = \mathbf{0}$ if and only if

$$(x_m, W_m(\boldsymbol{\lambda})x_m) = 0, \quad m = 1 \ldots k. \tag{3.3}$$

The set of nonzero x_m satisfying (3.3) for each fixed m is arcwise connected [**2**; Theorem 2.7.1]. Suppose $\boldsymbol{\lambda} = \alpha\boldsymbol{\delta}(x) = \beta\boldsymbol{\delta}(y) \neq \mathbf{0}$ — say $\lambda_n \neq 0$. Continuity of $\boldsymbol{\delta}$ along an arc joining x to y and the hypothesis $\alpha\beta \leqq 0$ force the contradiction $\lambda_n = 0$. Thus $\alpha\beta > 0$.

This enables us to attach the sign of α to any $\boldsymbol{\lambda} \in \mathbb{R}^{k+1}$ satisfying the conditions of Lemma 3.4. In particular, any eigenvalue satisfies those conditions under A (cf. [**2**; p. 174]) and we refer to $(\boldsymbol{i}, \sigma)$ as the *signed index* of $\boldsymbol{\lambda}$ if (1.4) holds and $\sigma = \operatorname{sgn}\alpha$. If A_0 holds then

$$\lambda_0 = \alpha\delta_0(x)$$

for any x_m satisfying (1.1), so since $\lambda_0 \neq 0$ by Theorem 3.3, we have

$$\sigma = \operatorname{sgn}(\lambda_0\delta_0(x)). \tag{3.4}$$

Similarly under A the sign is given by the formula

$$\sigma = \operatorname{sgn}[(\boldsymbol{\mu\lambda})(\boldsymbol{\mu\delta}(x))] \tag{3.5}$$

for fixed $\boldsymbol{\mu}$ whose existence is guaranteed by Theorem 3.1.

Our analogue of Theorem 1.3 for A will depend on the following uniqueness property.

Theorem 3.5. *Assuming A_0 [A], at most one eigenvalue $\boldsymbol{\lambda}$ of a given signed index, normalised to $|\lambda_0| = 1$ [$|\boldsymbol{\mu\lambda}| = 1$] can exist.*

Proof. We shall assume A_0—the proof under A is analogous. Suppose $\boldsymbol{\lambda}$ and $\boldsymbol{v}$ both have signed index $(\boldsymbol{i}, \sigma)$ and $|\lambda_0| = |v_0| = 1$. Let U_m denote the unit sphere of H_m and set $U = \times_{m=1}^{k} U_m$. Let y_m minimise $\boldsymbol{w}_m(u_m)\boldsymbol{\lambda} = (u_m, W_m(\boldsymbol{\lambda})u_m)$ over $u_m \in U_m$ and orthogonal to the first $i_m - 1$ eigenvectors of $W_m(\boldsymbol{v})$, i.e. those corresponding to $\rho_m^1(\boldsymbol{v}), \ldots, \rho_m^{i_m-1}(\boldsymbol{v})$.

The minimax principle yields

$$W(y)\boldsymbol{\lambda} \leqq \mathbf{0} \leqq W(y)\boldsymbol{v}$$

coordinatewise—cf. [**6**; p. 1057]. Writing $\boldsymbol{\lambda}^{\beta}$ for $(1-\beta)\boldsymbol{\lambda} - \beta\boldsymbol{v}$ we deduce

$$W(y)\boldsymbol{\lambda}^{\beta} \leqq \mathbf{0} \quad \text{whenever} \quad 0 \leqq \beta \leqq 1. \tag{3.6}$$

Similarly one may find $z \in U$ to reverse these inequalities, so

$$W(z)\boldsymbol{\lambda}^{\beta} \geqq \mathbf{0} \quad \text{whenever} \quad 0 \leqq \beta \leqq 1. \tag{3.7}$$

Define $S(\beta) = \times_{m=1}^{k} S_m(\beta) \subset U$ by the condition

$$u \in S(\beta) \Leftrightarrow W(u)\boldsymbol{\lambda}^{\beta} = \mathbf{0}. \tag{3.8}$$

Convexity of the numerical ranges of $W_m(\boldsymbol{\lambda}^{\beta})$, $m = 1 \dots k$, together with (3.6) and (3.7), guarantee $S(\beta) \neq \emptyset$ for each $\beta \in [0,1]$. If $\boldsymbol{\lambda}^{\beta} = \mathbf{0}$ then $\boldsymbol{\mu}$ and $\boldsymbol{v}$ are positively proportional, and the normalisation $|\lambda_0| = |v_0| = 1$ forces $\boldsymbol{\lambda} = \boldsymbol{v}$ as required.

