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# Poles of Intertwining Operators via Endoscopy; the Connection with Prehomogeneous Vector Spaces With an Appendix, 'Basic Endoscopic Data', by Diana Shelstad

*To the Memory of Magdy Assem*

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**Abstract.** In this paper, we determine the residues at poles of standard intertwining operators for parabolically induced representations of an arbitrary connected reductive quansisplit algebraic group over a $p$-acid field whenever the unipotent radical of the parabolic subgroup is Abelian. We then interpret these residues by means of the theory of endoscopy.

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**Key words:** induced representations, intertwining operators, endoscopy, $L$-functions.

## Introduction

The purpose of this paper is to set the foundation for a general treatment of the poles of standard intertwining operators and reducibility in the rank one case for an arbitrary quasisplit group by means of the theory of endoscopy [2, 7, 20, 21, 24, 37]. This is a problem whose solution has many applications in global and local theory and is equivalent to determining the nondiscrete tempered spectrum of these groups as well as certain local $L$-functions in the important context of endoscopy. In fact, in this paper, we shall show that in the Abelian unipotent radical case where there are only a finite number of orbits for adjoint action, the residue is always a finite sum of twisted orbital integrals and therefore everything is controlled by endoscopy and when the inducing data is generic, the poles of practically every standard $L$-function is determined by it.

What is important is that the presence of endoscopy persists even if the number of orbits are infinite and here is where some very fascinating examples show up, among them the symmetric cube of cusp forms on $GL_2$ which we hope our future

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work in this direction will shed some light on its existence. On the other hand, the finite orbit case fits well in the theory of prehomogeneous vector spaces and allows us to define our $L$-functions as values of certain Igusa zeta functions. We hope that this will lead us to a better understanding of possible connections of our work with invariant theory.

To be precise, let $\mathbf{G}$ be a quasisplit connected reductive group over a non-Archimedean local field $F$ of characteristic zero. Let $\mathbf{B} = \mathbf{TU}$ be a Borel subgroup of $G$ with a maximal torus $\mathbf{T}$, and let $\mathbf{P} = \mathbf{MN}$, $\mathbf{T} \subset \mathbf{M}$, $\mathbf{N} \subset \mathbf{U}$ be a parabolic subgroup with Levi factor $\mathbf{M}$. Let $\sigma$ be an irreducible unitary supercuspidal representation of $M = \mathbf{M}(F)$ and given $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, the complex dual of the real Lie algebra of the split component $\mathbf{A}$ of (the center) of $\mathbf{M}$, let $I(\nu, \sigma)$ be the corresponding induced representation. Assume $\mathbf{P}$ is maximal and let $\widetilde{w}_0$ be the longest element in the Weyl group of $\mathbf{A_0}$, the maximal split torus of $\mathbf{T}$, in $\mathbf{G}$ modulo that of $\mathbf{A_0}$ in $\mathbf{M}$. Let $A(\nu, \sigma, w_0)$ be the standard intertwining operator from $I(\nu, \sigma)$ into $I(w_0(\nu), w_0(\sigma))$, where $w_0$ is a representative for $\widetilde{w}_0$. Unless $w_0(\sigma) \cong \sigma$ which requires $w_0(\mathbf{M}) = \mathbf{M}$, the operator has no pole at $\nu = 0$ and $I(\sigma) = I(0, \sigma)$ is irreducible. Suppose $w_0(\sigma) \cong \sigma$. Then $I(\sigma)$ is reducible if and only if $A(\nu, \sigma, w_0)$ is holomorphic at $\nu = 0$ (cf. [13, 15, 32, 33, 38]).

In this paper we make the assumption that the unipotent radical of $\mathbf{N}$ is Abelian and therefore the action of $M$ on $\mathfrak{n}$, the Lie algebra of $N$, has a finite number of open orbits whose union is dense in $\mathfrak{n}$ [26]. This assumption covers a good number of cases which come from the theory of prehomogeneous vector spaces [18, 26, 28, 30, 31, 39]. Let $\{n_i\}$ denote a set of representatives for the corresponding orbits of $M$ in $N$. If $\mathbf{N}^-$ is the opposite of $\mathbf{N}$, choose $m_i \in M$ such that $w_0^{-1} n_i = m_i n_i' n_i^-$, $n_i' \in N$, $n_i^- \in N^-$. Let $\widetilde{A}$ be the center of $M$ and let $w_0(\widetilde{A})\widetilde{A}^{-1}$ be the subgroup of elements of the form $w_0(a)a^{-1}$, $a \in \widetilde{A}$. Denote by $\omega$ the central character of $\sigma$.

Let $\theta = \mathrm{Ad}(w_0)|\mathbf{M}$, a semisimple automorphism of $\mathbf{M}$. Clearly $\theta$ fixes the pair $(\mathbf{B} \cap \mathbf{M}, \mathbf{T})$ as in [20]. Moreover, changing $w_0$ by an appropriate element in $A_0$, we may assume $\theta$ also fixes the splitting in $U \cap M$. Given $f \in C_c^\infty(M)$ and $m_0 \in M$, let $\Phi_\theta(m_0, f) = \int_{M/M_{\theta, m_0}} f(\theta(m) m_0 m^{-1}) \,\mathrm{d}\dot{m}$ be the $\theta$-twisted orbital integral of $f$ at $m_0$. Here $M_{\theta, m_0}$ is the $\theta$-twisted centralizer of $m_0$ in $M$. Then the main result of our paper, Theorem 2.5, can be stated as

THEOREM 1. *Let* $\mathbf{G}$ *be an arbitrary quasisplit connected reductive algebraic group over* $F$ *with a Borel subgroup* $\mathbf{B} = \mathbf{TU}$. *Fix a maximal parabolic subgroup* $\mathbf{P} = \mathbf{MN}$, $\mathbf{T} \subset \mathbf{M}$, $\mathbf{N} \subset \mathbf{U}$. *Assume* $N$ *is abelian and therefore* $M$ *acts on* $\mathfrak{n}$ *by a finite number of open orbits whose union is dense in* $\mathfrak{n}$. *Suppose* $\sigma$ *is supercuspidal and irreducible, and* $w_0(\sigma) \cong \sigma$. *Then the intertwining operator* $A(\nu, \sigma, w_0)$ *has a (simple) pole at* $\nu = 0$ *or equivalently* $I(\sigma)$ *is irreducible if and only if*

$$\sum_i \int_{\widetilde{A}/w_0(\widetilde{A})\widetilde{A}^{-1}} \Phi_\theta(z m_i, f) \omega^{-1}(z) \,\mathrm{d}z \neq 0$$

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*for some* $f \in C_c^\infty(M)$ *defining a matrix coefficient of* $\sigma$ *by descent. Here* $m_i$*'s,* $m_i \in M$*, correspond to representatives* $\{n_i\}$ *of* $M$ *in* $N$ *as above.*

In most applications $\mathbf{A}_1 = \widetilde{\mathbf{A}}^0$, where $\mathbf{A}_1$ is the connected component of the subgroup $\widetilde{\mathbf{A}}_1$ of all $z \in \widetilde{\mathbf{A}}$ for which $\theta(z) = z^{-1}$. This is equivalent to $G$ being semisimple. Our Theorem 1 (Theorem 2.5 and Corollary 2.6) then simplifies as (Corollary 3.3):

THEOREM 2. *Suppose* $\mathbf{A}_1 = \widetilde{\mathbf{A}}^0$*. Then* $A(\nu, \sigma, w_0)$ *has no pole at* $\nu = 0$ *unless* $w_0(\sigma) \cong \sigma$ *and thus* $\omega^2 = 1$*. Suppose* $w_0(\sigma) \cong \sigma$ *and* $\omega_1 = \omega|A_1 \equiv 1$*. Then* $A(\nu, \sigma, w_0)$ *has a (simple) pole at* $\nu = 0$ *or equivalently* $I(\sigma)$ *is irreducible if and only if* $\sum_{\varepsilon \in F^*/(F^*)^2} \sum_i \Phi_\theta(\varepsilon m_i, \overline{f}) \neq 0$ *for some* $f \in C_c^\infty(M)$ *defining a matrix coefficient of* $\sigma$ *by descent. Here* $\overline{f}(m) = \sum_{\widetilde{A}/A_1} f(zm)\omega^{-1}(z)$.

Let $\{\overline{1}\}(F)$ be the $F$-rational points of the $\theta$-conjugacy class of 1 in $\mathbf{M}(\overline{F})$. In the event that

$$\bigcup_{\varepsilon} \bigcup_{i} \{\varepsilon m_i\} = \{\overline{1}\}(F), \quad \varepsilon \in F^*/(F^*)^2, \quad \text{and} \quad \mathbf{M}_i^0 = \mathbf{M}_1^0$$

which happens often and in our examples, Theorem 2 can be easily interpreted in the context of the theory of endoscopy of Kottwitz, Langlands, and Shelstad [20, 21, 24, 37] as follows. Here $\mathbf{M}_i$ is the $\theta$-twisted centralizer of $m_i$.

Let $\widehat{M}$ be the connected component of the $L$-group ${}^L M$ of $\mathbf{M}$. The automorphism $\theta$ can be transferred to an automorphism $\hat{\theta}$ of $\widehat{M}$. Let $\mathbf{H}$ be a $\theta$-twisted endoscopic group of $\mathbf{M}$ for which $\widehat{H} = \mathrm{Cent}_{\hat{\theta}}(1, \widehat{M})^0$ and ${}^L H = \hat{H} \propto W_F$ is $L$-embedded in ${}^L M = \hat{M} \propto W_F$ by inclusion, where $W_F$ is the Weil group. This gives a 'basic endoscopic data' in the sense of Shelstad ([37], the appendix to this paper) and we call $\mathbf{H}$ the basic endoscopic group attached to $(\mathbf{M}, \theta)$.

One of the fundamental assumptions of the theory of endoscopy is the existence (cf. [20, 24, 34, 37]) of a 'map' $f \mapsto f^{\mathbf{H}}$ from $C_c^\infty(M)$ into $C_c^\infty(H)$ such that $\Phi_\theta^{\mathrm{st}}(\gamma, f) = \Phi^{st}(\delta, f^{\mathbf{H}})$ for every strongly $\theta$-regular $\theta$-semisimple $\gamma \in M$, if $\delta \in H$ is the norm of $\gamma$, and $\Phi^{\mathrm{st}}(\delta, f^H) = 0$, otherwise (Assumption 4.2 of Section 4 here, paragraph 5.5 of [20], and [37]).

Now let $\sigma$ be an irreducible supercuspidal representation of $M$ such that $w_0(\sigma) \cong \sigma$. Let $\mathbf{H}$ be the basic endoscopic group attached to $(\mathbf{M}, \theta)$. (The subgroup $\mathbf{M}^\theta$ of elements of $\mathbf{M}$ fixed by $\theta$ has the largest dimension among those fixed by automorphisms in the class of $\theta$ in $\mathrm{Aut}(\mathbf{M})/\mathrm{Int}(\mathbf{M})$ which preserve $(\mathbf{B} \cap \mathbf{M}, \mathbf{T})$. Similarly for $\widehat{H} = (\widehat{M}^{\hat{\theta}})^0$.) Assume $\bigcup_\varepsilon \bigcup_i \{\varepsilon m_i\} = \{\overline{1}\}(F)$, $\mathbf{M}_i^0 = \mathbf{M}_1^0$ for all $i$, and that the map $f \mapsto f^{\mathbf{H}}$ exists. *We shall say* $\sigma$ *comes from* $H = \mathbf{H}(F)$ *by* $\theta$*-twisted endoscopic transfer, if there exists a function* $f \in C_c^\infty(M)$*, defining a matrix coefficient of* $\sigma$ *by descent, for which* $\overline{f}^{\mathbf{H}}(1) \neq 0$*, where* $\overline{f} \in C_c^\infty(M)$ *is defined as in Theorem 2.*

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The reader who is familiar with the theory of endoscopy realizes that conjecturally this is equivalent to the fact that the homomorphism of $W_F \to {}^L M$ which parametrizes $\sigma$ factors through ${}^L H$, the $L$-group of $\mathbf{H}$.

Our Theorem 2 can then be reformulated as follows (Theorem 4.5).

THEOREM 3. *With assumptions as in Theorem 2, suppose*

$$\bigcup_{\varepsilon} \bigcup_{i} \{\varepsilon m_i\} = \{\overline{1}\}(F), \mathbf{M}_i^0 = \mathbf{M}_1^0$$

*for all $i$, and that the 'map' $f \mapsto f^{\mathbf{H}}$ exists, where $\mathbf{H}$ is the basic endoscopic group attached to $(M, \theta)$. Let $\sigma$ be an irreducible supercuspidal representation of $M$ and that $w_0(\sigma) \cong \sigma$ which implies $\omega^2 = 1$. Suppose $\omega_1 = \omega|A_1 \equiv 1$. Then $I(\sigma)$ is irreducible if and only if $\sigma$ comes from $H = \mathbf{H}(F)$ by $\theta$-twisted endoscopic transfer.*

Observe how this generalizes the earlier results [9, 34] in the case of Siegel parabolic subgroups of classical groups. This was later interpreted in terms of $K$-types in [27]. We refer to [6] and [12] for further possible connections and applications.

Our examples are given in Section 5. Our Proposition 5.1 gives a quick and simple proof of Olšanskiĭ's result [28] for $\mathrm{GL}_n$ and shows that the residue is proportional to the inverse of the formal degree of the inducing representation. Propositions 5.2 and 5.3 determine the reducibility for representations of $\mathrm{SO}_n(F)$ induced from its $\mathrm{GL}_1(F) \times \mathrm{SO}_{n-2}(F)$ Levi subgroup. The most exotic of our examples is the case of parabolic induction from $\mathbf{P} = \mathbf{MN}$ of an exceptional group of type $E_7$ for which the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is of type $E_6$. When $w_0(\sigma) \cong \sigma$ and $\omega = 1$, $I(\sigma)$ is irreducible if and only if $\sigma$ comes from a group of type $F_4$, one of the two (in fact the larger) twisted endoscopic groups of $E_6$ (Proposition 5.4).

The case when $\mathbf{N}$ is not Abelian which includes all the cases when the number of orbits is infinite is harder and covers most rank one cases. The Lie algebra $\mathfrak{n}$ can no longer be a one step nilpotent Lie algebra [26, 30]. In fact, although still the action of $M$ on each step of $\mathfrak{n}$ has a finite number of open orbits and is a prehomogeneous vector space, the operator $A(\nu, \sigma, w_0)$ is obtained by integration over all of $N$ which in general will not have a finite number of open orbits under action of $M$, if $N$ is a multi-step nilpotent Lie group. In fact, this is precisely the case for an arbitrary maximal parabolic subgroup of a classical group, a problem which has been studied in [11, 35] with interesting conclusions. Clearly the automorphism $\theta$ of $\mathbf{M}$ still exists and plays an important role if the number of orbits is infinite, and as it did in the case of classical groups [11, 35], one expects that the theory of endoscopy will play a crucial role in general.

Finally our short discussion in Section 6 gives a new interpretation, in fact as an Igusa zeta function [3, 8, 16], for some of our $L$-functions (cf. [33]) when $\sigma$ is generic, giving a new context for global study. Almost all the standard $L$-functions that our method provides are among these.

Magdy Assem was one person whose work on and understanding of the theory of prehomogeneous vector spaces and Igusa zeta functions played a role in making me interested in the theory of prehomogeneous vector spaces (cf. [3–5]). His premature and sudden death left an empty space, both as a friend and as a colleague, and for that I would like to dedicate this paper to his memory.

## 1. Preliminaries

Let $F$ be a non-Archimedean field of characteristic zero. Denote by $O$ its ring of integers and let $P$ be the unique maximal ideal of $O$. Let $q$ be the number of elements in $O/P$ and fix a uniformizing element $\varpi$ for which $|\varpi|_F = q^{-1}$, where $|\ |_F = |\ |$ denotes an absolute value for $F$ normalized in this way.

Let $\mathbf{G}$ be a quasisplit connected reductive algebraic group over $F$. Fix a Borel subgroup $\mathbf{B}$ and write $\mathbf{B} = \mathbf{TU}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ and $\mathbf{T}$ is a maximal torus there. Let $\mathbf{A_0}$ be the maximal split torus of $\mathbf{T}$. Let $\Delta$ be the set of simple roots of $\mathbf{A_0}$ in the Lie algebra of $\mathbf{U}$.

Denote by $\mathbf{P} = \mathbf{MN}$ a standard parabolic subgroup of $\mathbf{G}$ in the sense that $\mathbf{N} \subset \mathbf{U}$. Assume $\mathbf{T} \subset \mathbf{M}$. Let $\theta \subset \Delta$ be the subset of $\Delta$ such that $\mathbf{M} = \mathbf{M}_\theta$.

As usual, we use $W = W(\mathbf{A_0})$ to denote the Weyl group of $\mathbf{A_0}$ in $\mathbf{G}$. Given $\widetilde{w} \in W$, we use $w$ to denote a representative for $\widetilde{w}$.

Let $X(\mathbf{M})_F$ be the group of $F$-rational characters of $\mathbf{M}$. Denote by $\mathbf{A}$ the split component of the center of $\mathbf{M}$. Then $\mathbf{A} \subset \mathbf{A_0}$. Let

$$\mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R}) = \mathrm{Hom}(X(A)_F, \mathbb{R})$$

be the real Lie algebra of $\mathbf{A}$. Set

$$\mathfrak{a}^* = X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R} = X(\mathbf{A})_F \otimes_{\mathbb{Z}} \mathbb{R}$$

and $\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ to denote its real and complex duals.

Given an algebraic group $\mathbf{H}$ over $F$, we will use $H = \mathbf{H}(F)$ to denote its subgroup of $F$-rational points, thus identifying $\mathbf{H} = \mathbf{H}(\overline{F})$. We then have $G$, $B$, $T$, $U$, $P$, $M$, $N$, $A$, $A_0$.

For $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ and $\sigma$ an irreducible admissible representation of $M$, let

$$I(\nu, \sigma) = \mathrm{Ind}_{MN\uparrow G}\sigma \otimes q^{\langle \nu, H_P() \rangle} \otimes 1,$$

where $H_P$ is the extension of the homomorphism $H_M : M \to \mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R})$ to $P$, extended trivially along $N$, defined by $q^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_F$ for all $\chi \in X(\mathbf{M})_F$.

Fix $\widetilde{w} \in W(\mathbf{A_0}) = W$ such that $\widetilde{w}(\theta) \subset \Delta$, where $\theta$ generates $\mathbf{M} = \mathbf{M}_\theta$. Let $\mathbf{N}^- = \mathbf{N}_{-\theta}$ be unipotent subgroup opposed to $\mathbf{N} = \mathbf{N}_\theta$. Set $\mathbf{N}_{\widetilde{w}} = \mathbf{U} \cap w\mathbf{N}^- w^{-1}$.

Let $V(\nu, \sigma)$ be the space of $I(\nu, \sigma)$. For $h \in V(\nu, \sigma)$, denote by

$$A(\nu, \sigma, w)h(g) = \int_{N_{\widetilde{w}}} h(w^{-1}ng)\, \mathrm{d}n, \tag{1.1}$$

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the standard intertwining operator from $I(\nu, \sigma)$ into $I(w(\nu), w(\sigma))$. The integral converges absolutely for $\nu$ in some cone in $\mathfrak{a}_{\mathbb{C}}^*$ and extends meromorphically to all of $\mathfrak{a}_{\mathbb{C}}^*$ (cf. [32, 38]).

When $\sigma$ is tempered, the cone of convergence of (1.1) equals to what one usually calls the positive Weyl chamber, denoted by $(\mathfrak{a}_{\mathbb{C}}^*)^+$. Every $\nu \in (\mathfrak{a}_{\mathbb{C}}^*)^+$ satisfies $\mathrm{Re}\langle \nu, H_\alpha \rangle > 0$ for every $\alpha \in \Delta - \theta$ and conversely, where $H_\alpha$ is the standard coroot attached to $\alpha$ and $\nu$ is realized as an element of $(\mathfrak{a}_0)_{\mathbb{C}}^*$. Here $\mathfrak{a}_0$ is the real Lie algebra of $\mathbf{A_0}$.

In the special case when $\sigma$ is also (unitary) supercuspidal, the poles of $A(\nu, \sigma, w)$ all lie on $\mathfrak{a}^*$ (cf. [38]).

For both local and global reasons it is very important to determine the poles of $A(\nu, \sigma, w)$ ([13, 19, 33]). It is well known (e.g. Theorem 2.1.1 of [32]) that $A(\nu, \sigma, w)$ can be written as a product of such operators for which the parabolic subgroups are maximal or of parabolic rank one. These are what one usually calls, *rank one operators*.

Let us concentrate on one important application of our knowledge of these poles. Assume $\mathbf{P} = \mathbf{MN}$ is maximal. If ${}^L M$ denotes the $L$-group of $\mathbf{M}$, it then acts by adjoint action, denoted by $r$, on the Lie algebra of ${}^L\mathfrak{n}$ of ${}^L N$, the $L$-group of $\mathbf{N}$. If we use $\oplus_{i=1}^m {}^L\mathfrak{n}_i$ to denote the gradation of ${}^L\mathfrak{n}$ under ${}^L M$ (cf. [22, 33, 36]), then each ${}^L\mathfrak{n}_i$ is invariant under ${}^L M$. Let $r_i = r|{}^L\mathfrak{n}_i$. Assume moreover that $\sigma$ is generic (cf. [32, 33]). For $s \in \mathbb{C}$ and each $i$, $1 \leqslant i \leqslant m$, let $L(s, \sigma, r_i)$ be the local $L$-function attached to $\sigma$ and $r_i$ in [33]. They are simply polynomials in $q^{-s}$ whose constant terms are normalized to be 1 and are the local components for the corresponding global functional equations satisfied by globally generic cusp forms.

Assume $\alpha$ is the unique simple root in $\mathbf{N}$ and let $\rho$ be half the sum of positive roots in $\mathbf{N}$. Denote by $\widetilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho \in \mathfrak{a}^*$. Then $s\widetilde{\alpha} \in \mathfrak{a}_{\mathbb{C}}^*$ for each $s \in \mathbb{C}$. Finally let $\widetilde{w}_0$ be the longest element in $W$ modulo that in the Weyl group of $\mathbf{A_0}$ in $\mathbf{M}$. Since $\mathbf{P}$ is maximal, this is the only element of $W$ which is of interest.

Now suppose $\sigma$ is (unitary) supercuspidal. It then follows from Lemma 7.5 of [33] that $L(s, \sigma, r_i) \equiv 1$ for $i \geqslant 3$. Moreover we can restate the following result from [33].

THEOREM 1.1. *Suppose $\sigma$ is an irreducible (unitary) generic supercuspidal representation of $M$. Then $\prod_{i=1}^2 L(is, \sigma, \widetilde{r}_i)^{-1} A(s\widetilde{\alpha}, \sigma, w_0)$ is a nonzero and holomorphic operator as a function of $s$, i.e., poles of $\prod_{i=1}^2 L(is, \sigma, \widetilde{r}_i)$ are precisely those of $A(s\widetilde{\alpha}, \sigma, w_0)$.*

As soon as the $L$-functions $L(s, \sigma, \widetilde{r}_i)$ are determined from Theorem 1.1 for supercuspidal $\sigma$, the machinery of $L$-functions developed in [33, 36] determines them for any irreducible admissible $\sigma$. In particular, this leads to a determination of the nondiscrete tempered spectrum of $G$ by means of the theory of $R$-groups (cf. [10, 13, 19]). We refer to [23], [25], and [33] for further applications of these $L$-functions.

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The purpose of this paper is to determine the poles of $A(\nu, \sigma, w)$ in the rank one case and with a supercuspidal inducing data (not necessarily generic), when the action of $M$ on the Lie algebra $\mathfrak{n}$ of $N$ has only a finite number of open orbits whose is dense in $\mathfrak{n}$. The case when $\mathfrak{n}$ is a one step nilpotent Lie algebra then falls in this class as a consequence of the theory of prehomogeneous vector spaces [18, 19, 26, 30, 39]. The results are interpreted in terms of the theory of twisted endoscopy [20, 21, 24, 37] and may be considered as a bridge between a number of deep and diverse disciplines such as number theory, harmonic analysis and representation theory of local group, invariant theory, theory of prehomogeneous vector spaces, and finally the theory of endoscopy.

The case of the infinite number of orbits which covers most cases is much harder. In the case of classical groups, we have already encountered the problem in [11, 35]. We hope to formulate the general case in a future paper. The possible hints that one may get towards some very important global problems makes the whole project quite worthwhile.

## 2. Poles of Operators

From now on assume $\mathbf{P} = \mathbf{MN}$ is maximal. Let $\sigma$ be an irreducible unitary supercuspidal representation of $M$. Fix $s \in \mathbb{C}$. Let $\widetilde{w}_0$ be the longest element in the Weyl group $W$ of $\mathbf{A_0}$ in $\mathbf{G}$ modulo that of the Weyl group of $\mathbf{A_0}$ in $\mathbf{M}$. Let $w_0$ denote a representative in $G$ for $\widetilde{w}_0$. In what follows we shall compute the poles of $A(\nu, \sigma, w_0)$ as a function of $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. Observe that to compute the operator $A(\nu, \sigma, w_0)$, $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, it is enough to take $\nu \in \mathfrak{a}^*_{\mathbb{C}}/\mathfrak{z}^*_{\mathbb{C}}$, where $\mathfrak{z}$ is the real Lie algebra of the connected center of $\mathbf{G}$ and therefore one would only need to determine the poles of $A(s\widetilde{\alpha}, \sigma, w_0)$ as a function of $s$.

The operator $A(\nu, \sigma, w_0)$ has a pole at $\nu = \nu_0$ if and only if $A(\nu, \sigma_{\nu_0}, w_0)$ has a pole at $\nu = 0$, where $\sigma_{\nu_0} = \sigma \otimes q^{\langle \nu_0, H_M() \rangle}$. It will therefore be enough to determine when one has a pole at $\nu = 0$. There are certain cases that one can dispose of immediately. The operator $A(\nu, \sigma, w_0)$ has a pole at $\nu = 0$ only if $\widetilde{w}_0(\theta) = \theta$ and $w_0(\sigma) \cong \sigma$. Thus from now on we shall assume $\mathrm{Ad}(w_0) : M \to M$, sending $\mathrm{Ad}(w_0) : A \to A$, and that $w_0(\sigma) \cong \sigma$.

It is then clear that $\mathbf{N}_{\widetilde{w}_0} = w_0 \mathbf{N}^- w_0^{-1} = \mathbf{N}$. By Lemma 4.1 of [34], it is enough to determine the pole of

$$\int_N h(w_0^{-1} n)\, \mathrm{d}n \tag{2.1}$$

at $\nu = 0$ for any $h$ in $V(\nu, \sigma)$ which is supported in $PN^-$, the open cell.

Given $m \in M$, from now on let $w_0(m) = w_0^{-1} m w_0$, i.e., $w_0(m) = \mathrm{Ad}(w_0)(m)$.

By the assumption on the support of $f$ we only need to integrate over part of $N$ for which $w_0^{-1} n \in PN^-$.

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Given $n_i \in N$, where at the moment $i$ is just an index to signify a specific element of $N$, for which $w_0^{-1}n_i \in PN^-$, write

$$w_0^{-1}n_i = p_in_i^- = m_in_i'n_i^-, \tag{2.2}$$

where $m_i \in M$, $n_i' \in N$, and $n_i^- \in N^-$.

Let $M$ act on $N$ and $N^-$ by adjoint action. Let $\mathrm{Cent}_M(n_i) = M_{n_i}$ be the centralizer of $n_i$ in $M$, i.e., $M_{n_i} = \{m \in M \mid \mathrm{Ad}(m)n_i = n_i\}$. Denote by $M_{n_i^-} = \mathrm{Cent}_M(n_i^-)$ and $M_{n_i'} = \mathrm{Cent}_M(n_i')$ centralizers of $n_i^-$ and $n_i'$ in $M$, respectively. They are all stabilizers of the action of $M$ at these points and, moreover, $\mathbf{M}_{n_i}(F) = M_{n_i}$ and so on, where $\mathbf{M}_{n_i}$ is the centralizer of $n_i$ in $\mathbf{M}$. Finally let $M_{m_i}^t = \mathrm{Cent}_M^t(m_i)$ be the twisted (by means of $w_0$) centralizer of $m_i$ in $M$; simply $M_{m_i}^t = \{m \in M \mid w_0(m)m_im^{-1} = m_i\}$ Again $\mathbf{M}_{m_i}^t(F) = M_{m_i}^t$. We start with the following lemma.

LEMMA 2.1. (a) *The groups* $\mathbf{M}_{n_i}$, $\mathbf{M}_{n_i^-}$, *and* $\mathbf{M}_{n_i'}$ *are all equal and are all contained in* $\mathbf{M}_{m_i}^t$.

(b) *Assume* $\mathbf{N}$ *is Abelian and therefore the adjoint action of* $\mathbf{M}$ *on* $\bar{\mathfrak{n}}$, *the Lie algebra of* $N$, *has a unique Zariski-dense* $\mathbf{M}$*-orbit* $\mathrm{Ad}(\mathbf{M})n_i$ (cf. [26, 39]), *then* $[\mathbf{M}_{m_i}^t : \mathbf{M}_{n_i}]$ *is finite and independent of* $n_i$ *in its orbit. Similar statements are true for* $[M_{m_i}^t : M_{n_i}]$.

*Proof*. The decomposition (2.2) determines $p_i$, $m_i$, $n_i'$, and $n_i^-$ uniquely as a function of $n_i$. Thus $\mathbf{M}_{n_i} \subset \mathbf{M}_{n_i^-}$, $\mathbf{M}_{n_i} \subset \mathbf{M}_{n_i'}$, and finally $\mathbf{M}_{n_i} \subset \mathbf{M}_{m_i}^t$, using

$$w_0^{-1}mn_im^{-1} = (w_0(m)p_im^{-1})(mn_i^-m^{-1}),$$

$m \in \mathbf{M}$ with $mn_im^{-1} \in N$.

Now write $n_i^- = p_i^{-1}w_0^{-1}n_i$ and apply the same uniqueness argument to get $\mathbf{M}_{n_i^-} \subset \mathbf{M}_{n_i}$ and so on, completing the proof of part (a).

For part (b) assume $[\mathbf{M}_{m_i}^t : \mathbf{M}_{n_i}]$ is not finite. Choose $X_i \in \mathfrak{n}$ such that $n_i = \exp(X_i)$. The orbit $\mathrm{Ad}(\mathbf{M})X_i$ is Zariski-open dense in $\bar{\mathfrak{n}}$ and may be realized as $\mathbf{M}/\mathbf{M}_{n_i}$. The quotient $\mathbf{M}_{m_i}^t/\mathbf{M}_{n_i}$ then imbeds in $\bar{\mathfrak{n}}$ through $\mathrm{Ad}(\mathbf{M}_{m_i}^t)X_i$. Since $\mathbf{M}_{m_i}^t$ is an algebraic group, dimension of $\mathbf{M}_{m_i}^t/\mathbf{M}_{n_i}$ must be positive. Choose a line $tX$ in $\bar{\mathfrak{n}}$ with an infinite intersection with $\mathbf{M}_{m_i}^t/\mathbf{M}_{n_i} = \mathrm{Ad}(\mathbf{M}_{m_i}^t)X_i$. We may and will assume $X \in \mathbf{M}_{m_i}^t/\mathbf{M}_{n_i}$. Take $a \in \mathbf{A}$ such that $\alpha(a) = t$ and $tX \in \mathbf{M}_{m_i}^t/\mathbf{M}_{n_i}$, where $\alpha$ is the simple root of $\mathbf{A}$ in $\mathfrak{n}$. Clearly $m(\exp(tX)) = m_i = m(\exp X)$. But $\exp(tX) = \mathrm{Ad}(a)(\exp X)$ implies that $m_i = \mathrm{Ad}^t(a)m_i = w_0(a)m_ia^{-1}$. We may assume $w_0(a) = a^{-1}$. It is therefore enough that $a^{-2} \neq 1$. But this is clearly possible by the infinitely many choices for $t$ that we have.