To complete the proof it will therefore suffice to contradict the assumption $\boldsymbol{\lambda}^{\beta} \neq \mathbf{0}$ for each $\beta \in [0,1]$. By Lemma 3.4 and (3.8), we may attach a well defined sign $\sigma(\beta)$ to each $\boldsymbol{\lambda}^{\beta}$. When $\beta = 0$, we use the fact that $\boldsymbol{\lambda}$ has signed index $(\boldsymbol{i}, \sigma)$ to deduce $\sigma(0) = \sigma$, and similarly we obtain $\sigma(1) = -\sigma$ from the sign of $\boldsymbol{v}$. Define

$$\beta' = \inf\{\beta : \sigma(\beta) = -\sigma\}.$$

If $\beta' > 0$, then we may choose $\beta^j \uparrow \beta'$ and $u^j \in S(\beta^j)$ (3.8) so that $u_m^j \to u_m$ as $j \to \infty$. Evidently $\boldsymbol{\lambda}^{\beta^j} \to \boldsymbol{\lambda}^{\beta}$ and $\boldsymbol{\delta}(u^j) \to \boldsymbol{\delta}(u)$ as $j \to \infty$. Thus if $\lambda_l^{\beta'} \neq 0$ then $\sigma(\beta') = \operatorname{sgn}(\lambda_l^{\beta'} \delta_0(u)) = \operatorname{sgn}(\lambda_l^{\beta^j} \delta_0(u^j)) = \sigma(\beta^j)$ for large j. It follows that $\sigma(\beta') = \sigma$, so $\beta' = 1$ contradicts $\sigma(1) = -\sigma$. If $0 \leqq \beta' < 1$ then we may use a sequence $\gamma^j \downarrow \beta'$ satisfying $\sigma(\gamma^j) = -\sigma$ to derive the contradiction $\sigma(\beta') = -\sigma$.

It is now straightforward to obtain our indicial consequence of A.

Corollary 3.6. *A_0 $[A]$ implies the existence, uniqueness and continuous dependence on the data T_m and V_{mn} of an eigenvalue $\boldsymbol{\lambda}$ for any signed index, normalised to $|\lambda_0| = 1$ $[|\boldsymbol{\mu\lambda}| = 1$ for fixed $\boldsymbol{\mu}]$.*

Proof. By Theorem 3.3 there are $\dim H^{\otimes}$ eigenvalues, repeated according to multiplicity. Again we shall assume A_0, so each eigenvalue $\boldsymbol{\lambda}$ may be normalised to $\lambda_0 = 1$ in the cited result, and each possesses a signed index by virtue of Lemma 3.4. Moreover the transformation $\boldsymbol{\lambda} \to -\boldsymbol{\lambda}$ changes the signed index $(\boldsymbol{i}, \sigma)$ to $(\boldsymbol{d} + \mathbf{1} - \boldsymbol{i}, -\sigma)$, where the d_m satisfy (1.2) and $\mathbf{1} = (1, 1, \dots, 1)$. Thus $2\dim H^{\otimes}$ signed indices can be derived this way from eigenvalues, and by Remark 2.5 there are precisely $2\dim H^{\otimes}$ signed indices. Existence and uniqueness now follow from Theorem 3.5.

Next we note from Lemma 3.4 that A is equivalent to $\|\boldsymbol{\delta}(u)\|$ being positively bounded below on $\|u_m\| = 1$, $m = 1 \dots k$. This condition obviously persists under small perturbations of the data, and the proof of continuous dependence of eigenvalues now follows essentially as for Theorem 2.3.

4. Indicial equivalents

It turns out that the implication of Corollary 3.6 is reversible. Moreover the full force of existence, uniqueness and continuous dependence of the $\boldsymbol{\lambda}^{(\boldsymbol{i},\sigma)}$ in not needed, and we shall explore some of the possibilities in the first part of this section. Let us begin by showing that existence alone, of eigenvalues $\boldsymbol{\lambda}$ of all possible indices $\boldsymbol{i}$ (Theorem 1.3 ff.) and admitting all possible signs of $\lambda_0\delta_0$, is not enough for A.

Example 4.1. Let $k=1$, $H_1=\mathbb{C}^2$, $T_1=0$ and $V_{11}(x^1,x^2)=(x^1,-x^2)$ for $x=(x^1,x^2)\in\mathbb{C}^2$.