Although $\mathrm{Ad}(w_o)$ is an inner automorphism for $\mathbf{G}$, its restriction to $\mathbf{M}$ may be outer. Let $\theta = \mathrm{Ad}(w_0)|\mathbf{M}$. Clearly $\theta$ is a semisimple automorphism of $\mathbf{M}$. It preserves the pair $(\mathbf{B} \cap \mathbf{M}, \mathbf{T})$, as required in [20], since $T = \mathbf{T}(F)$ and $\mathbf{A_0}$ share same Weyl groups. Moreover changing $w_0$ by an appropriate element in $A_0$, we may assume that $\theta$ also fixes the splitting in $\mathbf{U}\cap\mathbf{M}$. Observe that any change in $w_0$ by right translation by an element in $T$ will change $m_i$ in decomposition (2.2). But, up to conjugation by an element of $T$, $\mathbf{M}_{n_i}$ and $\mathbf{M}^t_{m_i}=\mathbf{M}_i$ will remain unchanged. We record this information as:

LEMMA 2.2. *The automorphism $\theta$ of $\mathbf{M}$ preserves the pair $(\mathbf{B}\cap\mathbf{M},\mathbf{T})$ and can be arranged so that it also fixes the splitting in $\mathbf{U}\cap\mathbf{M}$. Given $n_i\in N_i$, the $T$-conjugacy class of the group $\mathbf{M}_i$, the $\theta$-twisted centralizer of $m_i$ in $\mathbf{M}$, is independent of the choice of $w_0$ for $\widetilde{w}_0$ in his class modulo $T$.*

We shall now set out to compute the residue for the pole of (2.1) at $\nu=0$. When $\sigma$ is generic and $\nu=s\widetilde{\alpha}$, this determines the poles of $L(2s,\sigma,r_2)L(s,\sigma,r_1)$. We should point out that knowing the poles of a local $L$-function is equivalent to its full knowledge.

We may assume $h$ is supported in $PN^-$. The main assumption of this paper is that $\mathbf{N}$ is Abelian. Then $\mathbf{M}$ acts on $\overline{\mathfrak{n}}$, the Lie algebra of $\mathbf{N}$, with a Zariski-dense orbit $\overline{O}_0$ (cf. [26, 39]). Thus $\overline{O}_0(F)=\bigcup_i\{O_i\},\ O_i\subset\mathfrak{n}$ and the complement of $\bigcup_i\exp(O_i)$ in $N$ is of measure zero.

Given $n\in N$ with $w_0^{-1}n\in PN^-$, write $w_0^{-1}n=mn'n^-$, $m\in M$, $n'\in N$, and $n^-\in N^-$ as in (2.2). Define

$$\mathrm{d}^*n=q^{\langle\rho,H_M(m)\rangle}\,\mathrm{d}n,\tag{2.3}$$

where as before $\rho=\rho_P$ is half the sum of roots in $N$. We need:

LEMMA 2.3. *Given $i$, let $n_i\in\exp(O_i)$ denote an arbitrary element. Then the measure $d^*n_i$ is an invariant measure on $M/M_{n_i}$ and thus induces one on its quotient $M/M_i$* (Lemma 2.1.b).

*Proof.* Fix $m\in M$. We need to show $\mathrm{d}^*(m^{-1}n_im)=\mathrm{d}^*n_i$. We know that

$$d(m^{-1}n_im)=q^{\langle-2\rho,H_M(m)\rangle}\mathrm{d}n_i.\tag{2.3.1}$$

But

$$w_0^{-1}m^{-1}n_im=w_0(m^{-1})m_im\cdot m^{-1}n_i'm\cdot m^{-1}n_i^-m$$

gives the decomposition (2.2) for $m^{-1}n_im$. Thus

$$d^*(m^{-1}n_im)=q^{\langle\rho,H_M(w_0(m^{-1})m_im)\rangle}d(m^{-1}n_im).\tag{2.3.2}$$

Using the definition of $H_M$ we have:

$$q^{\langle\rho,H_M(w_0(m^{-1})m_im)\rangle}=q^{\langle\rho,H_M(w_0(m^{-1}))\rangle+\langle\rho,H_M(m)\rangle}\cdot q^{\langle\rho,H_M(m_i)\rangle}$$

and

$$q^{\langle 2\rho, H_M(w_0(m^{-1}))\rangle} = \left|\rho^2(w_0(m^{-1}))\right| = |\rho^2(m)| = q^{\langle 2\rho, H_M(m)\rangle},$$

where $\rho^2$ denotes $2\rho$ as a rational character of $\mathbf{M}$. The lemma is now a consequence of (2.3.1) applied to (2.3.2).

Finally given $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, let $\sigma_\nu = \sigma \otimes q^{\langle \nu, H_M(\ )\rangle}$. Then (2.1) can be written as

$$\begin{aligned}
&\sum_i \int_{\exp(O_i)} q^{\langle \rho, H_M(m_i)\rangle} \sigma_\nu(m_i) h(n_i^-)\, \mathrm{d}n_i \\
&\quad = \sum_i \int_{\exp(O_i)} q^{\langle \nu, H_M(m_i)\rangle} \sigma(m_i) h(n_i^-) q^{\langle \rho, H_M(m_i)\rangle}\, \mathrm{d}n_i \qquad (2.4)\\
&\quad = \sum_i \int_{\exp(O_i)} q^{\langle \nu, H_M(m_i)\rangle} \sigma(m_i) h(n_i^-)\, \mathrm{d}^* n_i,
\end{aligned}$$

where $w_0^{-1} n_i = m_i n_i' n_i^-$ according to decomposition (2.2).

Using Lemma 2.3, each $\mathrm{d}^* n_i$ induces a measure $\mathrm{d}\dot{m}$ on $M/M_i$ so that (2.4) can be written as

$$\begin{aligned}
&\sum_i \int_{M/M_i} \sum_{m_0 \in M_i/M_{n_i}} q^{\langle \nu, H_M(w_0(m) m_i m^{-1})\rangle} \times \\
&\quad \times \sigma(w_0(m) m_i m^{-1}) h(m m_0 n_i^- m_0^{-1} m^{-1})\, \mathrm{d}\dot{m}, \qquad (2.5)
\end{aligned}$$

where we have now fixed a representative $n_i$ for each orbit $O_i$. The representatives $m_i$ and $n_i^-$ are defined through decomposition (2.2).

For the purpose of computing the residue we may assume that there exists a Schwartz function $\Phi$ on $\mathfrak{n}^-$, the Lie algebra of $N^-$, such that $h(\exp X) = \Phi(X) h(e)$, where $X \in \mathfrak{n}^-$. Let $n_i^- = \exp X_i^-$, $X_i^- \in \mathfrak{n}^-$. Then

$$m m_0 n_i^- m_0^{-1} m^{-1} = \exp(\mathrm{Ad}(m^{-1})\, \mathrm{Ad}(m_0^{-1}) X_i^-),$$

and therefore (2.5) can be written as

$$\begin{aligned}
&\sum_i \int_{M/M_i} \sum_{m_0} \Phi(\mathrm{Ad}(m^{-1})\, \mathrm{Ad}(m_0^{-1}) X_i^-) q^{\langle \nu, H_M(w_0(m) m_i m^{-1})\rangle} \times \\
&\quad \times \sigma(w_0(m) m_i m^{-1}) h(e)\, \mathrm{d}\dot{m}. \qquad (2.6)
\end{aligned}$$

To study the poles, it is enough to evaluate an arbitrary element $\widetilde{v}$ in the contragredient space of $\sigma$ at (2.6). Given $m \in M$, let $\psi(m) = \langle \sigma(m) h(e), \widetilde{v}\rangle$ be the

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corresponding matrix coefficient. We have therefore arrived at

$$\sum_i \int_{M/M_i} \sum_{m_0} q^{\langle \nu, H_M(w_0(m)m_i m^{-1})\rangle} \times$$
$$\times \Phi(\mathrm{Ad}(m^{-1})\,\mathrm{Ad}(m_0^{-1})X_i^-)\psi(w_0(m)m_i m^{-1})\,\mathrm{d}\dot{m}. \tag{2.7}$$

What we have done up to now has required no use of the fact that $\sigma$ is supercuspidal which we shall invoke next. But it is good to record this as:

PROPOSITION 2.4. *Let $\sigma$ be an irreducible admissible representation of $M$ and assume that $M$ acts on $\mathfrak{n}$ by a finite union of open orbits $\{O_i\}$ which is dense in $\mathfrak{n}$. Then the poles of $A(\nu, \sigma, w_o)$ are the same as those of*

$$\sum_i \int_{M/M_i} \sum_{m_0 \in M_i/M_{n_i}} q^{\langle \nu, H_M(w_0(m)m_i m^{-1})\rangle} \Phi(\mathrm{Ad}(m^{-1})\,\mathrm{Ad}(m_0^{-1})X_i^-)\times$$
$$\times \psi(w_0(m)m_i m^{-1})\,\mathrm{d}\dot{m} \tag{2.4.1}$$

*as $\Phi$ ranges among Schwartz functions on $\mathfrak{n}^-$ and $\psi$ among matrix coefficients for $\sigma$, with absolute convergence for* (2.4.1) *for* $\mathrm{Re}\langle \nu, H_\alpha\rangle$ *sufficiently large.*

Now assume $\sigma$ is supercuspidal. Let $\widetilde{\mathbf{A}}$ be the center of $\mathbf{M}$. Then $\widetilde{\mathbf{A}}^0$ has $\mathbf{A}$ as its split component. Given a matrix coefficient $\psi$, there exists a function $f \in C_c^\infty(M)$ such that $\psi(m) = \int_{\widetilde{A}} f(am)\omega^{-1}(a)\,\mathrm{d}a$, where $\omega$ is the central character of $\sigma$. As a result (2.7), or equally (2.4.1), can now be written as

$$\sum_i \int_{\widetilde{A}} \int_{M/M_i} \sum_{m_0} \omega^{-1}(a) q^{\langle \nu, H_M(w_0(m)m_i m^{-1})\rangle} \Phi(\mathrm{Ad}(m^{-1})\,\mathrm{Ad}(m_0^{-1})X_i^-)\times$$
$$\times f(aw_0(m)m_i m^{-1})\,\mathrm{d}\dot{m}\,\mathrm{d}a. \tag{2.8}$$

Our manipulations being formal up to now will soon be justified.

Under our assumption that $w_0(\sigma) \cong \sigma$, we have $\omega(w_0(a)a^{-1}) = 1$ for all $a \in \widetilde{A}$ and therefore, up to the constant $[\widetilde{A}' \colon Z(G)]^{-1}$ in which $\widetilde{A}'$ is the subgroup of elements of $\widetilde{A}$ fixed by $w_0$, we can further invoke (2.8) as

$$\sum_i \int_{z \in \widetilde{A}/w_0(\widetilde{A})\widetilde{A}^{-1}} \int_{a \in \widetilde{A}/Z(G)} \int_{M/M_i} \sum_{m_0} \Phi(\mathrm{Ad}(m^{-1})\,\mathrm{Ad}(m_0^{-1})X_i^-)\times$$
$$\times q^{\langle \nu, H_M(w_0(m)m_i m^{-1})\rangle} \cdot f(zw_0(am)m_i(am)^{-1})\omega^{-1}(z)\,\mathrm{d}\dot{m}\,\mathrm{d}a\,\mathrm{d}z. \tag{2.9}$$

Here $Z(G)$ is the center of $G$. Changing $m$ to $a^{-1}m$, (2.9) can now be written as

$$\sum_i \int_{M/M_i} \int_{z \in \widetilde{A}/w_0(\widetilde{A})\widetilde{A}^{-1}} \theta_\nu(m) f(zw_0(m)m_i m^{-1})\times$$
$$\times q^{\langle \nu, H_M(w_0(m)m_i m^{-1})\rangle} \omega^{-1}(z)\,\mathrm{d}z\,\mathrm{d}\dot{m}, \tag{2.10}$$

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where

$$\theta_\nu(m) = \int_{\widetilde{A}/Z(G)} \sum_{m_0} \Phi(\mathrm{Ad}(am^{-1})\,\mathrm{Ad}(m_0^{-1})X_i^-)q^{\langle \nu, H_M(w_0(a^{-1})a)\rangle}\,\mathrm{d}a. \quad (2.11)$$

It follows from the compactness of support of $f$ that $z$ must belong to a compact subset of $\widetilde{A}/w_0(\widetilde{A})\widetilde{A}^{-1}$. In fact, one may identify $\widetilde{A}/w_0(\widetilde{A})\widetilde{A}^{-1}$ with a disjoint finite union $\bigcup_j a_j Z(G)^0$, where $Z(G)^0$ denotes the $F$–points of the connected component of the center of $G$. Changing $f$ with $R_{a_j} f$ for each $j$, we may assume that $z \in Z(G)^0$. If $z_\ell \in \{w_0(m)m_i m^{-1} | m \in M\} \cap Z(G)^0$, $\ell = 1, 2$, then $z_1$ and $z_2$ are $M - \theta$-conjugate, and consequently $z^2 = 1$, $z = z_1 z_2^{-1}$, using $w_0(z) = z$. It therefore follows that the above intersection is finite and thus $z$ must belong to a compact subset of $Z(G)^0$ and consequently $\widetilde{A}/w_0(\widetilde{A})\widetilde{A}^{-1}$.

On the other hand for each $i$, $m$ must belong to a bounded set (compact if $m_i$ is semisimple) in $M/M_i$ (cf. [1, 29]). Consequently $\mathrm{Ad}(a)X_i^-$ must belong to a compact set in $\mathfrak{n}^-$. Using the compactness of $\widetilde{A}/AZ(G)$, we may assume $a$ is in $A/A \cap Z(G)$. Since orbits of $n_i$ are open, $X_i^-$'s must all be nonzero and therefore $|\alpha(a)|$ or equally $|\rho(a)|$ must be bounded above.

To compute the residue, it would be enough to assume $|\rho(a)|$ is small enough to dispose of $\Phi$ in (2.11). Using the computation in the proof of Lemma 2.3 $q^{\langle \nu, H_M(w_0(a^{-1})a)\rangle} = q^{\langle 2\nu, H_M(a)\rangle}$. The pole therefore comes from

$$\int_{\substack{a \in A/Z(G)\cap A \\ |\rho(a)|<\kappa}} q^{\langle 2\nu, H_M(a)\rangle}\,\mathrm{d}a, \quad (2.12)$$

where $\kappa$ is some real bound.

Suppose $\nu = s\rho$, $s \in \mathbb{C}$. Then (2.12) can be written as

$$\int_{\substack{a \in A/Z(G)\cap A \\ |\rho(a)|<\kappa}} |\rho(a)|^{2s}\,\mathrm{d}a. \quad (2.13)$$

The integral (2.13) converges for $\mathrm{Re}(s) > 0$ and can in fact be computed as a geometric series in $|\rho(a_\alpha)|^{2s}$, where $a_\alpha \in A$, with $|\alpha(a_\alpha)| = q^{-2}$. In particular the pole is simple.

Looking back at (2.10) one can now easily conclude that the residue at $\nu = 0$, i.e. $s = 0$, is proportional to

$$[M_i : M_{n_i}] \sum_i \int_{z \in \widetilde{A}/w_0(\widetilde{A})\widetilde{A}^{-1}} \int_{M/M_i} f(zw_0(m)m_i m^{-1})\omega^{-1}(z)\,\mathrm{d}\dot{m}\,\mathrm{d}z. \quad (2.14)$$

The constant of proportionality depends only on **G** and **M** and in particular is independent of $\sigma$.

From (2.14), it is clear that the residue is a sum of certain integrals of twisted orbital integrals and must be formulated in the language of orbital integrals and

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endoscopy. Given $f \in C_c^\infty(M)$ and $m_0 \in M$, define the $\theta$-twisted orbital integral for $f$ at $m_0$ by

$$\Phi_\theta(m_0, f) = \int_{M/M_{\theta,m_0}} f(\theta(m)m_0m^{-1})\,\mathrm{d}\dot{m}, \tag{2.15}$$

where $M_{\theta,m_0} = M_{\theta,m_0}(F)$ is the $\theta$-twisted centralizer of $m_0$ in $M$. We refer to Lemma 2.2 for the properties of $\theta$. We have now arrived at

THEOREM 2.5. *Assume $\sigma$ is supercuspidal and $w_0(\sigma) \cong \sigma$. The intertwining operator $A(\nu, \sigma, w_0)$ has a (simple) pole at $\nu = 0$ if and only if*

$$\sum_i \int_{\widetilde{A}/w_0(\widetilde{A})\widetilde{A}^{-1}} \Phi_\theta(zm_i, f)\omega^{-1}(z)\,dz \neq 0, \tag{2.5.1}$$

*for some $f \in C_c^\infty(M)$ defining a matrix coefficient of $\sigma$ by descent. Here $m_i$'s correspond to representatives $\{n_i\}$ for the action of $M$ on $N$ through $w_0^{-1}n_i = m_i n_i' n_i^-$.*

COROLLARY 2.6. *Assume $\sigma$ is supercuspidal and $w_0(\sigma) \cong \sigma$. Let $I(\sigma) = I(0, \sigma)$. Then $I(\sigma)$ is irreducible if and only if (2.5.1) is nonzero for some choice of $f \in C_c^\infty(M)$ defining a matrix coefficient of $\sigma$ by descent.*

## 3. The Semisimple Case

Let $\overline{O}$ be the $\mathbf{M}$–orbit $\mathbf{M}/\mathbf{M}_i$ of $m_i$, i.e. the $\mathbf{M}-\theta$-conjugacy class of $m_i$. It will be the same for all $i$ and $\bigcup_i\{m_i\} \subseteq \overline{O}(F)$, where $\{m_i\}$ denotes the $M-\theta$-conjugacy class of $m_i$. Next, using $\widetilde{\mathbf{A}} = \widetilde{\mathbf{A}}^0 Z(\mathbf{G})$, observe that $w_0(\widetilde{\mathbf{A}})\widetilde{\mathbf{A}}^{-1} \subset \mathbf{A}_1$ and in fact $w_0(\widetilde{\mathbf{A}})\widetilde{\mathbf{A}}^{-1} = \mathbf{A}_1$. Here $\mathbf{A}_1$ is the connected component of the subgroup $\widetilde{\mathbf{A}}_1$ of all $z \in \widetilde{\mathbf{A}}$ for which $\theta(z) = z^{-1}$. It then follows from the finiteness of $H^1(Z(\mathbf{G}))$ that $[A_1 : w_0(\widetilde{A})\widetilde{A}^{-1}] < \infty$.

(3.1) *Suppose for each $z \in A_1$ and each $i$, $\{zm_i\} = \{m_j\}$ for some $j$. This is particularly the case if $\bigcup_i\{m_i\} = \overline{O}(F)$.* In fact, if $\bigcup_i\{m_i\} = \overline{O}(F)$, then given $z \in A_1$, choose $z_0 \in A_1$ such that $z = z_0^2$. Then $zm_i = \theta(z_0^{-1})m_i(z_0^{-1})^{-1}$ and therefore $\bigcup_i\{m_i\} = \overline{O}(F)$ implies $\{zm_i\} = \{m_j\}$ for some $j$. Moreover, if $\{zm_i\} = \{zm_j\}$, then $\{m_i\} = \{m_j\}$, and therefore under (3.1) the map $\{zm_i\} \to \{m_j\}$ is one–one and onto. The residue (2.5.1) of Theorem 2.5 can now be written as

$$\sum_{\varepsilon\in A_1/w_0(\widetilde{A})\widetilde{A}^{-1}} \omega_1(\varepsilon) \sum_i \int_{\widetilde{A}/A_1} \Phi_\theta(zm_i, f)\,\omega^{-1}(z)\,\mathrm{d}z, \tag{3.2}$$

where $\omega_1 = \omega|A_1$. Theorem 2.5 can now be stated as follows:

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PROPOSITION 3.1. *Let* $\mathbf{A}_1$ *be the connected component of* $\widetilde{\mathbf{A}}_1 = \{z \in \widetilde{\mathbf{A}} | \theta(z) = z^{-1}\}$. *Let* $\omega_1 = \omega | A_1$. *Assume for each* $z \in A_1$ *and each* $i$, $\{zm_i\} = \{m_j\}$ *for some* $j$. *This is in particular the case if* $\bigcup_i \{m_i\} = \overline{O}(F)$. *Then* $A(\nu, \sigma, w_0)$ *has no pole at* $\nu = 0$ *unless* $w_0(\sigma) \simeq \sigma$ *and* $\omega_1 \equiv 1$. *If* $w_0(\sigma) \simeq \sigma$ *and* $\omega_1 = 1$, *then* $A(\nu, \sigma, w_0)$ *has a (simple) pole at* $\nu = 0$ *if and only if*

$$\sum_i \int_{\widetilde{A}/A_1} \Phi_\theta(zm_i, f)\omega^{-1}(z)\,\mathrm{d}z \neq 0$$

*for some* $f \in C_c^\infty(M)$ *defining a matrix coefficient of* $\sigma$ *by descent.*

In most of the examples we shall encounter, $\mathbf{A}_1 = \widetilde{\mathbf{A}}^0$ which forces $\mathbf{G}$ to be semisimple, and conversely. The following corollary will then cover those cases. The corollary is quite important and will be referred to on several occasions, particularly in connection with endoscopy.

Observe that $A_1/w_0(\widetilde{A})\widetilde{A}^{-1} \simeq F^*/(F^*)^2$.

COROLLARY 3.2. *Assume* $\widetilde{\mathbf{A}}^0 = \mathbf{A}_1$. *Then* $A(\nu, \sigma, w_0)$ *has no pole at* $\nu = 0$ *unless* $w_0(\sigma) \simeq \sigma$ *and thus* $\omega^2 = 1$.

(a) *Assume* $w_0(\sigma) \simeq \sigma$ *and* $\omega_1 = \omega | A_1 \equiv 1$. *Let* $\overline{f} \in C_c^\infty(M)$ *be defined by* $\overline{f}(m) = \sum_{\widetilde{A}/A_1} f(zm)\omega^{-1}(z)$. *Then* $A(\nu, \sigma, w_0)$ *has a (simple) pole at* $\nu = 0$ *if and only if*

$$\sum_i \sum_{\varepsilon \in F^*/(F^*)^2} \Phi_\theta(\varepsilon m_i, \overline{f}) \neq 0 \tag{3.2.1}$$

*for some* $f \in C_c^\infty(M)$ *defining a matrix coefficient of* $\sigma$ *by descent.*

(b) *Suppose assumption* (3.1) *holds and* $\omega_1 \neq 1$. *Then* $A(\nu, \sigma, w_0)$ *has no poles at* $\nu = 0$.

COROLLARY 3.3. *Assume* $\widetilde{\mathbf{A}}^0 = \mathbf{A}_1$ *and* $w_0(\sigma) \simeq \sigma$ *leading to* $\omega^2 = 1$.

(a) *Suppose* $\omega_1 = \omega | A \equiv 1$. *Then* $I(\sigma) = I(0, \sigma)$ *is irreducible if and only if* (3.2.1) *is non–zero for some* $f \in C_c^\infty(M)$ *defining a matrix coefficient of* $\sigma$ *by descent.*

(b) *Assume* (3.1) *holds and* $\omega_1 \neq 1$. *Then* $I(\sigma)$ *is reducible.*

## 4. Connection with Endoscopy

Let $\mathbf{G}$ be a quasisplit connected reductive group over $F$. Throughout this section we shall freely use notation and results from [20] and [37] as well as [21] and [24]. Let $(\mathbf{B}, \mathbf{T})$ be a pair in $\mathbf{G}$, where $\mathbf{B}$ is a Borel subgroup with a maximal torus $\mathbf{T}$. Let $\theta$ be an automorphism of $\mathbf{G}$ fixing $(\mathbf{B}, \mathbf{T})$, i.e., $\theta(\mathbf{B}) = \mathbf{B}$ and $\theta(\mathbf{T}) = \mathbf{T}$. The group $\mathbf{G}$ being quasisplit, has an $F$-splitting. More precisely, there exists a collection $\{X\}$ of root vectors, one for each simple root of $\mathbf{T}$ in $\mathbf{B}$, such that the

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triple $(\mathbf{B}, \mathbf{T}, \{X\})$ is preserved by $\Gamma = Gal(\overline{F}/F)$. The automorphisms of $\mathbf{G}$ which preserve $(\mathbf{B}, \mathbf{T}, \{X\})$ then split the exact sequence

$$1 \to \mathrm{Int}(\mathbf{G}) \to \mathrm{Aut}(\mathbf{G}) \to \mathrm{Aut}(\mathbf{G}, \mathbf{B}, \mathbf{T}, \{X\}) \to 1.$$

We shall finally assume that $\theta$ preserves the splitting $(\mathbf{B}, \mathbf{T}, \{X\})$.

Let $(\widehat{G}, \rho, \eta_{\mathbf{G}})$ be a $L$–group data for $\mathbf{G}$. Then $\widehat{G}$ is a connected reductive group over $\mathbb{C}$, $\rho$ is an $L$–action of $\Gamma$ on $\widehat{G}$, and $\eta_{\mathbf{G}}\colon \Psi(\mathbf{G})^{\vee} \to \Psi(\widehat{G})$ is a $\Gamma$-bijection between canonical based root data (cf. [20]). The automorphism $\theta$ of $\mathbf{G}$ induces bijections $\theta\colon \Psi(\mathbf{G}) \to \Psi(\mathbf{G})$ and $\theta^{\vee}\colon \Psi(\mathbf{G})^{\vee} \to \psi(\mathbf{G})^{\vee}$. Let $\widehat{\theta}$ be an automorphism of $\widehat{G}$ which induces the bijection $\eta_{\mathbf{G}} \cdot \theta^{\vee} \cdot \eta_{\mathbf{G}}^{-1}$ on $\Psi(\widehat{G})$. Let $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ be a $\Gamma$-splitting of $\widehat{G}$ which we assume is preserved by $\hat{\theta}$. There is no harm in assuming $\mathcal{T} = \widehat{T}$ and we in fact will.

For the purpose of this discussion, we may assume $\mathbf{G}$ is simply connected. Then $\mathbf{G}^{\theta}$ and $\mathbf{T}^{\theta}$, i.e., the subgroups of $\mathbf{G}$ and $\mathbf{T}$, whose elements are fixed by $\theta$, are connected. Otherwise we need to take $(\mathbf{G}^{\theta})^0$ and $(\mathbf{T}^{\theta})^0$. If $R(\mathbf{G}, \mathbf{T})$ is the set of roots of $\mathbf{T}$ in $\mathbf{G}$, let $R_{\mathrm{res}}(\mathbf{G}, \mathbf{T}) = \{\alpha_{\mathrm{res}} = \alpha | \mathbf{T}^{\theta}; \alpha \in R(\mathbf{G}, \mathbf{T})\}$. Then by (1.3.4) of [20], the set of *indivisible* roots in $R_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$ coincides with $R(\mathbf{G}^{\theta}, \mathbf{T}^{\theta})$. Similarly we have $R_{\mathrm{res}}(\widehat{G}, \widehat{T})$ which can be identified with $\{(\alpha^{\vee})_{\mathrm{res}} = \alpha^{\vee} | \widehat{T}^{\hat{\theta}}; \alpha^{\vee} \in R^{\vee}(\mathbf{G}, \mathbf{T})\}$ since $\widehat{T}^{\hat{\theta}}$ is connected, $\widehat{G}$ being adjoint and $\hat{\theta}$ preserving a splitting. Here $R^{\vee}(\mathbf{G}, \mathbf{T})$ is the set of coroots of $\mathbf{T}$ in $\mathbf{G}$. Observe that by (1.3.8) of [20] $\alpha_{\mathrm{res}} \mapsto (\alpha^{\vee})_{\mathrm{res}}$ is a well-defined $\Gamma$-bijection between $R_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$ and $R_{\mathrm{res}}(\widehat{G}, \widehat{T})$.

Let $s = 1$ and let $\widehat{H}$ be the identity component of $\mathrm{Cent}_{\hat{\theta}}(1, \widehat{G})$ as in (2.1) of [20]. Then ${}^L H = \hat{H} \propto W_F$ is an $L$-group, $L$-embedded by inclusion in ${}^L G = \hat{G} \propto W_F$. We now refer to [37], where this particular case of twisted endoscopy, as called appropriately by Shelstad the 'basic endoscopic data', is studied in detail. Being an appendix to our paper, we shall freely refer to its definitions and results. In particular, we define:

DEFINITION 4.1. The group $\mathbf{H}$ whose $L$-group ${}^L H = \mathrm{Cent}_{\hat{\theta}}(1, \hat{G})^o \propto W_F$ is $L$-embedded by inclusion in ${}^L G$ is called the basic endoscopic group attached to $(\mathbf{G}, \theta)$.

Next we reformulate the discussion on transfer in [37] as

ASSUMPTION 4.2. *Let* $\mathbf{G}$ *and* $\theta$ *be as in Lemma* 4.1. *Let* $\mathbf{H}$ *be the basic endoscopic group attached to* $(\mathbf{G}, \theta)$. *Given* $f \in C_c^{\infty}(G)$, *there exists a function* $f^{\mathbf{H}} \in C_c^{\infty}(H)$ *such that*

$$\Phi_{\theta}^{\mathrm{st}}(\gamma, f) = \Phi^{\mathrm{st}}(\delta, f^{\mathbf{H}}), \tag{4.2.1}$$

*for every strongly* $\theta$*-regular* $\theta$*-semisimple* $\gamma \in \mathbf{G}(F)$ *if* $\delta \in H$ *is the norm of* $\gamma$, *and* $\Phi^{st}(\delta, f^{\mathbf{H}}) = 0$ *otherwise. Here* $\Phi_{\theta}^{st}(\gamma, f) = \sum_{\gamma' \sim \gamma} \Phi_{\theta}(\gamma', f)$, *where* $\gamma'$ *runs*

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*over representatives for all $\theta$-conjugacy classes $\{\gamma'\}$ which lie inside the stable $\theta$-conjugacy class of $\gamma$. Similarly are the stable ordinary orbital integrals $\Phi^{\mathrm{st}}(\delta, f^{\mathbf{H}})$ defined.*

We refer to paragraph (5.5) and Section 3 of [20] as well as all of [37] for the definition of norm and the detailed discussion of matching stated above.

We continue with the assumption that $\theta$ preserves the $F$-splitting $(\mathbf{B}, \mathbf{T}, \{X\})$. Moreover, we assume for a moment that $\mathbf{G}$ is simply connected. Then $\mathbf{G}^\theta$ and $\mathbf{T}^\theta$ are both connected. Since $\theta$ preserves $\{X\}$, simple roots in $R(\mathbf{B}^\theta, \mathbf{T}^\theta)$ are exactly the restriction to $\mathbf{T}^\theta$ of simple roots in $R(\mathbf{B}, \mathbf{T})$, identifying the Weyl group $\Omega(\mathbf{G}^\theta, \mathbf{T}^\theta)$ with $\Omega(\mathbf{G}, \mathbf{T})^\theta$. Consequently $\mathbf{G}^\theta$ has the largest dimension among those fixed by automorphisms in the class of $\theta$ in $\mathrm{Aut}(\mathbf{G})/\mathrm{Int}(\mathbf{G})$ which preserve the pair $(\mathbf{B}, \mathbf{T})$. In conclusion $\mathbf{G}^\theta$ ($\widehat{G}^{\hat{\theta}}$, respectively) which is the $\theta$-twisted centralizer of 1 in $\mathbf{G}$, a connected group, has the largest dimension (as a group over $\overline{F}$) for such $\theta$'s ($\hat{\theta}$'s, respectively) in their class preserving the pair $(\mathbf{B}, \mathbf{T})$. One can in fact remove the assumption that $\mathbf{G}$ is simply connected and conclude the same statement about the dimension of $\mathbf{G}^\theta$.

In particular, in the notation of Corollary 3.2, if it happens that for some (and thus all) $i$, $\mathbf{M}_i$ has the largest possible dimension for the elements in the class of $\theta$ which preserve $(\mathbf{B} \cap \mathbf{M}, \mathbf{T})$, where $\theta$ is assumed to fix an $F$-splitting, then the $\mathbf{M} - \theta$-conjugacy class of $m_i$ intersects the center of $\mathbf{M}$. In fact, let $t_i \in \mathbf{T}$ lie in this conjugacy class (cf. Lemma 3.2.A of [20] since $m_i$'s are $\theta$-semisimple). The $\theta$-twisted centralizer $\mathbf{M}^t_{t_i}$ of $t_i$ is isomorphic (over $\overline{F}$) with $\mathbf{M}_i = \mathbf{M}^t_{m_i}$. Since $\mathbf{M}^0_i = (\mathbf{M}^\theta)^0$, we will see that $\mathbf{M}^t_{t_i}$ has the same dimension as $\mathbf{M}^\theta$. But $\mathbf{M}^t_{t_i}$ is the fixed point set of $\mathrm{Int}\,(t_i) \circ \theta$ and for it to have the largest dimension which is that of $M^\theta$, $\mathrm{Int}\,(t_i)$ must be trivial. Now, multiplying $w_0$ by a central element if necessary, we may assume that the $\mathbf{M} - \theta$-conjugacy class of $m_i$ intersects $\tilde{\mathbf{A}}^0$. It therefore follows that $\bigcup_i \{m_i\} \subset \overline{\{1\}}(F)$, where $\overline{\{1\}}(F)$ is the $F$-rational points of $\overline{\{1\}}$.