Evidently $\boldsymbol{w}_1(1,1)=[0\,0]$ so A fails. On the other hand let $\boldsymbol{\lambda}=[1\,0]^T$. Then $W_1(\boldsymbol{\lambda})=0$ so $\boldsymbol{\lambda}$ is an eigenvalue for (1.1) with eigenspace $\mathbb{C}^2$. Moreover $\delta_0=v_{11}$ so $\delta_0(1,0)>0>\delta_0(0,1)$ and therefore all four possible combinations of index and $\operatorname{sgn}\lambda_0\delta_0$ are attained by $\boldsymbol{\lambda}$. Note that $\boldsymbol{\lambda}$ does *not* have a sign in the sense of (3.4).

It is time to detail some of the indicial conditions we shall need. By $\hat{b}$ we mean existence, for each $\boldsymbol{i}$, of an eigenvalue $\boldsymbol{\lambda}=\boldsymbol{\lambda}^i$ of index $\boldsymbol{i}$. If $\lambda_0\neq 0$ can be ensured (for some $\boldsymbol{\lambda}$ for each $\boldsymbol{i}$) then as before we write $\hat{b}_0$. By Example 4.1, $\hat{b}_0$ does not imply A, although $\hat{b}_0$ for all $\boldsymbol{T}$ implies $A_0^{\pm}$ by Theorem 2.3.

Two strenthenings of $\hat{b}$ that do imply A are now considered.

Firstly, we assume that there exists $\boldsymbol{\mu}\in\mathbb{R}^{k+1}$ such that $\boldsymbol{\mu\delta}(x)$ is non-zero for all eigenvectors x_m satisfying (1.1). Then, for the eigenvalue $\boldsymbol{\lambda}=\boldsymbol{\lambda}^i$, we can define a sign by (3.5). Indeed, elementary manipulation yields

$$\lambda_m^i\delta_n(x)=\lambda_n^i\delta_m(x)$$

—cf. [**2**, (6.8.7)]. Since $\boldsymbol{\mu\delta}(x)\neq 0$ and $\boldsymbol{\lambda}^i\neq 0$ it easily follows that $\boldsymbol{\mu\lambda}^i\neq 0$. We write b if, for each $\boldsymbol{i}$, at least one such signed eigenvalue $\boldsymbol{\lambda}^i$ exists. The sign may in principle depend on $\boldsymbol{\lambda}^i$, although in practice Corollary 3.6 limits the possibilities.

Secondly, we write $\hat{b}^s$ if the set of eigenvalues λ^i of index $\boldsymbol{i}$ is nonempty and depends continuously on the data T_m and V_{mn}. By Lemma 2.2, this is equivalent to lower semi-continuity, i.e. if (1.1) is embedded in a perturbation family then each $\boldsymbol{\lambda}^i$ is a limit point of perturbed eigenvalues of index $\boldsymbol{i}$. The various strenthenings of $\hat{b}$ may be applied simultaneously—e.g. b_0 means that signed eigenvalues $\boldsymbol{\lambda}^i$ exist satisfying $\lambda_0^i\neq 0$, and b^s means that the set of such eigenvalues λ^i as in b is nonempty and depends continuously on the data.

Our central result for A is as follows.

Theorem 4.2. *A, b, $\hat{b}^s$ and b^s are equivalent.*

Proof. By Corollary 3.6 and the trivial implications $b^s\Rightarrow b$ and $b^s\Rightarrow\hat{b}^s$, we need prove only (i) $\hat{b}^s\Rightarrow A$ and (ii) $b\Rightarrow A$.

Ad (i): we use the analogue of Theorem 1.4 for A. By [**2**; Theorem 9.8.1], if A fails then there exist $\boldsymbol{\sigma}\in\mathbb{R}^k$ with $|\sigma_m|=1$, nonnegative α_{n_m} and nonzero $x_{n_m}\in H_m$ such that

$$\sum_{m,\,n_m=1}^{k}\alpha_{n_m}\sigma_m\boldsymbol{w}_m(x_{n_m})=\boldsymbol{0},\quad \sum_{n_m=1}^{k}\alpha_{n_m}>0, \tag{4.1}$$

in the notation of (3.1). With $\boldsymbol{i}=\boldsymbol{i}(\boldsymbol{\sigma})$ as in the proof of Lemma 2.1, let us suppose that an eigenvalue $\boldsymbol{\lambda}$ exists with index $\boldsymbol{i}$.