Now assume

$$\bigcup_{\varepsilon} \bigcup_{i} \{\varepsilon m_i\} = \{\overline{1}\}(F), \quad \varepsilon \in F^*/(F^*)^2.$$

Moreover, assume $M^0_i = M^0_1$ for all $i$. We shall now reformulate Lemma 9 of [37] in our notation as follows:

PROPOSITION 4.3. *Suppose Assumption* 4.2 *is satisfied for the group* $\mathbf{M}$ *and the automorphism* $\theta$ *of* $\mathbf{M}$, *where* $\mathbf{G}$ *and* $\theta$ *are as in Corollary* 3.2. *In particular* $\mathbf{A}_1 = \widetilde{\mathbf{A}}^0$. *Assume further that*

$$\bigcup_{\varepsilon} \bigcup_{i} \{\varepsilon m_i\} = \overline{\{1\}}(F) \quad \textit{and} \quad M^0_i = M^0_1 \quad \textit{for all} \ \ i.$$

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*Let* $\mathbf{H}$ *be the basic endoscopic group attached to* $(\mathbf{M}, \theta)$. *Given* $f \in C_c^\infty(M)$, *define* $\overline{f}$ *as in Corollary* 3.2 *and let* $\overline{f}^{\mathbf{H}} \in C_c^\infty(H)$ *be as in Assumption* 4.2. *Then* $\sum_{\varepsilon\in F^*/(F^*)^2}\sum_i \Phi_\theta(\varepsilon m_i, \overline{f}) \neq 0$, *if and only if* $\overline{f}^H(1) \neq 0$.

DEFINITION 4.4. *Suppose* $\mathbf{G}$, $\mathbf{M}$ *and* $\theta$ *are as in Proposition* 4.3,

$$\bigcup_\varepsilon\bigcup_i\{\varepsilon m_i\} = \{\bar{1}\}(F), \qquad \mathbf{M}_i^0 = \mathbf{M}_1^0 \quad \text{for all } i,$$

*and that Assumption* 4.2 *is valid for* $\mathbf{M}$ *and* $\theta$. *Let* $\sigma$ *be an irreducible supercuspidal representation of* $M$ *such that* $w_0(\sigma) \cong \sigma$. *Assume* $\omega_1 = \omega|A_1 \equiv 1$. *We shall say* $\sigma$ *comes from* $H = \mathbf{H}(F)$ *by* $\theta$*-twisted endoscopic transfer, where* $\mathbf{H}$ *is the basic endoscopic group attached to* $(\mathbf{M}, \theta)$, *if there exists a function* $f \in C_c^\infty(M)$, *defining a matrix coefficient of* $\sigma$ *by descent, for which* $\overline{f}^H(1) \neq 0$.

We can now reformulate our result in the language of endoscopy as follows:

THEOREM 4.5. *Suppose* $\mathbf{G}$, $\mathbf{M}$, *and* $\theta$ *are as in Proposition* 4.3,

$$\bigcup_\varepsilon\bigcup_i\{\varepsilon m_i\} = \{\bar{1}\}(F), \qquad \mathbf{M}_i^0 = \mathbf{M}_1^0 \quad \text{for all } i,$$

*and that Assumption* 4.2 *is valid for* $\mathbf{M}$ *and* $\theta$. *Let* $\sigma$ *be an irreducible supercuspidal representation of* $M$ *and that* $w_0(\sigma) \cong \sigma$. *Then* $\omega^2 = 1$. *Suppose* $\omega_1 = \omega|A_1 \equiv 1$. *Then* $I(\sigma)$ *is irreducible if and only if* $\sigma$ *comes from* $H = \mathbf{H}(F)$ *by* $\theta$*-twisted endoscopic transfer.*

## 5. Examples and Connection with Prehomogeneous Vector Spaces

In this section we shall produce a good number of interesting examples where the situation of Theorem 2.5 happens. Most cases fall into the setting of Corollary 3.3 and Proposition 4.3 (under Assumption 4.2). Let us start with the most well known of all cases, the case of $\mathrm{GL}_n$ and reprove Olšanskiĭ's result [28]. The proof is remarkably simple and beautiful, and recaptures the inverse of the formal degree as the residue of the intertwining operator at $\nu = 0$.

PROPOSITION 5.1. *Fix positive integers* $m$, $n$ *and let* $\sigma_1$ *and* $\sigma_2$ *be irreducible unitary supercuspidal representations of* $\mathrm{GL}_m(F)$ *and* $\mathrm{GL}_n(F)$, *respectively. Let* $\sigma = \sigma_1 \otimes \sigma_2$. *Then* $w_o(\sigma) \cong \sigma$ *if and only if* $m = n$ *and* $\sigma_1 \cong \sigma_2$. *Assume* $m = n$ *and* $\sigma_1 \cong \sigma_2$. *Then* $A(\nu, \sigma, w_0)$ *always has a (simple) pole at* $\nu = 0$. *The residue (2.5.1) is proportional to the inverse of the formal degree of* $\sigma$.

*Proof.* We need to consider the Levi $\mathrm{GL}_n \times \mathrm{GL}_n$ inside $\mathrm{GL}_{2n}$. Let $h_1$ and $h_2$, $h_i \in C_c^\infty(\mathrm{GL}_n(F))$, define matrix coefficients of $\sigma_1$ and $\sigma_2$, respectively. Set $h =$

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$h_1 \otimes h_2$. Clearly $\mathbf{A} = \widetilde{\mathbf{A}} = \overline{F}^* \times \overline{F}^*$. Moreover $\mathbf{N} = M_n$ and there is one open orbit coming from $m_1 = I_n \in M_n(F)$. By Theorem 2.5, we need to calculate

$$\int_{A/w_0(A)A^{-1}} \Phi_\theta(z,h)\omega^{-1}(z)\,\mathrm{d}z, \tag{5.1.1}$$

where $\omega = \omega_1 \otimes \omega_2$ with $\omega_i$ the central character of $\sigma_i$, $i = 1, 2$.

Set $\mathbf{M} = \mathrm{GL}_n \times \mathrm{GL}_n$ and denote by $\mathbf{M}_1$ the twisted centralizer of $m_1 = I_n = I$ in $\mathbf{M}$. Then $\mathbf{M}_1 = \{(m, m) \mid m \in \mathrm{GL}_n\}$ and therefore $M/M_1$ can be identified with $\mathrm{GL}_n(F)$ which we realize as

$$M/M_1 \cong \{(I, m) \mid m \in \mathrm{GL}_n(F)\}. \tag{5.1.2}$$

Write $A = \{(z_1, z_2) \mid z_i \in Z_n(F) \cong F^*\}$. Then

$$w_0(A)A^{-1} = \{(z, z^{-1}) \mid z \in Z_n(F)\}. \tag{5.1.3}$$

Using identification (5.1.2), (5.1.1) can be written as

$$\begin{aligned}&\int_{\overline{(z_1,z_2)}\in A/w_0(A)A^{-1}} \int_{m\in GL_n(F)} h_1(z_1m)h_2(z_2m^{-1})\times\\ &\quad\times\omega_1^{-1}(z_1)\omega_2^{-1}(z_2)\,\mathrm{d}m\, d\overline{(z_1,z_2)}.\end{aligned} \tag{5.1.4}$$

Breaking the integration over $\mathrm{GL}_n(F)$ to one over $Z_n(F)$ and another over $Z_n(F)\backslash \mathrm{GL}_n(F)$, (5.1.4) equals

$$\begin{aligned}&\int_{\overline{(z_1,z_2)}\in A/w_o(A)A^{-1}} \int_{z\in Z_n(F)} \int_{\dot m\in Z_n(F)\backslash \mathrm{GL}_n(F)} \times\\ &\quad\times h_1(z_1 z m)h_2(z_2 z^{-1}m^{-1})\omega_1^{-1}(z_1)\omega_2^{-1}(z_2)\,\mathrm{d}\dot m\,\mathrm{d}z\, d\overline{(z_1,z_2)}.\end{aligned} \tag{5.1.5}$$

But identifying integration over $Z_n(F)$ with $w_0(A)A^{-1}$ via (5.1.3), one can telescope the first two integrals in (5.1.5) to imply

$$\begin{aligned}&\int_{Z_n(F)\backslash \mathrm{GL}_n(F)} \left(\int_{z_1\in Z_n(F)} h_1(z_1m)\omega_1^{-1}(z_1)\,\mathrm{d}z_1\right)\times\\ &\quad\times\left(\int_{z_2\in Z_n(F)} h_2(z_2m^{-1})\omega_2^{-1}(z_2)\,\mathrm{d}z_2\right)\mathrm{d}\dot m.\end{aligned} \tag{5.1.6}$$

If

$$\langle\sigma_1(m)v_1, \widetilde{v}_1\rangle = \int_{Z_n(F)} h_1(zm)\omega_1^{-1}(z)\,\mathrm{d}z$$

and

$$\langle\sigma_2(m)v_2, \widetilde{v}_2\rangle = \int_{Z_n(F)} h_2(zm)\omega_2^{-1}(z)\,\mathrm{d}z,$$

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denote the corresponding matrix coefficients defined by $h_1$ and $h_2$, respectively, then (5.1.6) can be written as

$$\int_{Z_n(F)\backslash \mathrm{GL}_n(F)} \langle \sigma_1(m)v_1, \widetilde{v}_1\rangle \langle \sigma_2(m^{-1})v_2, \widetilde{v}_2\rangle \, \mathrm{d}\dot{m}, \tag{5.1.7}$$

which is precisely the Schur orthogonality relation of Harish-Chandra [14]. It is simply equal to zero unless $\sigma_1 \cong \sigma_2$ in which case equals $\mathrm{d}(\sigma_1)^{-1}\langle v_1, \widetilde{v}_2\rangle\langle v_2, \widetilde{v}_1\rangle$, where $\mathrm{d}(\sigma_1)$ is the formal degree of $\sigma_1$. Observe that this nonvanishing is precisely equivalent to $w_0(\sigma) \cong \sigma$.

Another reductive case when Theorem 2.5 can be applied is the case of Siegel parabolics for unitary groups. We refer to [9] for this case.

Our remaining examples are taken from the semisimple case to which we can apply Corollary 3.2. Moreover, $\bigcup_\varepsilon \bigcup_i \{\varepsilon m_i\} = \{\overline{1}\}(F)$ and $\mathbf{M}_i^0 = \mathbf{M}_1^0$ for all $i$ in each case (as we verify them individually) and we can therefore apply our results from Section 4. The only exception is the case of Proposition 5.2. We leave out the case of Siegel parabolic for the group $\mathrm{SO}_{2n}$ as it was treated earlier in [34]. With the exception of Proposition 5.2, in each case ${}^L\mathfrak{n} = {}^L\mathfrak{n}_1$ (notation as in first section), i.e., $m = 1$ and $r = r_1$ is irreducible.

There are two new cases of classical groups which fall immediately into this category. We shall treat them first.

Let $\mathbf{G} = \mathrm{SO}_m$ and $\mathbf{M} = \mathrm{GL}_1 \times \mathrm{SO}_{m-2}$. Let $\mathbf{P} = \mathbf{MN}$ be the corresponding standard parabolic subgroup. The $F$-points of $\mathbf{N}$ can be identified with $F^{m-2}$ and $\mathrm{GL}_1(F) \times \mathrm{SO}_{m-2}(F)$ acts on $F^{m-2}$ by $N \ni X \mapsto aXh^{-1}$, $a \in \mathrm{GL}_1(F), h \in \mathrm{SO}_{m-2}(F)$.

We first consider the case of $m = 2n+1$. Then $\mathbf{M} = \mathrm{GL}_1 \times \mathrm{SO}_{2n-1}$ and $\mathbf{M}_D = \mathrm{SO}_{2n-1}$ is adjoint and has no outer automorphisms. In this case $\mathbf{M}_i^0 = \mathrm{SO}(2n-2)$ (cf. [31] and Lemma 2.1(b)) and since $\mathrm{SO}(2n-1)$ has no outer automorphisms, $m_i$'s will not be central and therefore the interpretation in terms of twisted endoscopy given in Theorem 4.5 will not apply. On the other hand the twisted orbital integrals in (2.5.1) now become basically ordinary ones and the nonvanishing condition (5.2.1) of the next proposition may now be handled by ordinary endoscopy [24]. We leave this to a future paper. But when $\sigma$ is generic, i.e. has a Whittaker model, the theory of $L$–functions developed in [33, 36] applies. In fact, in this case $m = 2$ and $L(s, \sigma, r_2) = L(s, \omega^2) = L(s, \mathbf{1})$ which always has a pole at $s = 0$. Thus if $\sigma$ is generic supercuspidal, then $I(\sigma)$ is always irreducible. We reformulate Theorem 2.5 and the above observation here as follows:

PROPOSITION 5.2. (a) *Let* $\mathbf{P} = \mathbf{MN}$ *be the standard parabolic subgroup of* $\mathrm{SO}_{2n+1}$ *whose Levi subgroup* $\mathbf{M} \simeq \mathrm{GL}_1 \times \mathrm{SO}_{2n-1}$. *Let* $\sigma$ *be an irreducible unitary supercuspidal representation of* $M$. *Suppose* $w_0(\sigma) \simeq \sigma$. *Then* $I(\sigma)$ *is irreducible if and only if*

$$\sum_i \sum_{\varepsilon \in F^*/(F^*)^2} \omega(\varepsilon)\Phi_\theta(\varepsilon m_i, f) \neq 0 \tag{5.2.1}$$

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*for some* $f \in C_c^\infty(M)$ *defining a matrix coefficient of* $\sigma$ *by descent.*

(b) *Assume, moreover, that* $\sigma$ *is generic. Then* $I(\sigma)$ *is always irreducible.*

Now suppose $m = 2n$. Then $\mathbf{M} = \mathrm{GL}_1 \times \mathrm{SO}_{2n-2}$. Let $\{\alpha_1, \cdots, \alpha_n\}$ be the simple roots of $SO_{2n}$. Then $\mathbf{M}$ is generated by $\{\alpha_2, \cdots, \alpha_n\}$. Suppose $n$ is even, then $w_0^{\mathbf{G}}(\alpha_i) = -\alpha_i$, $1 \leqslant i \leqslant n$, where $w_0^{\mathbf{G}}$ is the longest element in the Weyl group of $\mathbf{A}_0$ in $\mathbf{G}$. On the other hand, $w_0^{\mathbf{M}}(\alpha_i) = -\alpha_i$, $2 \leqslant i \leqslant n-2$, while $w_0^{\mathbf{M}}(\alpha_{n-1}) = -\alpha_n$ and $w_0^{\mathbf{M}}(\alpha_n) = -\alpha_{n-1}$, where again $w_0^{\mathbf{M}}$ has the same meaning for $\mathbf{M}$. Consequently $w_0(\alpha_i) = \alpha_i, 2 \leqslant i \leqslant n-2$, $w_0(\alpha_{n-1}) = \alpha_n$, and $w_0(\alpha_n) = \alpha_{n-1}$. We therefore conclude that $\theta$ is defined by the graph automorphism of the Dynkin diagram of $\mathrm{SO}_{2n-2}$. The case of odd $n$ leads to a similar result. The graph automorphism is the one which sends $\alpha_{n-1}$ to $\alpha_n$ and vice versa, while fixing other simple roots. (Unless $n = 5$, the Dynkin diagram of $\mathrm{SO}_{2n-2}$ always has a unique nontrivial automorphism.) To apply Theorem 4.5 or Corollary 3.3, we need to assume $\theta$ fixes, say, the standard splitting of $\mathrm{SO}_{2n-2}$. Observe that $\widehat{H} = \mathrm{Cent}_{\hat{\theta}}(1, \widehat{M})^0 = \mathrm{SO}_{2n-3}(\mathbb{C})$ (cf. [15]). Then $\mathbf{H} = \mathrm{Sp}_{2n-4}$.

Next we need to verify other conditions of Theorem 4.5 or Corollary 3.3. We will first show that open orbits of $N$ under the action of $\widetilde{M} = \mathrm{GL}_1(F) \times \mathrm{O}_{2n-2}(F)$ are parametrized by classes $\{\varepsilon\}$ in $F^*/(F^*)^2$, $\varepsilon \in F^*$. With notation as in [35], $N$ is parametrized by pairs $(X, Y)$, $XX' = 2Y$, $X \in F^{2n-2}$, $Y \in F$. Here ${}^tX' = -Xw_{2n-2}$ and for each positive integer $r$, $w_r \in M_r(F)$ has nonzero entries which are equal to 1 only on its second diagonal. The open orbits come from $Y \in F^*$ and from now on we assume $Y \neq 0$. If $(X, Y)$ and $(X_1, Y_1)$ are in the same orbit of $\widetilde{M}$, then $Y$ and $Y_1$ must be in the same class $\{\varepsilon\}$. Now suppose $Y = a^2Y_1$, $a \in F^*$. Write $X = aX_1h$ with $h \in \mathrm{GL}_{2n-2}(F)$. Then $XX' = 2Y$ and $X_1X_1' = 2Y_1$ imply $X_1hh'X_1' = X_1X_1'$ and by Witt's Theorem we may assume $h \in O_{2n-2}(F)$. (See Lemma 4.1 of [35]). Since $XX'$ is a regular isotropic quadratic form, it is universal. Therefore any such orbit appears in $N$.