We claim that perturbations W_m^{ε} of the W_m exist such that at least one of the

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$\sigma_m W_m^\varepsilon(\boldsymbol{v})$ fails to be nonnegative definite for all $\boldsymbol{v}$ satisfying

$$\boldsymbol{v}^T\boldsymbol{\lambda} > 0. \tag{4.2}$$

Note that (4.2) forces $\boldsymbol{v}$ to belong to an ε-independent neighbourhood of $\boldsymbol{\lambda}$.

Let

$$T_m^\varepsilon = T_m - \varepsilon\sigma_m\lambda_0 I_m, \quad V_{mn}^\varepsilon = V_{mn} - \varepsilon\sigma_m\lambda_n I_m$$

where I_m is the identity on H_m, and set

$$W_m^\varepsilon = \lambda_0 T_m^\varepsilon - \sum_{n=1}^{k} \lambda_n V_{mn}^\varepsilon.$$

Then we have

$$w_m^\varepsilon(x_m) = w_m(x_m) - \varepsilon\sigma_m\boldsymbol{\lambda}\|x_m\|^2,$$

so

$$\sum_{m,n_m} \alpha_{n_m}\sigma_m(x_{n_m}, W_m^\varepsilon(\boldsymbol{v})x_{n_m})$$

$$= -\varepsilon \sum_{m,n_m} \alpha_{n_m}\boldsymbol{v}^T\boldsymbol{\lambda}\|x_{n_m}\|^2$$

by (4.1). For $\varepsilon > 0$ and any $\boldsymbol{v}$ satisfying (4.2), this expression is negative. Thus a neighbourhood of $\boldsymbol{\lambda}$ exists containing no eigenvalue $\boldsymbol{v}$ of index $\boldsymbol{i}$, and therefore $\hat{b}^s$ must fail.

Ad (ii): again let an eigenvalue $\boldsymbol{\lambda}$ exist with index $\boldsymbol{i}(\boldsymbol{\sigma})$. By scaling and an axis rotation, if necessary, we may assume $\boldsymbol{\lambda} = \boldsymbol{e}_0$, so $\sigma_m W_m(\boldsymbol{\lambda}) = \sigma_m T_m$ has minimal eigenvalue zero. Let R_σ^* be defined as for R_σ (1.5) but for nonzero $x_m \in \mathcal{N}(T_m)$. The condition that $\boldsymbol{\lambda}$ possesses a sign forces δ_0 not to vanish on $\times_{m=1}^k \mathcal{N}(T_m)$ by (3.5), and by Theorem 1.4 this implies $\mathbf{0} \notin R_\sigma^*$.

On the other hand, suppose that A fails. It is convenient now to restrict the index set of n_m in (4.1) so that each α_{n_m} is *positive*. If

$$\alpha_{n_m}\sigma_m w_m(x_{n_m})\boldsymbol{\lambda} = \alpha_{n_m}\sigma_m t_m(x_{n_m})$$

is positive for some m and n_m in (4.1) then the sum of the other terms in (4.1) is negative. This contradicts nonnegativity of the $\sigma_m t_m$, so it follows that each term $t_m(x_{n_m})$ vanishes in (4.1). Thus each $x_{n_m} \in \mathcal{N}(T_m)$, and so (4.1) forces the contradiction $\mathbf{0} \in R_\sigma^*$.

Corollary 4.3. *A_0, b_0, $\hat{b}_0^s$ and b_0^s are equivalent.*

Proof. By Corollary 3.6 and the previous result, it is enough to assume A and either b_0 or $\hat{b}_0^s$, and then to deduce A_0. As in the proof of Corollary 3.6, A yields precisely $2\dim H^\otimes$ eigenvalues $\boldsymbol{\lambda}$, normalised to $|\boldsymbol{\mu\lambda}| = 1$, with $\boldsymbol{\lambda}$ as in Theorem 3.1. Thus either $\hat{b}_0^s$ or b_0 forces *each* eigenvalue $\boldsymbol{\mu}$ to satisfy $\lambda_0 \neq 0$.

By [2; Theorem 6.7.2], the operators

$$\Gamma_n = \Delta(\boldsymbol{\mu})^{-1}\Delta_n, \quad n = 0\ldots k \tag{4.3}$$

commute. In particular, $\mathcal{N}(\Gamma_0) = \mathcal{N}(\Delta_0)$ is invariant for each Γ_n. If $\mathcal{N}(\Gamma_0)$ is nontrivial then the Γ_n have a common eigenvector with eigenvalues v_n, say, where $v_0 = 0$. By [2; Theorem 6.8.1], $\boldsymbol{v}$ is an eigenvalue for (1.1) so $v_0 = 0$ gives a contradiction. It follows that $\mathcal{N}(\Delta_0)$ must be trivial.

We turn now to Atkinson's "Definiteness Condition II" [2; p. 121] which we label A^+ here. This requires the existence of $\boldsymbol{\mu} \in \mathbb{R}^{k+1}$ such that

$$\delta(\boldsymbol{\mu}, x) > 0 \quad \text{for all nonzero} \quad x_m \in H_m, \quad m = 1\ldots k.$$

The corresponding analogue of Theorem 1.1 is [2; Theorem 7.8.2] that A^+ implies positive definiteness of $\Delta(\boldsymbol{\mu})$ on $H^{\otimes}$. When $\mu = \pm e_0$, we write $A_0^{\pm}$ for A^+, in agreement with Sections 1 and 2.

As we have seen, if b holds then so does A, and the eigenvalues $\boldsymbol{\lambda}$ may be partitioned into two subsets by the sign of either $\boldsymbol{\mu\lambda}$ or $(\boldsymbol{\mu\lambda})(\boldsymbol{\mu\delta}(x))$, $\boldsymbol{\mu}$ as in Theorem 3.1, x_m as in (1.1). Our specialisation b^+ of b for A^+ requires these methods of partitioning to coincide. In other words, b^+ means for each $\boldsymbol{i}$ the existence of at least one eigenvalue $\boldsymbol{\lambda}$ of index $(\boldsymbol{i}, +)$ such that $\boldsymbol{\mu\lambda} > 0$, $\boldsymbol{\mu}$ being independent of $\boldsymbol{i}$. Suffix $_0$ continues to refer to $\boldsymbol{\mu} = \boldsymbol{e}_0$, and superfix ^{s} to continuous dependence on the data.

Corollary 4.4. *A^+, b^+ and b^{+s} are equivalent.*

Proof. By an axis rotation, we may assume $\boldsymbol{\mu} = \boldsymbol{e}_0$, and by Corollary 4.3, A_0^+ implies b_0. As in the proof of Corollary 3.6, there is a (unique) eigenvalue $\boldsymbol{\lambda}$ of signed index $(\boldsymbol{i}, +)$. With x_m as in (1.1), we have

$$\begin{aligned} \operatorname{sgn}\lambda_0 &= \operatorname{sgn}(\lambda_0\delta_0(x)) \quad && \text{by } A_0^+ \\ &= 1 && \text{by definition.} \end{aligned}$$

Thus b_0^+ holds, and continuous dependence comes from Corollary 3.6.

Conversely, b_0^+ implies A by Corollary 4.3. If $\boldsymbol{\lambda}$ has index $(\boldsymbol{i}, +)$ and $\lambda_0 = 1$ then

$$\operatorname{sgn}\delta_0(x) = \operatorname{sgn}(\lambda_0\delta_0(x)) = 1$$

so $\delta_0(x) > 0$. The proof of Corollary 2.6 then completes the argument.

We are now in a position to reexamine RD, and first we summarise some of its equivalents. By $b_0^{\pm}$ we mean b^+ for $\boldsymbol{\mu} = \pm\boldsymbol{e}_0$ and $b_0^{\pm s}$ is analogous.

Corollary 4.5. *$A_0^{\pm}$ is equivalent to* (i) *$\hat{b}_0$ for all $\boldsymbol{T}$* (ii) *b for $\boldsymbol{T} = \boldsymbol{0}$* (iii) *$b_0^{\pm}$ for any fixed $\boldsymbol{T}$* (iv) *$b_0^{\pm s}$ for all $\boldsymbol{T}$.*

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Proof. We note that $A_0^{\pm}$ is equivalent to A when $\boldsymbol{T}=\mathbf{0}$. Thus the result follows from Theorems 2.3 and 4.2 and Corollary 4.4.

Formally, (iv) includes the other consequences of RD, and even includes uniqueness of the normalised eigenvalues (cf. b_{0+}^s in Theorem 2.3), by virtue of Corollary 3.6. Let us compare (i), (ii) and (iii) as sufficient conditions for RD. (i) involves checking all eigenvalues for all $\boldsymbol{T}$ (actually for the 2^k choices $T(\boldsymbol{\sigma})$ in the proof of Lemma 2.1). (ii) and (iii) both involve checking two scalars for all eigenvalues (actually for 2^k indices $\boldsymbol{i}(\boldsymbol{\sigma})$) for a single choice of $\boldsymbol{T}$). These checks are significantly easier than those in (i).