We first show that if $\omega_1 \neq 1$, then $I(\sigma)$ is reducible. Given $n_1 = n(X_1, Y_1)$ and $n_2 = n(X_2, Y_2)$ both in $N$ with $Y_i \in F^*$ and $\varepsilon = Y_2Y_1^{-1}$ nonsquare in $F^*$, choose $m \in \mathbf{M}$ such that $n_2 = mn_1m^{-1}$. Let $m_i \in M$ be as before so that $w_0^{-1}n_i = m_in_i'n_i^-$, $i = 1, 2$. Write $m = \mathrm{diag}(a, m_0, a^{-1})$, $a \in \overline{F}^*$, $m_0 \in \mathrm{SO}_{2n-2}(\overline{F})$. Then $a^2 = \varepsilon$ and $m_2 = \theta(m)m_1m^{-1}$. If $m_1 = \mathrm{diag}(a_1, m_1', a_1^{-1})$, then $m_2 = \mathrm{diag}(\varepsilon^{-1}a_1, \theta(m_0)m_1'm_0^{-1}, \varepsilon a_1^{-1})$ and since $a_1$ and $\varepsilon^{-1}a_1$ have different classes modulo squares in $F^*$, one concludes that $\{m_1\} \neq \{m_2\}$. Soon we will show $\overline{\{1\}}(F) = \bigcup_\varepsilon \{\varepsilon\}$. Since $\bigcup_i \{m_i\} \subset \overline{\{1\}}(F)$, this will imply $\bigcup_i \{m_i\} = \bigcup_\varepsilon \{\varepsilon\}$. Applying Assumption (3.1) this implies that if $\omega_1 \neq 1$, then $I(\sigma)$ is reducible.

As we discussed before open orbits of $N$ under $\widetilde{M}$ are parametrized by hypersurfaces $XX' = \varepsilon Z^2$, $X \in F^{2n-2}$, $Z \in F^*$, $\varepsilon \in F^*/(F^*)^2$, or the $F$-equivalence classes of quadratic forms $Q(X, Z) = XX' - \varepsilon Z^2$ which are quadratic forms in $2n-1$ variables, $\varepsilon \in F^*/(F^*)^2$. Observe that $Q$ is then equivalent to $Q_1(X, Z) = \varepsilon(XX' - Z^2)$, using the equivalence of $XX'$ and $\varepsilon XX'$. Its orthogonal group is split $\mathrm{O}_{2n-1}(F)$. Let us call the corresponding open orbit, the $\varepsilon$-orbit.

Let $(X_0, Z_0)$ be a point in the $\varepsilon$-orbit. Then $Q(X_0, Z_0) = 0$ implies that it is $Q$-isotropic. Observe that if $aX_0h^{-1} = X_0$, $a \in F^*$, $h \in \mathrm{O}_{2n-2}(F)$, then

$$Q(aX_0h^{-1}, Z) = a^2Q(X_0, Za^{-1}) = Q(X_0, Z).$$

Since $X_0X_0' \neq 0$, this implies $a^2 = 1$ or $a = \pm 1$. Thus to determine the $\widetilde{M}$-stabilizer $\widetilde{M}_\varepsilon$ of $X_0$, we need to find $h \in \mathrm{O}_{2n-2}(F)$ such that $(\pm 1, h)X_0 = X_0$. Define $(\pm 1, h)(X, Z) = (\pm Xh^{-1}, Z)$. We shall first find all $h \in \mathrm{O}_{2n-2}(F)$ for which $(1, h)(X_0, Z_0) = (X_0, Z_0)$.

Consider $\mathrm{O}_{2n-1}(F)$ and $\mathrm{O}_{2n-2}(F)$ as orthogonal groups for matrices $\mathrm{diag}(-\varepsilon w_{2n-2}, -\varepsilon)$ and $-\varepsilon w_{2n-2}$, respectively. It then gives an embedding of $\mathrm{O}_{2n-2}(F)$ into $\mathrm{O}_{2n-1}(F)$ as well as $U \subset V$, corresponding quadratic spaces. By an appropriate change of coordinates in $U$, the matrix $-\varepsilon w_{2n-2}$ can be written as $\mathrm{diag}(-\varepsilon, -\varepsilon w_{2n-4}, \varepsilon)$. This gives an embedding of split $\mathrm{O}_{2n-3}(F)$ into $\mathrm{O}_{2n-2}(F)$. Choose $X_0 = (0, \cdots, 0, 1) \in U = F^{2n-2}$. Then identifying $X_0$ with its images under equivalences, $Q(X_0, 1) = 0$. The stabilizer of $X_0$ in $\mathrm{O}_{2n-2}(F)$ is now $\mathrm{O}_{2n-3}(F)$, the split orthogonal group in $2n-3$ variables. Since $h$ and $\{\pm 1\}$ are both in $\mathrm{O}_{2n-3}(F)$, we may disregard $\pm 1$ as the stabilizer will not change. The group $\widetilde{M}_\varepsilon$ is therefore the split orthogonal group in $2n-3$ variables, independent of $\varepsilon$. Observe that $M_i = \widetilde{M}_\varepsilon \cap M = M_1$ for all $i$ and $\varepsilon$. Moreover $\bigcup_i \{m_i\} = \{\bar{1}\}(F)$. In fact the $M-\theta$-conjugacy classes in $\{\bar{1}\}(F)$ are parametrized by elements of

$$\ker(H^1(\mathrm{O}(2n-3)) \to H^1(\mathrm{SO}(2n-2))),$$

where the groups are both split and therefore of highest Witt index. Identifying elements of $H^1(\mathrm{O}(2n-3))$ with equivalence classes of quadratic forms in $2n-3$ variables and those in $H^1(\mathrm{SO}(2n-2))$ with equivalence classes of quadratic forms in $2n-2$ variables with same discriminant but different Witt indices, we see that no form with a Witt index less than $n-1$ can be in the kernel. The corresponding $M-\theta$-conjugacy classes then have split $\mathrm{O}(2n-3)$ as stabilizers and are parametrized by $F^*/(F^*)^2$ to account for different discriminants. It is easily checked that if $\varepsilon = \mathrm{diag}(\varepsilon, I, \varepsilon^{-1})$ is $M-\theta$-conjugate to $\varepsilon' = \mathrm{diag}(\varepsilon', I, \varepsilon'^{-1})$, then $\varepsilon$ and $\varepsilon'$ have the same class modulo $(F^*)^2$. Thus $\{\bar{1}\}(F) = \bigcup_\varepsilon \{\varepsilon\} = \bigcup_i \{m_i\}$. Applying Theorem 4.5 and Corollary 3.3 and taking into account that $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}_1 = \mathrm{GL}_1 \times \{\pm 1\}$ and $\mathbf{A} = \mathbf{A}_1 = \mathrm{GL}_1$, we have

PROPOSITION 5.3. *Let $\mathbf{P} = \mathbf{MN}$ be the standard parabolic subgroup of $\mathrm{SO}_{2n}$ whose Levi subgroup $\mathbf{M} \cong \mathrm{GL}_1 \times \mathrm{SO}_{2n-2}$. Let $\sigma = \omega_1 \otimes \tau$ be an irreducible unitary supercuspidal representation of $M$ whose central character is $\omega = \omega_1 \otimes \omega_\tau$. The representation $I(\sigma)$ is irreducible unless $w_0(\sigma) \cong \sigma$. Suppose $w_0(\sigma) \cong \sigma$. Then $\omega^2 = \omega_1^2 = 1$. Suppose $\omega_1 \neq 1$. Then $I(\sigma)$ is reducible. Assume $\omega_1 = 1$. Then $I(\sigma)$ is irreducible if and only if $\tau$ comes from $\mathrm{Sp}_{2n-4}(F)$ by $\theta$-twisted endoscopic transfer.*

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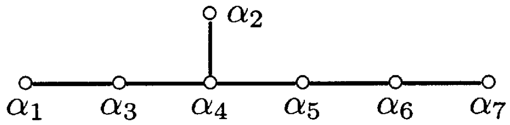
We conclude our examples with the exotic case of an exceptional group. Let $\mathbf{G}$ be an exceptional group of type $E_7$, either simply connected or adjoint. Let $\mathbf{M}$ be the Levi subgroup of $\mathbf{G}$ generated by the roots $\alpha_1, \cdots, \alpha_6$, where the roots are as in the Dynkin diagram of $E_7$ as follows:




In both cases $\mathbf{M} \cong (\mathrm{GL}_1 \times E_6)/\langle \zeta_3 \rangle$, where $\zeta_3$ is a primitive 3rd root of 1 and $\mathbf{M}_D$ is the simply connected $E_6$.

Since $w_0^{\mathbf{G}}(\alpha_i) = -\alpha_i$, $1 \leqslant i \leqslant 7$, while

$$w_0^{\mathbf{M}}(\alpha_1) = -\alpha_6, \qquad w_0^{\mathbf{M}}(\alpha_2) = -\alpha_2, \qquad w_0^{\mathbf{M}}(\alpha_3) = -\alpha_5,$$

$$w_0^{\mathbf{M}}(\alpha_4) = -\alpha_4, \qquad w_0^{\mathbf{M}}(\alpha_5) = -\alpha_3, \quad \text{and} \quad w_0^{\mathbf{M}}(\alpha_6) = -\alpha_1,$$

one has

$$w_0(\alpha_1) = \alpha_6, \qquad w_0(\alpha_2) = \alpha_2, \qquad w_0(\alpha_3) = \alpha_5,$$

$$w_0(\alpha_4) = \alpha_4, \qquad w_0(\alpha_5) = \alpha_3, \qquad w_0(\alpha_6) = \alpha_1.$$

Thus, $\theta$ is defined by the unique nontrivial graph automorphism of the Dynkin diagram of $E_6$. If we again assume that $\theta$ fixes a splitting, then $\widehat{H} = \mathrm{Cent}_{\hat{\theta}}(1, \widehat{M})^0 \cong F_4(\mathbb{C})$ (cf. [15], page 514, or Proposition 47 of [31]). Consequently, $\mathbf{H} = F_4$.

One needs to check the remaining condition of Theorem 4.5. We start with a general discussion. In the general setting of Corollary 3.3 in which $\widetilde{\mathbf{A}}^0 = \mathbf{A}_1$ is the only restriction, we consider the projection $\mathbf{M} \to \mathbf{M}/\widetilde{\mathbf{A}}^0 = \overline{\mathbf{M}}$, where $\widetilde{\mathbf{A}}^0$ is the connected component of $\widetilde{\mathbf{A}}$. We shall *assume* $M/\widetilde{A}^0 = \overline{M} = \overline{\mathbf{M}}(F)$. This is the case if $\widetilde{\mathbf{A}}$ is connected, i.e. $\widetilde{\mathbf{A}} = \widetilde{\mathbf{A}}^0$, using standard lemmas (cf. [22]).

If $m \in M$ and $O_m$ denotes its $M-\theta$-conjugacy class, then $\overline{O}_m$ gives the $\overline{M}-\overline{\theta}$-conjugacy class of $\overline{m}$, where $\overline{O}_m$ and $\overline{m}$ denote images of $O_m$ and $m$ under the projection $M \to \overline{M}$, i.e. $\overline{O}_m = O_{\overline{m}}$. Here $\overline{\theta}$ is the automorphic of $\overline{\mathbf{M}}$ induced from $\theta$. Moreover, for $m_i \in M$, $i = 1, 2$, if $O_{m_1} \cap O_{m_2} \neq \phi$, then $\overline{m}_1 = \overline{m}'_2$ for some $m'_2 \in O_{m_2}$. Similar statements are true for $\mathbf{M}-\theta$-conjugacy classes and those of $\overline{\mathbf{M}} - \overline{\theta}$-classes.

One can check that if $m \in M$, then $\overline{\mathbf{M}^t_m} = \overline{\mathbf{M}}^t_{\overline{m}}$, where $\mathbf{M}^t_m$ and $\overline{\mathbf{M}}^t_{\overline{m}}$ are the $\theta$-twisted and $\overline{\theta}$-twisted centralizers of $m$ and $\overline{m}$ in $\mathbf{M}$ and $\overline{\mathbf{M}}$, respectively.

Suppose $O$ is a $\mathbf{M} - \theta$-conjugacy class in $\mathbf{M}$ and $O(F) = \bigcup_j O_j(F)$, where $O_j(F)$'s are $M - \theta$-conjugacy classes. Let $\overline{O}$ be the image of $O$ and *assume* that $\overline{O}(F)$ consists of a single $\overline{M} - \overline{\theta}$-conjugacy class, then $\overline{O_j(F)} = \overline{O}(F)$ for all $j$. Let $O_1(F) = \{m'_1\}, m'_1 \in M$. Then $\overline{O}(F) = \{\overline{m'_1}\}$. By the previous observations,

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we can choose representatives $m'_j \in O_j(F)$, $j = 2, \ldots$, such that $\overline{m'_j} = \overline{m'_1}$. Thus $m'_j = a_j \cdot m'_1$, $a_j \in A$. Changing $m'_j$ in its orbit, we may assume $m'_j = \varepsilon m'_1$ for some $\varepsilon \in F^*/(F^*)^2$. Thus $O(F) = \bigcup_{\varepsilon \in F^*/(F^*)^2} \{\varepsilon m'_1\}$. Moreover $\overline{\mathbf{M}^t_{m'_j}} = \overline{\mathbf{M}}^t_{\overline{m'_j}} = \overline{\mathbf{M}}^t_{\overline{m'_1}}$ for all $j$.

In the present case of $\mathbf{M} = (\mathrm{GL}_1 \times E_6)/\langle\zeta_3\rangle$ inside $E_7$, $\mathbf{A} = \widetilde{\mathbf{A}}$ and therefore $\overline{\mathbf{M}}$ is adjoint $E_6$. Thus $M/A = \overline{\mathbf{M}} = \overline{M}(F)$. Let $O$ be the $M - \theta$-conjugacy class of $m_1$. The arguments of Sato–Kimura in Proposition 47 and Example 39 of [31] are valid for an algebraically closed $p$-adic field, if one appeals to [17] for minor appropriate changes for $p$–adic fields. The generic stabilizer of $n_1$ must have $F_4$ as its connected component. By part (b) of Lemma 2.1, $\mathbf{M}^0_1 = \mathbf{M}^0_{m_1} = F_4$. As before let $\overline{m}_1 \in \overline{O}$ be the image of $m_1$. By our earlier comments, $\overline{\mathbf{M}^t_{m_1}} = \overline{\mathbf{M}}_j = \overline{\mathbf{M}}^t_{\overline{m}_j}$ will have a connected component of type $F_4$. But $\overline{M}^t_{\overline{m}_1}$ is just the fixed point set of $\mathrm{Int}(\overline{m}_1) \circ \overline{\theta}$ and since it has the largest dimension, it must fix a splitting in $\overline{\mathbf{M}}$. Since $\overline{\mathbf{M}}$ is adjoint, the fixed point set $\overline{M}^t_{\overline{m}_1}$ of such an automorphism must be connected (cf. the discussion at the end of Section 1.1 of [20]). Thus $\overline{\mathbf{M}}^t_{\overline{m}_1} = F_4$. Now using the triviality of $H^1(F_4)$, $F_4$ being simply connected, $\overline{O}(F) = \{\overline{m}_1\}$, the $\overline{M} - \overline{\theta}$-conjugacy class of $\overline{m}_1$. As before $\{\overline{m}_1\} = \{\overline{1}\}$. By the previous discussion

$$O(F) = \bigcup_{\varepsilon \in F^*/(F^*)^2} \{\varepsilon m_1\} = \overline{\{m_1\}}(F) = \{\overline{1}\}(F) = \bigcup_{\varepsilon}\bigcup_{i} \{\varepsilon m_i\}.$$

Here $\overline{\{m_1\}}$ is as earlier just the $\mathbf{M} - \theta$-conjugacy class and does not denote the projection modulo $\mathbf{A}$. Finally observe that $\overline{\mathbf{M}^t_{m_i}} = \overline{\mathbf{M}}_i = \overline{\mathbf{M}}^t_{\overline{m}_i} = \overline{\mathbf{M}}^t_{\overline{m}_1} = F_4$ for all $i$. This implies $\mathbf{M}^0_i = \mathbf{M}^0_1$ for all $i$ as needed. We can now apply Theorem 4.5 to get:

PROPOSITION 5.4. *Let $\mathbf{G}$ be a group of type $E_7$ and let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup whose Levi component has a derived group of type $E_6$. Fix $(\mathbf{B}, \mathbf{T})$ as before and in particular such that $\mathbf{B} \subset \mathbf{P}$ and $\mathbf{T} \subset \mathbf{M}$. Let $\sigma$ be an irreducible unitary supercuspidal representation of $M$ with central character $\omega$. The representation $I(\sigma)$ is irreducible unless $w_0(\sigma) \cong \sigma$. Suppose $w_0(\sigma) \cong \sigma$. Then $\omega^2 = 1$. Assume $\omega = 1$. Then $I(\sigma)$ is irreducible if and only if $\sigma$ comes from $F_4(F)$ by $\theta$-twisted endoscopic transfer. (One expects that if $w_0(\sigma) \cong \sigma$, then $\sigma$ either comes from $F_4(F)$ or $\mathrm{SO}_9(F)$.)*

*Remark.* The stabilizer $\mathbf{M}^\theta$ of $\theta$ in $\mathbf{M}$ is $\{\pm 1\} \cdot F_4$. This follows from paragraph 1.1 of [20] which implies $\mathbf{M}^\theta = \mathbf{A}^\theta(\mathbf{M}^\theta)^0$.

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## 6. *L*-Functions as Igusa Zeta Functions

With notation as in Section 1, let $r$ be the action of ${}^LM$ on ${}^L\mathfrak{n}$ and write $r = \oplus_{i=1}^m r_i$. With the exception of Proposition 5.2, in all the examples of Section 5, $m = 1$, and when $\sigma$ is generic, the $L$-function $L(s, \sigma, \widetilde{r}_1)$ is precisely the normalized inverse polynomial which gives the poles of $A(s\widetilde{\alpha}, \sigma, w_0)$ (Theorem 1.1). Observe that for the case of odd orthogonal groups (Proposition 5.2), the standard $L$–function is always trivial as $L(s, \sigma, \tilde{r}_2) = L(s, \omega^2)$ has always a pole since $\omega^2 = 1$ and the poles of intertwining operators are simple. On the other hand in all our examples, the poles of $A(s\tilde{\alpha}, \sigma, w_0)$ are obtained by integrating an appropriate test function over the union of open orbits of action of $M$ on the $F$-vector space $\mathfrak{n}$, the Lie algebra of $N$. Thus the $L$-function $L(s, \sigma, \widetilde{r}_1)$ is the Igusa zeta-function (cf. [3. 8, 16]) attached to open orbits of adjoint action of $M$ on the prehomogeneous vector space $\mathfrak{n}$ and an appropriate test function (function $\Phi\psi$ in Equation (2.4.1)). We state our result as follows.

THEOREM 6.1. *Let $\sigma$ be an irreducible unitary generic supercuspidal representation of $M$, where $P = MN$ is as in Propositions* 5.1, 5.3 *and* 5.4. *Then $L(s, \sigma, \widetilde{r}_1)$ is the Igusa zeta-function attached to the open orbits of action of $M$ on $\mathfrak{n}$, the Lie algebra of $N$, and an appropriate test function. In each case, the $L$-function is the standard $L$-function. On the other hand, if one is in the situation of Proposition* 5.2, *then* $L(s, \sigma, \widetilde{r}_1) \equiv 1$.


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# Appendix: Basic Endoscopic Data

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**Abstract.** We make some remarks about the simplest example of a set of endoscopic data in general twisted endoscopy. We call it the *basic* set of data. It is associated with purely stable transfer. Our purpose here is simply to describe some immediate consequences and simplifications of the general constructions for this example.

**Mathematics Subject Classifications (1991):** 11F72, 11R34, 22E35, 22E50, 22E55.

**Key words:** twisted endoscopy, orbital integrals, transfer factors.

The ingredients for twisted endoscopy ([KS]) are a connected reductive algebraic group $G$ over a field $F$ (here local non-Archimedean, characteristic zero), an $F$-automorphism $\theta$ of $G$, and a cocycle which we can ignore since our interest is in representations $\pi$ for which $\pi$ is exactly equivalent to $\pi \circ \theta$. Modulo an inner twisting of both automorphism and group (see Section 3.1 of [KS]), we have that $G$ is quasisplit over $F$ and $\theta$ preserves an $F$-splitting of $G$. *These will be our assumptions throughout*, although often they are unnecessarily restrictive or a simple modification yields the general case.

There is a set of endoscopic data attached to $(G, \theta)$ that is *basic* in several ways. First, we expect a stable transfer of orbital integrals, one that is as invariant as possible. Second, the definition of transfer is as simple as possible, the transfer factors being essentially trivial on the most regular elements. At the same time, the construction of transfer factors for a general endoscopic group ([KS]) measures, in a certain sense, the variation from this simple case (see especially the fundamental term $\Delta_{\mathrm{III}}$ in Section 4.4 of [KS]). For the example of cyclic base change for $\mathrm{GL}(n)$ the basic set is essentially the only set of endoscopic data. It also appears significant in applications such as [Sha] which is the motivation for our final observation (Lemma 9).

## 1. Definitions

To form the basic set of endoscopic data for $(G, \theta)$, we start with the strong invariants, that is, the identity component $(G^\wedge)^1$ of the group of invariants, of $\theta^\wedge$ in $G^\wedge$. This group is preserved under the action of the Weil group $W_F$ on $G^\wedge$, because $\theta^\wedge$ is constructed to preserve a $\Gamma$-splitting of $G^\wedge$ and so commutes with the action of $\Gamma$. Further, from the $\Gamma$-splitting of $G^\wedge$ we may construct a $\Gamma$-splitting of $(G^\wedge)^1$, with $\Gamma$ acting by restriction of the action on $G^\wedge$. Namely, for the Borel subgroup in $(G^\wedge)^1$ we take the intersection of the Borel subgroup in $G^\wedge$ with $(G^\wedge)^1$, for the maximal torus in $(G^\wedge)^1$ the intersection of that in $G^\wedge$ with $(G^\wedge)^1$ and then construct the root vectors in the usual way, following Steinberg (see Section 1.1 in [KS]). We have then that ${}^LG^1 = (G^\wedge)^1 \propto W_F$ is an $L$-group, $L$-embedded by inclusion in ${}^LG = G^\wedge \propto W_F$. We will call ${}^LG^1$ the *$L$–group of strong invariants for* $\theta^\wedge$. Let $G_1$ be a dual quasisplit group over $F$. We shall refer to $G_1$ as the *coinvariant* group for $G$.

In the case $G$ is a torus, the coinvariant group $G_1$ is the torus of coinvariants of $\theta$. In general, a maximal torus in $G_1$ is naturally isomorphic to the coinvariants of $\theta$ in a $\theta$–admissible maximal torus in $G$, etc. In some cases, such as cyclic base change, $G_1$ is naturally isomorphic to the (strong) invariants of $\theta$ in $G$, but even in these cases it is convenient to work expressly with the coinvariant group. On the other hand, if $G$ is $\mathrm{GL}(n)$, with $n$ odd, and $\theta$ is transpose-inverse (followed by a suitable inner automorphism, since we are insisting here that $\theta$ preserve an $F$-splitting), then the coinvariant group $G_1$ is symplectic, while the group $G^1$ of strong invariants is special orthogonal, and $G_1$, $G^1$ are of dual type.

The *basic* set of endoscopic data for $(G,\theta)$ is $(G_1, {}^LG^1, \mathrm{id}, \mathrm{incl})$, that is, the tuple consisting of the coinvariant group for $\theta$, the $L$-group of strong invariants for $\theta^\wedge$, the identity element of $G^\wedge$, and the inclusion of the $L$-group of strong invariants for $\theta^\wedge$ in ${}^LG$. The defining properties for a set of endoscopic data ((2.1) in [KS]) are readily verified.

## 2. Relative Transfer Factors

For the basic set of endoscopic data, passage to a $z$-pair as in Section 2.3 of [KS] is unnecessary, because the datum ${}^LG^1$ is an $L$-group. The transfer factor $\Delta(\gamma,\delta)$ is then defined in [KS] for $\gamma$ strongly regular in $G_1(F)$ and $\delta$ strongly $\theta$-regular in $G(F)$. Note that we have replaced *strongly $G$-regular* in [KS] by *strongly regular*. This is allowed by Lemma 2 below. Recall that $\Delta(\gamma,\delta) = 0$ unless $\gamma$ is a norm of $\delta$. We shall start with the canonical *relative* transfer factor attached to two norm pairs.

LEMMA 1. *The relative transfer factors for $G_1$ are trivial, that is $\Delta(\gamma,\delta;\gamma',\delta') = 1$ for all strongly regular $\gamma,\gamma'$ in $G_1(F)$ that are norms of strongly $\theta$-regular $\delta,\delta'$, respectively, in $G(F)$.*

*Proof.* $\Delta$ is the product of four terms, three of which depend on additional choices in general. We will show that in the present case all four terms are equal to 1, whatever those additional choices may be. First, $\Delta_{\mathrm{I}}$ is a quotient, each term of which is defined by a certain pairing in (Abelian) Galois cohomology (see[KS], Section 4.2 for definitions). This amounts to evaluating a multiplicative character at some element in a finite abelian group. For $G_1$, the element is the identity, since our endoscopic datum $s$ is the identity element of $G^\wedge$.

The term $\Delta_{\mathrm{II}}$ is again a quotient, and we use Lemma 4.3.A of [KS] to evaluate each term in this quotient. Observe that every restricted root $\alpha_{\mathrm{res}}$ of types $\mathrm{R}_1$ and

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$R_2$ is *from* $H(= G_1)$, whereas *none* of the restricted roots of type $R_3$ is *from* $H$ (see Section 1.1 of [KS] for a summary of the relevant facts due to Steinberg). We see then from the cited lemma that there are no nontrivial contributions to $\Delta_{\mathrm{II}}$. The same remark about the types of restricted roots also implies that the discriminant term $\Delta_{IV}$ is trivial (see Lemma 4.5.A of [KS]).

We are then left with the one genuinely relative term $\Delta_{\mathrm{III}}$. Because it is not necessary to pass to $z$-pairs we can use the constructions of the first part of Section 4.4 in [KS]. The term $\Delta_{\mathrm{III}}$ is defined by a certain pairing (of Galois hypercohomology classes) and again it is enough to show that one of them, in this case the class $A$ represented by the hypercocycle labelled $(A^{-1}, s_U)$, is the identity element. The element $s_U$ is the identity, because our endoscopic datum $s$ is the identity element in $G^\wedge$. For $A$ we recall the paragraph in [KS] before Lemma 4.4.B. Observe that the $L$-group of strong invariants for $\theta^\wedge$ appears in the construction whatever the endoscopic group $H$, and $A$ measures how embeddings in ${}^LG$ of $L$-groups of maximal tori in $H$ differ from those of the $L$-groups of (isomorphic) maximal tori in $G_1$. Following the actual construction shows that for the basic set of endoscopic data each $A(w)$, $w \in W_F$, is the identity element. Note that the last datum, the inclusion homomorphism in ${}^LG$ of the $L$-group of strong invariants for $\theta^\wedge$, is significant here.

This completes the proof of the lemma.

## 3. Transfer Factors

The transfer factor $\Delta(\gamma, \delta)$ for $G_1$ may now be normalized so that $\Delta(\gamma, \delta) = 1$ if $\gamma$ is strongly regular and a norm of strongly $\theta$-regular $\delta$, and $\Delta(\gamma, \delta) = 0$ if $\gamma$ is strongly regular and not a norm of strongly $\theta$-regular $\delta$ (see Section 5.1 of [KS]).

Before continuing, we record the following:

LEMMA 2. *A strongly regular element in $G_1(F)$ is strongly $G$-regular.*

*Proof.* This is a supplement to Lemma 3.3.C of [KS]. We use the notation from that lemma without further explanation. We assume that the element $\gamma$ is strongly regular in $H(F) = G_1(F)$ but not strongly $G$-regular. Then there is an element, say $\omega$, of the Weyl group $\Omega^\theta(G, T)$ realized in $\mathrm{Cent}_\theta(\delta^*, G)$. Recall that in the present setting we have $G = G^*$, $\theta = \theta^*$; the element $\delta^*$ in the $\theta$-admissible maximal torus $T$ is not, however, to be identified with $\delta$, the given element with norm $\gamma$ (the definition of *norm* in Section 3.3 of [KS] extends naturally to strongly regular $\gamma$). But any element of $\Omega^\theta(G, T)$ is realized in the $\theta$-invariants. A short calculation then shows that $\omega(\delta^*) \equiv \delta^*(\mathrm{mod}(1-\theta)\mathrm{T})$. This then implies that $\omega(\gamma) = \gamma$. That is impossible because $\Omega(H, T_H)$, a subgroup of $\Omega^\theta(G, T)$ under our various identifications, coincides with $\Omega^\theta(G, T)$ in the case $H$ is $G_1$ (see Section 1.1 of [KS]). Thus $\gamma$ is strongly $G$-regular and the assertion of the lemma is proved.

*Remark* 1. The lemma is true for any set of endoscopic data that is *large* in the sense of Remark 2 below.

## 4. Norms in $G_1$

The definition of norm in [KS] does not guarantee that a strongly $\theta$-regular element has a norm in a given endoscopic group. However we do have the following:

LEMMA 3. *Every strongly $\theta$-regular element in $G(F)$ has a (strongly regular) norm in $G_1(F)$.*

*Proof.* We return to Lemma 3.3.B of [KS] in which a maximal torus $T_H$ over $F$ in an endoscopic group $H$ is shown to embed over $F$ as the coinvariants $T_\theta$ in some $\theta$-admissible maximal torus over $F$ in $G$. What we need to show now is that:

(a) given $\delta$ strongly $\theta$-regular in $G(F)$ $(= G^*(F))$ there is a $\theta$-admissible maximal torus $T$ over $F$ in $G$ and an element $\delta^*$ in $T(\overline{F})$ such that $\delta^*$ is $\theta$-conjugate to $\delta$ and the image of $\delta^*$ in $T_\theta$ is $F$-rational, i.e. $\sigma(\delta^*) \equiv \delta^* \mathrm{mod}(1-\theta)T$, $\sigma \in \Gamma$, and

(b) there is a maximal torus $T_H$ over $F$ in $H$ which embeds, in the manner of the lemma, as the coinvariants of the maximal torus $T$ given in (a).

Then $\delta$ evidently has a norm in $T_H(F)$, completing the proof of the lemma.

To prove (a) we first choose an arbitrary $\theta$-stable pair $(B', T')$ in $G$ with $T'$ defined over $F$ and then take $g \in G(\overline{F})$ such that $(B(\delta), T(\delta))^g = (B', T')$, where on the left we have chosen some $\mathrm{Int}(\delta) \circ \theta$-stable pair. Observe that $\delta' = g^{-1}\delta\theta(g)$ lies in $T'(\overline{F})$ and that for any $\sigma$ in $\Gamma$ we have $\sigma(g)^{-1}g$ normalizes $T'$ and $(T')^\theta$, and acts as an element $\omega_\sigma$ of $\Omega^\theta(G, T')$ such that $\sigma(\delta') = \omega_\sigma(\delta')$ and $\sigma(\varepsilon) = \omega_\sigma(\varepsilon)$ for $\varepsilon$ in $g^{-1}G^{\delta\theta}(F)g \subset T'(\overline{F})^\theta$. We can then apply the usual argument with Steinberg's Theorem (on rational elements in semisimple conjugacy classes in a simplyconnected quasisplit group) to get $h$ in $G^\theta(\overline{F})$ such that $\sigma(h)^{-1}h$ normalizes $T'$ and $(T')^\theta$, and acts as $\omega_\sigma$ on them. We then set $B = hB'h^{-1}$, $T = hT'h^{-1}$ and $\delta^* = h\delta'h^{-1} = h\delta'\theta(h)^{-1}$, and observe that the statement of (2a) is true with these choices.

For (b) we again use Steinberg's Theorem, this time for $H = G_1$ (or, more precisely, its simply-connected cover). To follow the usual argument we need to know that any element of $\Omega^\theta(G, T)$ lies in $\Omega(H, T_H)$, (if $T_H$ is embedded as $T_\theta$ over $\overline{F}$), as is true.

This completes the proof of the lemma.

*Remark* 2. The assertion (b), and hence also the lemma, is true for any large set of endoscopic data, by which we mean the Weyl group for $H$ is the full set of $\theta$-invariants in the Weyl group for $G$. If the system of restricted roots associated to $\theta$ is *reduced* then $H$ must be the coinvariant group $G_1$, but in even in the simplest nonreduced example $G = \mathrm{GL}(3)$ with $\theta$ transpose-inverse (followed by an inner autorphism in order to preserve an $F$-splitting), both $G_1 = \mathrm{SL}(2)$ and $H = \mathrm{PGL}(2)$ are attached to large sets of data.