Remark 4.6. The formal transcription of Corollary 4.5 to A^+ involves replacing $\boldsymbol{T}$ by a linear combination of the columns in W, so $\boldsymbol{T}$ loses its special status. This loss is not explicit in Corollary 4.5, but is implicit because λ_0 plays a special rôle. In particular, $\lambda_0=0$ changes the nature of (1.1) as far as $\boldsymbol{T}$ is concerned. One can circumnavigate this by suppressing such eigenvalues, which correspond to $\mathcal{N}(\Delta_0)$—see (3.2). Then we may scale λ_0 to unity, and write $\hat{\boldsymbol{\lambda}}$ for the "inhomogeneous" eigenvalue $[\lambda_1 \dots \lambda_k]^T$. We may also rotate $\boldsymbol{\mu}$ to say $\boldsymbol{e}_k$ by what is now an affine transformation of $\hat{\boldsymbol{\lambda}}$ space—cf. [**4**; Lemma 5.1]. Then $H_k^{\otimes}$ may be constructed as $H^{\otimes}$ under the inner product $(x,y)_k=(x,\Delta_k y)$, and the Γ_n (4.3) are Hermitian in $H_k^{\otimes}$. Commutativity of the Γ_n implies that the eigenvectors $x^{\otimes}$ corresponding to "inhomogeneous" eigenvalues span $\mathcal{N}(\Gamma_0)^{\perp}=\mathcal{N}(\Delta_0)^{\perp}$ in $H_k^{\otimes}$. The sign of an eigenvalue $\hat{\boldsymbol{\lambda}}$ can be obtained from the sign of $\delta_0(x)$, since

$$\operatorname{sgn}\delta_0(x)=\operatorname{sgn}(\lambda_0\delta_0(x))=\operatorname{sgn}(\lambda_k\delta_k(x)).$$

Thus an indicial theory can be constructed for A^+, via Corollary 4.4, preserving the special status of $\boldsymbol{T}$ (and λ_0 and δ_0). This extends a corresponding construction for the left definite case—see [**5**] where a signed index was introduced for inhomogeneous eigenvalues, based on δ_0.

We conclude with an indicial equivalent for left definiteness, which is the combination [**3**; p. 321], [**8**; pp. 62–3] of:

(i) (definiteness of $\boldsymbol{T}$) all T_m are nonnegative definite, with at least one positive definite.

(ii) (cofactor condition) for some $\boldsymbol{\mu}\in\mathbb{R}^{k+1}$ and for each $m=1\dots k$, the determinant $\delta_0(x)$, with mth row replaced by $[\mu_1\dots\mu_k]$, is positive for all nonzero $x_m\in H_m$. It is convenient to set the arbitrary μ_0 to zero.

Corollary 4.7. *The cofactor condition is equivalent to b^+ for all definite $\boldsymbol{T}$, with the proviso that $\boldsymbol{\mu}$ in the definition of b^+ may be chosen with $\mu_0=0$ and independently of $\boldsymbol{T}$.*

Proof. A simple computation [**3**; equation (1.6)] shows that left definiteness implies A^+, with $\boldsymbol{\mu}$ as in the cofactor condition, so b^+ for all definite $\boldsymbol{T}$ follows from Corollary 4.4. Conversely, b^+ for all definite $\boldsymbol{T}$ implies A^+ for all definite $\boldsymbol{T}$ by the same result, with $\boldsymbol{\mu}$ as in the definition of b^+.

We claim that the cofactor condition holds for the same $\boldsymbol{\mu}$. Indeed if not then

$$\sum_{n=1}^{k}\mu_n\delta_{0ln}(x)\leqq 0$$

for some l and some nonzero $x_m \in H_m$, where $\delta_{0ln}(x)$ is the (l, n) cofactor of $\delta_0(x)$. Choose T_l positive definite, with the other $T_m = 0$. Then

$$\delta(\boldsymbol{\mu}, x) = \mu_0 \delta_0(x) + \sum_{m,n=1}^{k} t_m(x) \mu_n \delta_{0mn}(x)$$

$$= t_l(x) \sum_{n=1}^{k} \mu_n \delta_{0ln}(x) \leqq 0$$

and this contradicts A^+ for the given choice of $\boldsymbol{\mu}$.

Remark. Further equivalences involving uniqueness and continuous dependence of the eigenvalues of index $(\boldsymbol{i}, +)$ follow from Corollaries 3.6 and 4.4.

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