We also note the following simple corollary of Lemma 3.3.B of [KS]:

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LEMMA 4. *Let $T_H$ be a maximal torus over $F$ in $H$. Then the strongly $G$-regular elements in $T_H(F)$ that are norms form the strongly $G$-regular elements in a neighborhood of the identity in $T_H(F)$.*

*Proof.* Here $H$ can be arbitrary but $(G, \theta)$ must be as we have assumed. We choose $T$ as in the cited lemma and observe that because the restriction of $\theta$ to the derived group of $G$ is semisimple [St], the image of $Z(F)T^1(F)$ under the natural projection $T \to T_\theta$ is open in $T_\theta(F)$, where $Z$ denotes the center of $G$ and $T^1 = T \cap G^1$. The lemma then follows.

## 5. Transfer

We recall the expected transfer of orbital integrals associated with the basic set of endoscopic data as:

CONJECTURE. Given $f \in C_c^\infty(G(F))$ there exists $f_1 \in G_1(F)$ such that $O_{st}(\gamma, f_1) = O_{st}^\theta(\delta, f)$ if strongly regular $\gamma \in G_1(F)$ is a norm of (strongly $\theta$-regular) $\delta \in G(F)$, and $O_{st}(\gamma, f_1) = 0$ if strongly regular $\gamma \in G_1(F)$ is not a norm.

Here $O_{st}(\gamma, f_1)$ is simply the sum of the integrals of $f_1$ along the conjugacy classes in the stable conjugacy class of $\gamma$, and $O_{st}^\theta(\delta, f)$ is the sum of the integrals of $f$ along the $\theta$-twisted conjugacy classes in the stable $\theta$-twisted conjugacy class of $\theta$. Invariant measures are normalized in the usual way; we will say a little more about this below.

The conjecture is known to be true for archimedean $F$ ([RS]). In the present case, $F$ nonarchimedean and of characteristic zero, it amounts to some familiar problems about the behavior of orbital integrals around the identity; we forgo a more detailed discussion of this. What we will do here is simply to *assume that the conjecture is true near the identity in $G_1(F)$*. This means we have an equality of functions $O_{\mathrm{st}} = O_{\mathrm{st}}^\theta$ on the strongly regular elements around the identity in $G_1(F)$. Here the function $O_{\mathrm{st}}^\theta$ is defined by $O_{\mathrm{st}}^\theta(\gamma) = O_{\mathrm{st}}^\theta(\delta, f)$ if $\gamma$ is a norm of $\delta$, and $O_{\mathrm{st}}^\theta(\gamma) = 0$ if $\gamma$ is not a norm. We remark in passing that the equality is extended, with just a little care, to all regular elements.

## 6. Germ Expansion I

Let $D_1(\gamma)$ be the usual normalizing factor for the (unstabilized) orbital integral $O(\gamma, f_1)$ and $D_G(\delta)$ be that for the $\theta$-twisted orbital integral $O^\theta(\delta, f)$. Then $D_1(\gamma) = D_G(\delta)$ if $\delta$ has norm $\gamma$; this was the assertion $\Delta_{IV} = 1$ in Lemma 1. Because $D_1$ is stably invariant and $D_G$ is stably $\theta$-twisted invariant, we can replace $O_{\mathrm{st}} = O_{\mathrm{st}}^\theta$ by an equality of normalized integrals which we write as $\Phi_{\mathrm{st}} = \Phi_{\mathrm{st}}^\theta$.

Each side of $\Phi_{\mathrm{st}} = \Phi_{\mathrm{st}}^\theta$ has a Shalika germ expansion around the identity in each Cartan subgroup $T_1(F)$ of $G_1(F)$. We shall compare *constant terms (that is,*

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*the contributions from identity elements* $\varepsilon_{G_1}$ on the left and $\varepsilon_G$ on the right) in the case $T_1$ is elliptic, and work on a neighborhood of the identity sufficiently small that all its strongly regular elements are norms of elements in $Z(F)T^1(F)$, with notation as in the proof of Lemma 4 In fact, to shorten arguments we will later assume that the restriction of $\theta$ to $Z$ is semisimple, allowing us, in particular, to omit $Z(F)$ from the last sentence.

Each term in $\Phi_{st}(\gamma)$ is a normalized orbital integral $\Phi(\gamma', f_1)$, where $\gamma'$ is a representative sufficiently close to the identity for a conjugacy class in the stable conjugacy class of $\gamma$. It therefore contributes $c(\gamma')f_1(\varepsilon_{G_1})$ to the constant term in the germ expansion of $\Phi_{st}$, where the constant $c(\gamma')$ depends on the choice of invariant measures defining the orbital integral. By Rogawski's Theorem, which describes the constant explicitly, we can choose measures in such a way as to have $c(\gamma') = c(\gamma)$ (see [K]). We then conclude that the constant term for the expansion of $\Phi_{st}$ is $c_0 f_1(\varepsilon_{G_1})$, where $c_0$ is nonzero.

By definition, $\Phi^{\theta}_{st}(\gamma)$ is the sum, over representatives $\delta$ for the $\theta$-twisted conjugacy classes of elements in $G(F)$ with $\gamma$ as norm, of the normalized $\theta$-twisted orbital integrals $\Phi^{\theta}(\delta, f)$. Some of these elements $\delta$ are near the identity in $G(F)$ and we can immediately do a uniform version of the usual Harish Chandra descent around the identity element $\varepsilon_G$ in $G(F)$ for these $\delta$. For general $\delta$, however, we need some preparation.

## 7. A Stable $\theta$-Twisted Conjugacy Class

Observe that $\varepsilon_G$ is $\theta$-semisimple [KS] since $Int(\varepsilon_G) \circ \theta = \theta$ is a quasi-semisimple automorphism [St]. More general considerations then lead us to define the *stable $\theta$-twisted conjugacy class* of $\varepsilon_G$ to be the set of all elements in $G(F)$ that are $\theta$-twisted conjugate to $\varepsilon_G$ in $G(\overline{F})$, that is, to consist of all elements $\varepsilon$ in $G(F)$ of the form $\varepsilon = g^{-1}\theta(g)$, with $g \in G(\overline{F})$. Then $\mathrm{Int}(g)$ maps $\mathrm{Cent}_\theta(\varepsilon, G)^0$ to $G^1 = \mathrm{Cent}_\theta(\varepsilon_G, G)^0 = (G^\theta)^0$ and moreover:

LEMMA 6. $\mathrm{Int}(g) : \mathrm{Cent}_\theta(\varepsilon, G)^0 \to G^1$ *is an inner twist.*

*Proof.* Let $\sigma \in \Gamma$. Then $g\sigma(g)^{-1}$ is fixed by $\theta$. But because $\theta$ preserves a splitting of $G$ we have that $G^\theta = Z^\theta G^1$ (see [KS, Section 1.1]), where $Z$ is the center of $G$. The lemma then follows.

In general, the $\theta$-twisted conjugacy classes in the stable $\theta$-conjugacy class of $\varepsilon_G$ are parametrized by the classes in $H^1(\Gamma, G^\theta)$ which vanish in $H^1(\Gamma, G)$ under the map given by attaching the cocycle $\sigma \to g\sigma(g)^{-1}$ to $g^{-1}\theta(g)$. In particular, they are finite in number. We shall consider the case in [Sha]. Namely, we assume that *each $\theta$-twisted conjugacy class in the stable $\theta$-twisted conjugacy class of $\varepsilon_G$ contains an element $\varepsilon$ such that* $\mathrm{Cent}_\theta(\varepsilon, G)^0$ *is quasisplit over $F$*. Then all attached cohomology classes have trivial image under the map induced by the projection $G^\theta \to (G^\theta)_{\mathrm{ad}} = G^1_{\mathrm{ad}}$. As we shall see, the $\theta$–twisted conjugacy classes are then parametrized (with multiplicity) by the kernel of $H^1(\Gamma, Z^\theta) \to H^1(\Gamma, Z)$.

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LEMMA 7. *Suppose that $\varepsilon \in G(F)$ is of the form $g^{-1}\theta(g)$, $g \in G(\overline{F})$, and that* $\mathrm{Cent}_\theta(\varepsilon, G)^0$ *is quasisplit over $F$. Then, after replacing $\varepsilon$ by a $\theta$-twisted conjugate element if necessary, we may assume that* $\mathrm{Cent}_\theta(\varepsilon, G)^0$ *coincides with $G^1$ and that both $\varepsilon$ and $g$ are central in $G$.*

*Proof.* Take $\varepsilon$ as in the statement of the lemma. Then $\mathrm{Int}(g) : \mathrm{Cent}_\theta(\varepsilon, G)^0 \to G^1$, an inner twist of quasisplit groups, must be an $F$-isomorphism. Multiplying $g$ on the left by an element of $G^1$, as we may, we can then assume that $\mathrm{Int}(g)$ induces a map between given pairs $(B', T')$ and $(B, T)$ in $G$ such that all four groups $B', T', B, T$ are defined over $F$ and $(B^1 = B \cap G^1, T^1 = T \cap G^1)$ is part of an $F$-splitting for $G^1$. We now multiply $g$ on the right by a suitable element of $G(F)$ and assume that $g$ lies in $T$. But then $N\alpha(\varepsilon) = N\alpha(g^{-1}\theta(g)) = 1$ for all roots $\alpha$ of $T$ in $G$ and so $\mathrm{Cent}_\theta(\varepsilon, G)^0 = G^1$ (see [KS, Section 1.3]).

Second, we multiply $g$ by an element of $T^1$ to assume $\mathrm{Int}(g)$ preserves an $F$-splitting of $G^1$. Then, examining the action of $g$ on root vectors in $G^1$, we find that $\alpha(g) = \alpha(\theta(g))$ for all roots $\alpha$ of $T$ in $G$. Thus $\varepsilon = g^{-1}\theta(g)$ is central in $G$.

The last step is to show that $g$ lies in $G^1Z$. Let $g_{\mathrm{ad}}$ be the image of $g$ under the natural projection of $G$ onto its adjoint group $G_{\mathrm{ad}}$. Then $\theta_{\mathrm{ad}}(g_{\mathrm{ad}}) = g_{\mathrm{ad}}$, where $\theta_{\mathrm{ad}}(g_{\mathrm{ad}}) = (\theta(g))_{\mathrm{ad}}$ as usual, and so $g_{\mathrm{ad}}$ lies in $(G_{\mathrm{ad}})^{\theta_{\mathrm{ad}}}$. But this group is connected (see [KS, Section 1.1]) and so it is the image of $G^1$ under the natural projection. This implies that $g$ lies in $G^1Z$, and so again we can multiply $g$ on the left by an element of $G^1$ to get central $g$ such that $\varepsilon = g^{-1}\theta(g)$. This completes the proof of the lemma.

If we now set

$$Z_1 = Z(F) \cap \{z^{-1}\theta(z) : z \in Z(\overline{F})\} \quad \text{and}$$

$$Z_2 = Z(F) \cap \{g^{-1}\theta(g) : g \in G(F)\}$$

then, arguing as in the lemma, we have that $Z_2$ is contained in $Z_1$. Moreover a set of representatives (*complete and irredundant* will always be assumed in this terminology) for the cosets of $Z_2$ in $Z_1$ provides us with a set of representatives for the $\theta$-twisted conjugacy classes in the stable $\theta$-twisted conjugacy class of the identity element $\varepsilon_G$. We write a representative as $\varepsilon_i = z_i^{-1}\theta(z_i)$, with $z_1 = \varepsilon_1 = \varepsilon_G$.

*Remark* 3. If we want to allow redundancy in counting the $\theta$-twisted conjugacy classes then we can set $Z_3 = \{z^{-1}\theta(z) : z \in Z(F)\}$, so that $Z_3 \subset Z_2$ and $Z_1/Z_3$ is isomorphic to $\mathfrak{K} = Ker(H^1(\Gamma, Z^\theta) \to H^1(\Gamma, Z))$ in the usual manner. The group $\mathfrak{K}$ then yields $[Z_2 : Z_3]$ representatives for each of the $\theta$-twisted conjugacy classes.

## 8. Germ Expansion II

We return to the germ expansion of $\Phi^\theta_{st}(\gamma)$ for strongly regular $\gamma$ sufficiently close to the identity in an elliptic Cartan subgroup $T_1(F)$. As promised, to make the arguments a little shorter we shall assume the restriction of $\theta$ to $Z$ is semisimple.

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Then we can choose strongly $\theta$-regular $\delta$ in $T^1(F)$ near $\varepsilon_G$ with $\gamma$ as norm, and do it in such a way that $\gamma \to \delta$ is smooth. The element $\delta$ is strongly regular in $G^1$. Choose a set of representatives $w_j$ for the conjugacy classes in the stable conjugacy class (no twisting) of $\delta$ in $G^1(F)$, with $w_1 = \varepsilon_G$.

LEMMA 8. *$\{w_j^{-1}\varepsilon_i\delta w_j\varepsilon_i, w_j$ as above $\}$ is a set of representatives for the $\theta$-conjugacy classes of elements in $G(F)$ with norm $\gamma$.*

*Proof.* Suppose $\delta' = w^{-1}\delta\theta(w)$ is an arbitrary $\theta$-twisted stable conjugate of $\delta$. Then $w\sigma(w)^{-1}$ lies in $T^\theta = \mathrm{Cent}_\theta(\delta, G)$, $\sigma \in \Gamma$. Thus $w^{-1}\theta(w)$ lies in $G(F)$ and so is stably $\theta$-twisted conjugate to $\varepsilon_G$. Then there is $g$ in $G(F)$ and some $i$ such that $g^{-1}w^{-1}\theta(w)\theta(g) = \varepsilon_i = z_i^{-1}\theta(z_i)$ and so $w' = wgz_i^{-1}$ lies in $G^\theta = Z^\theta G^1$. Write $w' = zw^1$, accordingly. Then $\delta'$ is $\theta$-twisted conjugate to $(w^1)^{-1}\varepsilon_i\delta w^1$ and moreover $w^1\sigma(w^1)^{-1}$ lies in $T^1$, $\sigma \in \Gamma$. It is now easy to complete the argument that each $\theta$–twisted conjugacy class has a representative as in the statement of the lemma, and check there is no redundancy. Thus the lemma is proved.

We now apply Harish Chandra's Compactness Principle to descend uniformly from $G(F)$ to $G^1(F)$ (see Section 1 of [LS] for similar arguments). This yields functions $f_i \in C_c^\infty(G^1(F))$ such that

$$\sum_j \Phi(w_j^{-1}\delta w_j, f_i) = \sum_j \Phi^\theta(w_j^{-1}\varepsilon_i\delta w_j, f)$$

for each $i$. The left side is a normalized stable orbital integral $\Phi_{\mathrm{st}}(\delta, f_i)$ for $G^1(F)$. Its germ expansion around the identity element has constant term $c_i f_i(\varepsilon_{G^1}) = c_i O^\theta(\varepsilon_i, f)$, where the term on the right is the integral of $f$ along the $\theta$-twisted conjugacy class of $\varepsilon_i$. Rogawski's Theorem again shows that measures can be normalized so that all $c_i$ are the same and nonzero. We then conclude that the constant term in the germ expansion of $\Phi^\theta_{\mathrm{st}}(\gamma)$ is $c_1 \sum_i O^\theta(\varepsilon_i, f)$. This sum is a stable distribution, and so we write it as $O^\theta_{\mathrm{st}}(\varepsilon_G, f)$. To finish our comparison of the constant terms we have:

LEMMA 9. *There is a nonzero constant $c$ such that $c.f_1(\varepsilon_{G_1}) = O^\theta_{\mathrm{st}}(\varepsilon_G, f)$.*

*Remark* 4. A closer look at the various constants shows that we can normalize measures so that $c = 1$ (see [K]). Here we use $q(G_1) = q(G^1)$, where $q(*)$ denotes the $F$-rank of the derived group of $*$.

*Remark* 5. It is no more difficult to handle the general case, that is, to drop the assumption from [Sha] on the structure of the stable $\theta$-twisted conjugacy class of the identity. However, to define $O^\theta_{st}(\varepsilon_G, f)$ we must then insert the sign $(-1)^{q(\varepsilon_i)}$ in front of each term $O^\theta(\varepsilon_i, f)$ before summing, where $q(\varepsilon_i) = q((\mathrm{Cent}_\theta(\varepsilon_i, G))^0)$.

*Remark* 6. An analogous result for $F$ archimedean is shown using a limit formula of Harish Chandra in place of Shalika germs.

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